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MODULI OF EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER AFFINE CONES WITH ONE DIMENSIONAL QUOTIENT

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Introduction

Let G be a reductive complex algebraic group. We consider on the base field $\mathbf{C}$ of complex numbers. Let X be an affine G-variety with a G-fixed base point $x_0 \in X$ and Q be a G-module. We denote by $Vec_G(X,Q)$ the set of algebraic G-vector bundles over X whose fiber at x_0 is Q and by $VEC_G(X,Q)$ the set of G-isomorphism classes in $Vec_G(X,Q)$. The set $VEC_G(X,Q)$ has the distinguished element represented by the product bundle $\Theta_Q := X \times Q$. We denote by $[E]$ the isomorphism class of $E \in Vec_G(X,Q)$.

The study of $VEC_G(X,Q)$ is especially interesting when X is a G-module P (see e.g. [2]). In this case we take the origin as the G-fixed base point. When G is trivial, the Serre conjecture, which was proved by Quillen and Suslin independently, implies that $VEC_G(P,Q)=\{*\}$ (the trivial set consisting of the distinguished element) for any P and Q. However, only few facts are known when G is non-trivial. One approach is to require that the quotient space $P//G$ be of small dimension. It is easy to see that $VEC_G(P,Q)=\{*\}$ if $\dim P//G=0$. But, $VEC_G(P,Q)$ is not trival in genèral. Schwarz [11] (see [5] for the details) has shown that if $\dim P//Q=1$, $VEC_G(P,Q)$ has a structure of finite dimensional vector group and it can be non-trivial. Later, many other families of non-trivial examples have been produced by Knop [4], Masuda-Petrie [9] and Masuda-Moser-Petrie [7] when P has a higher dimensional quotient. However it remains open to classify elements in $VEC_G(P,Q)$ when $\dim P//G \geq 2$.

If $\dim P//G \geq 1$, there is a non-zero point $x \in P$ whose orbit is closed. The closure of the orbit of the line spanned by x is an affine cone with G-action whose quotient is one dimensional (but not necessarily isomorphic to affine line). Masuda-Moser-Petrie [8] noticed that elements of $VEC_G(P,Q)$ can be often distinguished by restricting to the cone. This led them to the notion of *weighted G-cones with smooth one dimensional quotient* (see §1). Note that a G-module with one dimensional quotient is an example of a weighted G-cone with smooth one dimensional quotient.

In this paper, we extend the main results of Schwarz [11] to the case that the base space X is a weighted G-cone with smooth one dimensional quotient, i.e. we prove

Theorem. *Let X be a weighted G-cone with smooth one dimensional quotient and H be a principal isotropy group of X. Let Q, Q_1 and Q_2 be G-modules.*

(1) *$VEC_G(X,Q)\cong(\mathbf{C}^p,+)$ ($\mathbf{C}^p$ as a vector group under addition) for some non-negative integer p. Moreover, there is a G-vector bundle $\mu:\mathfrak{B}\to X\times VEC_G(X,Q)$ such that $\mu^{-1}(X\times[E])\cong E$ for every $E\in Vec_G(X,Q)$.*

(2) *Whitney sum induces an epimorphism of vector groups*

$$WS: VEC_G(X,Q_1)\times VEC_G(X,Q_2)\to VEC_G(X,Q_1\oplus Q_2).$$

If $\mathrm{Hom}(Q_1,Q_2)^H=\{0\}$, *then WS is an isomorphism.*

(3) *Let $E_1,E_2\in Vec_G(X,Q)$. Then $E_1\oplus E_2\cong E_3\oplus\Theta_Q$ where $[E_3]:=[E_1]+[E_2]$.*

(4) *The stabilization map*

$$Stab: VEC_G(X,Q)\to VEC_G(X,Q\oplus Q)$$
$$[E]\mapsto[E\oplus\Theta_Q]$$

is an isomorphism.

Schwarz [11] (or Kraft-Schwarz [5]) proved the theorem above when X is a G-module with one dimensional quotient and basically we follow his argument. However our argument is considerably simplified and made elementary at several points. The key fact to enable it is Equivariant Nakayama Lemma, which implies that $VEC_G(X,Q)\cong VEC_G(Y,Q)$ if Y is a closed G-subvariety of X containing all closed orbits in X. We take Y to be the minimal one among those G-subvarieties. Such Y is called the *closed orbit closure* of X and denoted by X_{cl} (cf. [1]). It turns out that X_{cl} is also a weighted G-cone with smooth one dimensional quotient. The advantage of taking X_{cl} is that the generic fiber F of the quotient map $\pi_{cl}: X_{cl}\to X_{cl}//G\cong A$ is a closed orbit. This fact makes the proofs much simpler.

The organization of this paper is as follows. We define a closed orbit closure in §1 and a weighted G-cone with smooth one dimensional quotient in §2 and discuss their properties. In §3, we show that every G-vector bundle over X_{cl} is trivial when restricted to $X_{cl}-\pi_{cl}^{-1}(0)$. This reduces $VEC_G(X_{cl},Q)$ to the double coset of the group of transition functions. In order to deform the double coset to a calculable form, we prove the decomposition property for $\mathrm{Mor}(F,\mathrm{GL}Q)^G$ (the group of G-equivariant morphisms from F to $\mathrm{GL}Q$) and the approximation property for the semisimple part of $\mathrm{Mor}(F,\mathrm{GL}Q)^G$ in §§4 and 5. These properties are established in [5] in full generality, but thanks to the fact that F is a closed orbit, it suffices to prove them in a special case and we give them rather elementary proofs in that case. The main theorem above is proved in §§6 and 7. In §8, we make an explicit computation of the dimension of $VEC_G(X, Q)$ for an example treated in [8].

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1. Closed orbit closure

Let G be a reductive algebraic group and Z be an affine G-variety (reduced but not necessarily irreducible). We denote by $\mathcal{O}(Z)$ the ring of regular functions on Z and by $\mathcal{O}(Z)^G$ the G-invariant subring of $\mathcal{O}(Z)$. The algebraic quotient space of Z by G, denoted by $Z//G$, is defined to be $\operatorname{Spec}\mathcal{O}(Z)^G$. The algebraic quotient map $\pi: Z \to Z//G$ is defined to be the morphism corresponding to the inclusion $\mathcal{O}(Z)^G \subsetneq \mathcal{O}(Z)$.

DEFINITION ([1]). The minimal closed G-subvariety of Z containing all closed orbits of Z is called *the closed orbit closure* of Z and denoted by Z_{cl}.

REMARK. If $Z//G$ is irreducible, then it follows from Luna's slice theorem [6] that there exist a maximal open dense subset $U \subset Z//G$ and a reductive subgroup $H \subseteq G$ such that the isotropy groups of points of closed orbits in $\pi^{-1}(U)$ are all conjugate to H and $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \to U$ is a G-fiber bundle. The group H is the minimal one among isotropy groups of points of closed orbits in Z up to conjugation. The group H is called a *principal isotropy group* of Z and U is called the *principal stratum* of $Z//G$. We call the fiber over U the *generic fiber* of π. One sees that $Z_{cl} = \overline{G\cdot(\pi^{-1}(U))^H}$. In fact, it is clear that $Z_{cl} \supset \overline{G\cdot(\pi^{-1}(U))^H}$. Since π maps a G-closed set to a closed set ([3, p.96]), $\pi(\overline{G\cdot(\pi^{-1}(U))^H}) \supset \overline{\pi(G\cdot(\pi^{-1}(U))^H)} = \overline{U} = Z//G$. This means that $\overline{G\cdot(\pi^{-1}(U))^H}$ contains all closed orbits. Hence $Z_{cl} = \overline{G\cdot(\pi^{-1}(U))^H}$. Note that a principal isotropy group of Z_{cl} is also H up to conjugation.

Lemma 1.1. *The closed orbit closure Z_{cl} satisfies the following properties*:
(1) *The restriction map $\mathcal{O}(Z)^G \to \mathcal{O}(Z_{cl})^G$ is an isomorphism.*
(2) *If $Z//G$ is irreducible, then the generic fiber of $\pi|_{Z_{cl}}: Z_{cl} \to Z_{cl}//G = Z//G$ is isomorphic to G/H where H is a principal isotropy group of Z.*

Proof. (1) The injectivity follows from the fact that Z_{cl} contains all closed orbits of Z and the surjectivity follows from the fact that Z_{cl} is closed and G is reductive.

(2) Let $(\pi|_{Z_{cl}})^{-1}(\xi)$ be the generic fiber and U be the principal stratum of $Z//G$. Since generic fibers are isomorphic to each other, we may think $\xi \in U$. The

fiber $\pi^{-1}(\xi)$ contains a unique closed orbit Gz such that the isotropy group of $z \in Z$ is H and $\pi^{-1}(\xi)=\{z' \in Z | \overline{Gz'} \ni z\}$. It is clear that $(\pi|_{Z_{cl}})^{-1}(\xi)=\pi^{-1}(\xi) \cap Z_{cl}$ contains Gz. We show that $(\pi|_{Z_{cl}})^{-1}(\xi)=Gz$. From the remark above, $Z_{cl}=\overline{G \cdot (\pi^{-1}(U))^H}$. Hence

$$\begin{aligned}\dim (\pi|_{Z_{cl}})^{-1}(\xi) &= \dim Z_{cl} - \dim Z_{cl} // G \\ &= \dim G \cdot (\pi^{-1}(U))^H - \dim Z // G \\ &= \dim G \cdot (\pi^{-1}(\xi))^H + \dim U - \dim Z // G \\ &= \dim G \cdot (\pi^{-1}(\xi))^H.\end{aligned}$$

Let $z' \in \pi^{-1}(\xi)$ and $z' \notin Gz$. Since $\dim Gz' \gneqq \dim Gz$, the dimension of the isotropy group of z' is strictly smaller than that of H. Thus $(\pi^{-1}(\xi))^H=(Gz)^H$ and $\dim (\pi|_{Z_{cl}})^{-1}(\xi) = \dim G \cdot (\pi^{-1}(\xi))^H = \dim Gz$. While, if $z' \in (\pi|_{Z_{cl}})^{-1}(\xi)$ then $\dim (\pi|_{Z_{cl}})^{-1}(\xi) \gneqq \dim Gz$. This means that $(\pi|_{Z_{cl}})^{-1}(\xi)=Gz \cong G/H$. □

The next lemma is the key fact used in this paper.

Equivariant Nakayama Lemma ([1]). *Let Z be an affine G-variety, $W \subset Z$ be a closed G-subvariety and let E and E' be G-vector bundles over Z.*

(1) *Every G-vector bundle homomorphism $\Phi: E|_W \to E'|_W$ extends to a G-vector bundle homomorphism $\bar{\Phi}: E \to E'$.*

(2) *If W contains all closed orbits of Z and Φ is an isomorphism, then the extension $\bar{\Phi}$ is also an isomorphism.*

Corollary 1.2. *The restriction map $VEC_G(Z,Q) \to VEC_G(Z_{cl},Q)$ is injective for any G-module Q.*

2. Weighted G-cone with smooth one dimensional quotient

Let X be a $G \times \mathbf{C}^*$-affine variety. The $\mathbf{C}^*$-action defines a (integer-valued) grading on $\mathcal{O}(X)$, i.e. we say that $f \in \mathcal{O}(X)$ has degree r iff

$$f(\lambda x)=\lambda^r f(x) \qquad \text{for all} \qquad \lambda \in \mathbf{C}^* \qquad \text{and} \qquad x \in X.$$

DEFINITION ([8]). An affine $G \times \mathbf{C}^*$-variety X is called *a weighted G-cone with smooth one dimensional quotient* if it satisfies the following conditions:

(1) $\mathcal{O}(X)^{\mathbf{C}^*}=\mathbf{C}$ and $\mathcal{O}(X)$ is positively graded with respect to the $\mathbf{C}^*$-action.

(2) $\mathcal{O}(X)^G=\mathbf{C}[t]$ where $t \in \mathcal{O}(X)^G$ is homogeneous.

REMARK. A G-module admits the $\mathbf{C}^*$-action defined by scalar multiplication, which satisfies condition (1). Since a G-module whose quotient is one dimensional satisfies condition (2) (see [5, II]), it is an example of a weighted G-cone with smooth one dimensional quotient.

From now on X will denote a weighted G-cone with smooth one dimensional quotient unless otherwise stated. Condition (2) means that $X//G$ is isomorphic to the affine line $A=\operatorname{Spec}\mathbf{C}[t]$. Through an isomorphism between $X//G$ and A the quotient map $\pi: X \to X//G \cong A$ is nothing but the function t. Let $d := \deg t > 0$. Note that the $\mathbf{C}^*$-action on X induces a $\mathbf{C}^*$-action on $X//G \cong A$ which is d-th power scalar multiplication. It follows from condition (1) that X has a unique closed $\mathbf{C}^*$-ordit, in fact, a $G \times \mathbf{C}^*$-fixed point (see [8, 2.1]), which we denote by x_0.

Lemma 2.1. *For any $x \in X$ such that $t(x) \neq 0$, $\lim_{\lambda \to 0} \lambda x = x_0$ where $\lambda \in \mathbf{C}^*$.*

Proof. Since X has a unique closed $\mathbf{C}^*$-orbit $\{x_0\}$, one easily sees that $\overline{\mathbf{C}^* x} = \mathbf{C}^* x \cup \{x_0\}$. This implies that x_0 equals to $\lim_{\lambda \to 0} \lambda x$ or $\lim_{\lambda \to \infty} \lambda x$. If $x_0 = \lim_{\lambda \to \infty} \lambda x$, then

$$t(x_0) = t(\lim_{\lambda \to \infty} \lambda x) = \lim_{\lambda \to \infty} t(\lambda x) = \lim_{\lambda \to \infty} \lambda^d t(x).$$

Since $d > 0$ and $t(x) \neq 0$, the identity above cannot hold. Hence $x_0 = \lim_{\lambda \to 0} \lambda x$. □

We consider the closed orbit closure X_{cl} of X (as an affine G-variety). Let H be a principal isotropy group of X and $x \in X - \pi_{(0)}^{-1}$ be a point whose isotropy group is H. Then $X_{cl} = \overline{(G \times \mathbf{C}^*)x}$, in particular X_{cl} is a $G \times \mathbf{C}^*$-variety. In fact, X_{cl} is also a weighted G-cone with smooth one dimensional quotient because condition (1) is obviously satisfied and condition (2) follows from Lemma 1.1 (1). We abbreviate the quotient map $\pi|_{X_{cl}}: X_{cl} \to X_{cl}//G \cong A$ by π_{cl}. Let $F = \pi_{cl}^{-1}(1)$, which is a generic fiber. For affine G-varieties (or schemes) Y and Z, we denote by $\operatorname{Mor}(Y, Z)$ the set of morphisms from Y to Z. With this understood

Lemma 2.2.
(1) *$F \cong G/H$.*
(2) *For any G-module V, $\operatorname{Mor}(X_{cl}, V)^G$ is a free $\mathcal{O}(X_{cl})^G$-module of rank $\dim V^H$. Moreover the restriction map $\operatorname{Mor}(X_{cl}, V)^G \to \operatorname{Mor}(F, V)^G \cong V^H$ is surjective.*

Proof. The first statement follows from Lemma 1.1 (2) and the second one is proved in [8, 2.3]. □

3. Triviality over the principal stratum

In this section, we show that every G-vector bundle over X_{cl} is trivial when restricted to $\dot{X}_{cl} := X_{cl} - \pi_{cl}^{-1}(0)$. We identify $X_{cl}//G$ with A so that the induced

$\mathbf{C}^*$-action on $X_{cl}//G=\boldsymbol{A}$ is d-th power scalar multiplication. The group of d-th roots of unity, denoted by Γ, acts trivially on $\boldsymbol{A}$, so the generic fiber $F=\pi_{cl}^{-1}(1)$ is invariant under the Γ-action. Let $\boldsymbol{B}=\operatorname{Spec}\mathbf{C}[s]$ where $t=s^d$. We define a Γ-action on $\boldsymbol{B}$ by scalar multiplication. Then $\boldsymbol{B}/\Gamma=\boldsymbol{A}$. We denote by $\boldsymbol{B}*^{\Gamma}F$ the quotient of $\boldsymbol{B}\times F$ by Γ where $\gamma\in\Gamma$ acts on $\boldsymbol{B}\times F$ by $(b,f)\to(b\gamma,\gamma^{-1}f)$, and define a G-action on $\boldsymbol{B}*^{\Gamma}F$ by $g\cdot[b,f]=[b,gf]$ for $g\in G$. There is a G-morphism $\dot{\boldsymbol{B}}*^{\Gamma}F\to X_{cl}$ mapping $[b,f]$ to bf where $b\in\dot{\boldsymbol{B}}:=\boldsymbol{B}-\{0\}$ is identified with $\mathbf{C}^*$ so that bf makes sense. This can be extended to a G-map $\varphi:\boldsymbol{B}*^{\Gamma}F\to X_{cl}$ by defining $\varphi([0,f])=x_0$.

Lemma 3.1. *The map $\varphi:\boldsymbol{B}*^{\Gamma}F\to X_{cl}$ is a G-morphism which restricts to an isomorphism from $\dot{\boldsymbol{B}}*^{\Gamma}F$ to $\dot{X}_{cl}$.*

Proof. Since $\varphi|_{\dot{\boldsymbol{B}}*^{\Gamma}F}$ is a morphism, to see that φ is a morphism from $\boldsymbol{B}*^{\Gamma}F$, it suffices to show that the image of $\varphi^*:\mathcal{O}(X_{cl})\to\mathcal{O}(\dot{\boldsymbol{B}}*^{\Gamma}F)=(\mathcal{O}(\dot{\boldsymbol{B}})\otimes\mathcal{O}(F))^{\Gamma}$ is contained in $(\mathcal{O}(\boldsymbol{B})\otimes\mathcal{O}(F))^{\Gamma}$. This is equivalent to showing that $\lim_{b\to0}(\varphi^*h)([b,f])$ exists for any $h\in\mathcal{O}(X_{cl})$. From Lemma 2.1 we have

$$\lim_{b\to0}(\varphi^*h)([b,f])=\lim_{b\to0}h(bf)=h(\lim_{b\to0}bf)=h(x_0).$$

Hence φ is a morphism from $\boldsymbol{B}*^{\Gamma}F$ to X_{cl}.

Clearly $\varphi|_{\dot{\boldsymbol{B}}*^{\Gamma}F}:\dot{\boldsymbol{B}}*^{\Gamma}F\to\dot{X}_{cl}$ is a bijective morphism. Note that $\dot{X}_{cl}$ consists of one $G\times\mathbf{C}^*$-orbit, so $\dot{X}_{cl}$ is normal at every point. Therefore $\varphi|_{\dot{\boldsymbol{B}}*^{\Gamma}F}$ is an isomorphism by Richardson's lemma (see [3, p.106]). $\square$

Let $E\in Vec_G(X_{cl},Q)$ and $\tilde{E}$ be the pull-back of E by the map $\boldsymbol{B}\times F\to\boldsymbol{B}*^{\Gamma}F\xrightarrow{\varphi}X_{cl}$. Then $\tilde{E}$ is a $G\times\Gamma$-vector bundle over $\boldsymbol{B}\times F$.

Lemma 3.2. *$\tilde{E}$ is isomorphic to the trivial bundle $\boldsymbol{B}\times F\times Q\to\boldsymbol{B}\times F$ as a G-vector bundle.*

Proof. We identify F with G/H and set $E_0:=\tilde{E}|_{\boldsymbol{B}\times\{eH\}}$. Then $\tilde{E}$ is isomorphic to $G*^{H}E_0$ and E_0 is isomorphic to a trivial H-vector bundle since the base space is a trivial H-module (cf. [2, 2.1]). Let $\Phi:\boldsymbol{B}\times\{eH\}\times Q\cong E_0$ be an H-vector bundle isomorphism over $\boldsymbol{B}\times\{eH\}$. It induces a G-vector bundle isomorphism $\tilde{\Phi}$ over $\boldsymbol{B}\times G/H$

$$\tilde{\Phi}:\boldsymbol{B}\times G/H\times Q\to\tilde{E}\cong G*^{H}E_0$$
$$(b,gH,q)\mapsto g\Phi(b,eH,g^{-1}q).\qquad\square$$

Set $M:=\operatorname{Mor}(F,\mathrm{GL}Q)^G$. We define an action of $\gamma\in\Gamma$ on M by

$$(\gamma m)(f) = m(\gamma^{-1} f) \quad \text{for } m \in M \text{ and } f \in F$$

and on $M(\boldsymbol{B}) := \mathrm{Mor}(\boldsymbol{B}, M)$ by

$$(\gamma\mu)(b) = \gamma(\mu(b\gamma)) \quad \text{for } \mu \in M(\boldsymbol{B}) \text{ and } b \in \boldsymbol{B}.$$

Theorem 3.3. *Let X be a weighted G-cone with smooth one dimensional quotient and Q be a G-module.*

(1) *For every $E \in Vec_G(X_{cl}, Q)$, $E|_{\dot{X}_{cl}}$ is isomorphic to a trivial G-vector bundle.*

(2) *The restriction map $VEC_G(X, Q) \to VEC_G(X_{cl}, Q)$ is bijective.*

Proof. (1) Let $\tilde{E}$ be the same as in Lemma 3.2. By Lemma 3.2, we may assume $\tilde{E} = \boldsymbol{B} \times F \times Q$ as a G-vector bundle. Since $E|_{\dot{X}_{cl}}$ is isomorphic to the quotient of $\tilde{E}|_{\dot{\boldsymbol{B}} \times F}$ by the Γ-action, we investigate the Γ-action on $\tilde{E}|_{\dot{\boldsymbol{B}} \times F} = \dot{\boldsymbol{B}} \times F \times Q$.

The action of $\gamma \in \Gamma$ on $\tilde{E} = \boldsymbol{B} \times F \times Q$ can be expressed as

$$\boldsymbol{B} \times F \times Q \to \boldsymbol{B} \times F \times Q$$
$$(b, f, q) \mapsto (b\gamma, \gamma^{-1} f, (\tilde{h}_\gamma(b)(f))(q))$$

with $\tilde{h}_\gamma \in M(\boldsymbol{B})$. One easily verifies that

$$\tilde{h}_{\gamma\gamma'} = \gamma \tilde{h}_{\gamma'} \tilde{h}_\gamma \quad \text{for} \quad \gamma, \gamma' \in \Gamma.$$

Hence elements $h_\gamma := \tilde{h}_\gamma^{-1}$ satisfy the 1-cocycle condition $h_{\gamma\gamma'} = h_\gamma(\gamma h_{\gamma'})$ and give rise to an element of a group cohomology set $H^1(\Gamma, M(\boldsymbol{B}))$. Since $H^1(\Gamma, M(\dot{\boldsymbol{B}})) = \{*\}$ from [5, IV 5.6], there exists $\phi \in M(\dot{\boldsymbol{B}})$ such that $h_\gamma|_{\dot{\boldsymbol{B}}} = \phi^{-1}(\gamma\phi)$ for all $\gamma \in \Gamma$. Then the map

$$\tilde{E}|_{\dot{\boldsymbol{B}} \times F} = \dot{\boldsymbol{B}} \times F \times Q \to \dot{\boldsymbol{B}} \times F \times Q$$
$$(b, f, q) \mapsto (b, f, (\phi(b)(f))(q))$$

is a $G \times \Gamma$-equivariant vector bundle isomorphism, the Γ-action on Q at the target being trivial. This shows that $E|_{\dot{X}_{cl}}$ is isomorphic to a trivial G-vector bundle.

(2) By Corollary 1.2 it suffices to prove the surjectivity. Let $E \in Vec_G(X_{cl}, Q)$. It is trivial over $\dot{X}_{cl}$ by the above (1) and there is an open neighborhood U of $0 \in A$ such that E is trivial over $\pi_{cl}^{-1}(U)$ ([2, 6.2]). Let ψ be a transition function of E with respect to trivializations over $\dot{X}_{cl}$ and $\pi_{cl}^{-1}(U)$. It can be viewed as an equivariant vector bundle automorphism of the trivial bundle over $\pi_{cl}^{-1}(U) \cap \dot{X}_{cl}$ with fiber Q. Let $\dot{X} := X - \pi^{-1}(0)$. As is easily seen, $\pi^{-1}(U) \cap \dot{X}$ is an affine G-variety and $\pi_{cl}^{-1}(U) \cap \dot{X}_{cl}$ is its closed G-subvariety containing all colsed G-orbits of $\pi^{-1}(U) \cap \dot{X}$; so ψ extends to an equivariant vector bundle automorphism $\bar{\psi}$ over $\pi^{-1}(U) \cap \dot{X}$ by Equivariant Nakayama Lemma. Let $\bar{E}$ be the G-vector bundle over X obtained from $\bar{\psi}$. Clearly $\bar{E}$ restricts to E, proving the surjectivity. □

REMARK. The statement (1) in Theorem 3.3 holds for X (instead of X_{cl}) since the restriction map $VEC_G(\dot{X},Q) \to VEC_G(\dot{X}_{cl},Q)$ is injective by Corollary 1.2.

By virtue of Theorem 3.3 (2) we may take X_{cl} as the base space instead of X. We set

$$\dot{A} = A - \{0\}(= \operatorname{Spec} \mathbf{C}[t,t^{-1}]), \quad \tilde{A} = \operatorname{Spec} \mathbf{C}[t]_0, \quad \tilde{\dot{A}} = \operatorname{Spec} \mathbf{C}(t)$$

where $\mathbf{C}[t]_0$ denotes the localized ring at 0, i.e. $\mathbf{C}[t]_0 = \{f(t)/g(t) \mid f(t),g(t) \in \mathbf{C}[t], g(0) \neq 0\}$ and $\mathbf{C}(t)$ the quotient field of $\mathbf{C}[t]$. Note that $\tilde{\dot{A}}$ is the schematic intersection $\dot{A} \cap \tilde{A}$.

Theorem 3.4.

$$VEC_G(X_{cl},Q) \cong \tilde{D}\mathfrak{P} := \mathfrak{P}(\dot{A}) \backslash \mathfrak{P}(\tilde{\dot{A}}) / \mathfrak{P}(\tilde{A}),$$

where $\mathfrak{P}(Z) := \operatorname{Mor}(Z \times_A X_{cl}, \mathrm{GL}Q)^G$ *for an A-scheme Z.*

Proof. This is a direct result from Theorem 3.3 (1). Let $E \in Vec_G(X_{cl},Q)$. There exist an open neighborhood U of $0 \in A \cong X_{cl}//G$ and a trivialization $\psi_U : E|_{\pi_{cl}^{-1}(U)} \cong \pi_{cl}^{-1}(U) \times Q$ as remarked in the proof of Theorem 3.3 (2). By Theorem 3.3 (1) there is a trivialization $\dot{\psi} : E|_{\dot{X}_{cl}} \cong \dot{X}_{cl} \times Q$. Then $\dot{\psi} \circ \psi_U^{-1}$ defines an element $\tilde{\dot{\alpha}} \in \mathfrak{P}(\tilde{\dot{A}})$ by

$$\dot{\psi} \circ \psi_U^{-1}(x,q) = (x, \tilde{\dot{\alpha}}(x)q) \qquad \text{for} \quad x \in \dot{X}_{cl} \cap \pi_{cl}^{-1}(U),\ q \in Q.$$

Take another open neighborhood V of $0 \in A$ together with a trivialization ψ_V over $\pi_{cl}^{-1}(V)$ and another trivialization $\dot{\psi}'$ over $\dot{X}_{cl}$. Then $\dot{\psi}' \circ \psi_V^{-1}$ defines an element $\tilde{\dot{\alpha}}' \in \mathfrak{P}(\tilde{\dot{A}})$. We also have $\dot{\alpha} \in \mathfrak{P}(\dot{A})$ and $\tilde{\alpha} \in \mathfrak{P}(\tilde{A})$ defined by $\dot{\psi}' \circ \dot{\psi}^{-1}$ and $\psi_U \circ \psi_V^{-1}$, respectively. Then $\tilde{\dot{\alpha}}' = \dot{\alpha}\tilde{\dot{\alpha}}\tilde{\alpha}$ and this proves the theorem. $\square$

Since the morphism $\varphi : \boldsymbol{B} *^{\Gamma} F \to X_{cl}$ is an isomorphism over $\dot{A}$ from Lemma 3.1, it induces the following isomorphisms:

$$\varphi_*^{\cdot} : \mathfrak{P}(\dot{A}) \xrightarrow{\sim} M(\dot{\boldsymbol{B}})^{\Gamma} \qquad \text{and} \qquad \varphi_* : \mathfrak{P}(\tilde{\dot{A}}) \xrightarrow{\sim} M(\tilde{\dot{\boldsymbol{B}}})^{\Gamma}$$

where $\tilde{\dot{\boldsymbol{B}}} = \operatorname{Spec} \mathbf{C}(s)$. Thus we obtain an isomorphism

$$\tilde{D}\mathfrak{P} = \mathfrak{P}(\dot{A}) \backslash \mathfrak{P}(\tilde{\dot{A}}) / \mathfrak{P}(\tilde{A}) \underset{\sim}{\overset{\varphi_*}{\to}} M(\dot{\boldsymbol{B}})^{\Gamma} \backslash M(\tilde{\dot{\boldsymbol{B}}})^{\Gamma} / \varphi_* \mathfrak{P}(\tilde{A}).$$

In the following sections we analyze the latter double coset.

4. The decomposition property

Decompose

$$Q \cong \oplus_{i=1}^{q} n_i W_i \qquad \text{as } H\text{-modules}$$

where W_i are mutually non-isomorphic irreducible H-modules and n_i is the multiplicity of W_i. Then since $F \cong G/H$, we have

$$M = \operatorname{Mor}(F, \mathrm{GL}Q)^G \cong \mathrm{GL}(Q)^H \cong \prod_{i=1}^{q} GL_{n_i}.$$

Note that $M(\boldsymbol{B})$ has a natural grading induced from $\mathcal{O}(\boldsymbol{B})$. We define

$$M(\boldsymbol{B})_r := \{\mu \in M(\boldsymbol{B}) \mid \mu = I + O(s^r)\}$$
$$M(\boldsymbol{B})_r^{\Gamma} := M(\boldsymbol{B})^{\Gamma} \cap M(\boldsymbol{B})_r$$

where I denotes the constant map to the unit element of M. We set

$$\hat{\boldsymbol{B}} = \operatorname{Spec} \mathbf{C}[[s]], \quad \hat{\dot{\boldsymbol{B}}} = \operatorname{Spec} \mathbf{C}((s))$$

where $\mathbf{C}[[s]]$ denotes the ring of formal power series and $\mathbf{C}((s))$ the ring of finite Laurent series. We define $M(\hat{\boldsymbol{B}})_r$ and $M(\hat{\boldsymbol{B}})_r^{\Gamma}$ etc. similarly to $M(\boldsymbol{B})_r$ and $M(\boldsymbol{B})_r^{\Gamma}$. The main purpose of this section is to prove

Theorem 4.1. (The decomposition property).

$$M(\hat{\dot{\boldsymbol{B}}})^{\Gamma} = M(\dot{\boldsymbol{B}})^{\Gamma} M(\hat{\boldsymbol{B}})_1^{\Gamma} \qquad \text{and} \qquad M(\tilde{\dot{\boldsymbol{B}}})^{\Gamma} = M(\dot{\boldsymbol{B}})^{\Gamma} M(\tilde{\boldsymbol{B}})_1^{\Gamma}.$$

First, we show that M has the decomposition property if we forget the Γ-action, i.e.

Proposition 4.2. $M(\hat{\dot{\boldsymbol{B}}}) = M(\dot{\boldsymbol{B}}) M(\hat{\boldsymbol{B}})_1$ *and* $M(\tilde{\dot{\boldsymbol{B}}}) = M(\dot{\boldsymbol{B}}) M(\tilde{\boldsymbol{B}})_1$.

Proof. Since $M(\hat{\boldsymbol{B}}) = M \cdot M(\hat{\boldsymbol{B}})_1$, it suffices to show that $M(\hat{\dot{\boldsymbol{B}}}) = M(\dot{\boldsymbol{B}}) M(\hat{\boldsymbol{B}})$. Furthermore since M is isomorphic to the product of general linear groups, it is sufficient to prove the proposition when $M = GL_n$. We prove that

$$GL_n(\hat{\dot{\boldsymbol{B}}}) = GL_n(\dot{\boldsymbol{B}}) GL_n(\hat{\boldsymbol{B}})$$

by induction on n. Note that an element of $GL_n(\hat{\dot{\boldsymbol{B}}})$ (resp. $GL_n(\dot{\boldsymbol{B}})$, $GL_n(\hat{\boldsymbol{B}})$) is an invertible matrix whose entries are in $\mathbf{C}((s))$ (resp. $\mathbf{C}[[s, s^{-1}]]$, $\mathbf{C}[[s]]$).

The above identity is clear for $n = 1$. Suppose $n \geq 2$. Take $A(s) = (a_{ij}(s)) \in GL_n(\hat{\dot{\boldsymbol{B}}})$ where $a_{ij}(s) \in \mathbf{C}((s))$. By permuting the columns, we may assume that $a_{11}(s)$ is a non-zero finite Laurent series whose order at 0 is the smallest among entries in the first row. Multiplying the first column by an appropriate element of $\mathbf{C}[[s]]$ and adding it to j-th column, we can make $a_{1j}(s)=0$ for $j>1$. This procedure is done by operating $GL_n(\hat{\boldsymbol{B}})$ from the right hand side.

By operating $GL_n(\dot{\boldsymbol{B}})$ from the left hand side, we can make the order of $a_{j1}(s)$ $(2\leq j\leq n)$ at 0 as large as we want without changing $a_{ij}(s)$ $(2\leq i,j\leq n)$. In fact, this can be done by multiplying the first row by an appropriate element of $\mathbf{C}[s,s^{-1}]$ and adding it to j-th row. Applying the induction hypothesis to the matrix $A_{n-1}(s):=(a_{ij}(s))_{2\leq i,j\leq n}$, there exist $B_{n-1}(s)\in GL_{n-1}(\dot{\boldsymbol{B}})$ and $C_{n-1}(s)\in GL_{n-1}(\hat{\boldsymbol{B}})$ such that

$$A_{n-1}(s)=B_{n-1}(s)C_{n-1}(s).$$

Define a column vector $c(s)$ by

$$c(s):=B_{n-1}^{-1}(s)(a_{j1}(s))_{2\leq j\leq n}.$$

From the above observation, we may assume that each entry of $c(s)$ belongs to $\mathbf{C}[[s]]$. Let r be the order of $a_{11}(s)$ at 0 and set

$$B(s):=\begin{pmatrix} s^r & 0 \\ 0 & B_{n-1}(s) \end{pmatrix}\in GL_n(\dot{\boldsymbol{B}}),\quad C(s):=\begin{pmatrix} s^{-r}a_{11}(s) & 0 \\ c(s) & C_{n-1}(s) \end{pmatrix}\in GL_n(\hat{\boldsymbol{B}}).$$

Then one sees $A(s)=B(s)C(s)$.

The identity $M(\tilde{\boldsymbol{B}})=M(\dot{\boldsymbol{B}})M(\tilde{\boldsymbol{B}})_1$ can be proved in a similar way. □

Proof of Theorem 4.1. For any $\hat{\hat{A}}\in M(\hat{\boldsymbol{B}})^\Gamma$ there exist $\dot{A}\in M(\dot{\boldsymbol{B}})$ and $\hat{A}\in M(\hat{\boldsymbol{B}})_1$ such that $\hat{\hat{A}}=\dot{A}\hat{A}$ by Proposition 4.2. Define a map $\tilde{A}:\Gamma\to M(\boldsymbol{B})$ by $\tilde{A}(\gamma)=\dot{A}^{-1}(\gamma\dot{A})$ for $\gamma\in\Gamma$. Clearly $\tilde{A}$ satisfies the 1-cocycle condition. Since $\hat{\hat{A}}$ is Γ-invariant,

$$\tilde{A}(\gamma)=\dot{A}^{-1}(\gamma\dot{A})=\hat{A}(\gamma\hat{A})^{-1}\in M(\dot{\boldsymbol{B}})\cap M(\hat{\boldsymbol{B}})_1=M(\boldsymbol{B})_1.$$

Thus, $\tilde{A}$ defines an element of $H^1(\Gamma,M(\boldsymbol{B})_1)$. Since $H^1(\Gamma,M(\boldsymbol{B})_1)=\{*\}$ ([5, IV 6.3]), there exists $A\in M(\boldsymbol{B})_1$ such that $\tilde{A}(\gamma)=A^{-1}(\gamma A)$. Thus, $\dot{A}A^{-1}\in M(\dot{\boldsymbol{B}})^\Gamma$ and $A\hat{A}\in M(\hat{\boldsymbol{B}})_1^\Gamma$. Hence $\hat{\hat{A}}=(\dot{A}A^{-1})(A\hat{A})\in M(\dot{\boldsymbol{B}})^\Gamma M(\hat{\boldsymbol{B}})_1^\Gamma$.

The identity $M(\tilde{\boldsymbol{B}})^\Gamma=M(\dot{\boldsymbol{B}})^\Gamma M(\tilde{\boldsymbol{B}})_1^\Gamma$ can be proved in a similar way. □

Finally we make an observation on the Γ-action on M, which will be used in the next section. Take a point $f_0\in F$ whose isotropy group is H. Evaluation at f_0 gives an isomorphism

$$\Psi: M=\mathrm{Mor}(F,\mathrm{GL}Q)^G\to\mathrm{GL}(Q)^H.$$

Recall that the action of $\gamma\in\Gamma$ on M is given by $m\to m\circ\gamma^{-1}$. Since the Γ-action on F is G-equivariant and the isotropy group of f_0 is H, $\gamma^{-1}f_0=gf_0$ with some element g in the normalizer of H in G. Therefore we have

$$(\gamma m)(f_0)=m(\gamma^{-1}f_0)=m(gf_0)=\rho(g)m(f_0)\rho(g)^{-1}$$

where $\rho:G\to \mathrm{GL}Q$ is the homomorphism (i.e. rational representation) associated with Q. This shows that the action of γ on M corresponds to the conjugation by $\rho(g)\in \mathrm{GL}Q$ on $\mathrm{GL}(Q)^H$ through the above isomorphism Ψ. Note that the conjugation by $\rho(g)^d$ is the identity since $\gamma^d=1$.

Remember the decomposition $Q\cong\oplus_{i=1}^q n_iW_i$ as H-modules. If $q=1$ i.e. $Q\cong nW$ for some irreducible H-module W, then $M\cong GL_n$.

Lemma 4.3. *Suppose $Q\cong nW$ as H-modules. Then the action of $\gamma\in\Gamma$ on $M\cong GL_n$ is equivalent to the conjugation by a diagonal matrix of GL_n with elements of d-th roots of unity.*

Proof. We choose $f_0\in F$ and fix the isomorphism $\Psi:M=\mathrm{Mor}(F,\mathrm{GL}Q)^G\to \mathrm{GL}(Q)^H$ and $g\in G$ such that $\gamma^{-1}f_0=gf_0$ for a generator $\gamma\in\Gamma$. Furthermore, we fix an H-equivariant isomorphism $\phi:Q\to nW$ and identify $\mathrm{GL}(Q)^H$ with $\mathrm{GL}(nW)^H\cong GL_n$ through the isomorphism ϕ. The γ-action on $\mathrm{GL}(nW)^H$ corresponds to the conjugation by $\tilde\rho(g):=\phi\rho(g)\phi^{-1}\in \mathrm{GL}(nW)$. Hence the γ-action is an automorphism of $\mathrm{GL}(nW)^H\cong GL_n$ which fixes the center of GL_n. It is known that $\mathrm{Aut}(GL_n)/\mathrm{Int}(GL_n)\cong \boldsymbol{Z}/2\boldsymbol{Z}$ $(n\ge 2)$ and the non-trivial element is represented by $\iota\in\mathrm{Aut}(GL_n)$ where $\iota(A)={}^tA^{-1}$ for $A\in GL_n$ (cf. [10, p.298]). However, ι is not identity on the center of GL_n, thus the γ-action on GL_n is an inner automorphism of GL_n. Hence we may think of $\tilde\rho(g)$ as an element of GL_n.

Since the conjugation by $\tilde\rho(g)^d$ is the identity, $\tilde\rho(g)^d$ is a scalar matrix. Hence there is $S\in GL_n$ such that $S\tilde\rho(g)S^{-1}$ is diagonal and the i-th diagonal entry of $S\tilde\rho(g)S^{-1}$ is written as $\lambda_i\alpha$, where λ_i is a d-th root of unity and α is a complex number independent of i. The conjugation by $\tilde\rho(g)$ is equivalent to that by $diag(\lambda_1,\cdots,\lambda_n)$ so the lemma has been proved. □

5. The approximation property

Let M' be the commutator subgroup of M, which is the semisimple part of M and isomorphic to $\Pi_i SL_{n_i}$. Note that M' is invariant under the Γ-action. In this section we prove the approximation property for M' and deduce a few consequences from it.

Theorem 5.1. (The approximation property).

$$M'(\hat{\boldsymbol{B}})_1^\Gamma=M'(\boldsymbol{B})_1^\Gamma M'(\hat{\boldsymbol{B}})_r^\Gamma \qquad \textit{for all } r\geqq 1.$$

The interaction between Lie groups and Lie algebras is necessary to prove the theorem above. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the Lie algebras of M and M' respectively. Then $\mathfrak{m}=\mathrm{Mor}(F,\mathrm{End}Q)^G\cong \mathrm{End}(Q)^H\cong\oplus_i\mathfrak{gl}_{n_i}$ and $\mathfrak{m}'\cong\oplus_i\mathfrak{sl}_{n_i}$. Note that $\mathfrak{m}'$ is Γ-invariant. The key result to prove Theorem 5.1 is

Lemma 5.2. *Let $r \geq 1$ and let $A \in \mathfrak{m}'$ such that $s^r A \in \mathfrak{m}'(\boldsymbol{B})^\Gamma$. Then there exists $g(s) \in M'(\boldsymbol{B})_1^\Gamma$ such that $g(s) = I + s^r A + O(s^{r+1})$.*

We take this lemma for granted for a while and prove Theorem 5.1.

Proof of Theorem 5.1. It suffices to show that for any $\hat{g}(s) \in M'(\hat{\boldsymbol{B}})_r^\Gamma$, $r \geq 1$, there exists $g(s) \in M'(\boldsymbol{B})_1^\Gamma$ such that $g(s)^{-1}\hat{g}(s) \in M'(\hat{\boldsymbol{B}})_{r+1}^\Gamma$. Write $\hat{g}(s) = I + s^r A + O(s^{r+1}) \in M'(\hat{\boldsymbol{B}})_r^\Gamma$. Then $A \in \mathfrak{m}'$ and $s^r A \in \mathfrak{m}'(\boldsymbol{B})^\Gamma$. Hence, the theorem follows from Lemma 5.2. $\square$

Proof of Lemma 5.2. Let γ be a generator of Γ. We may reduce to the case where $\mathfrak{m}' = \mathfrak{m}'_1 \oplus \cdots \oplus \mathfrak{m}'_l$, $\mathfrak{m}'_i \cong \mathfrak{sl}_n$ for each i and $\gamma\mathfrak{m}'_1 = \mathfrak{m}'_2, \cdots, \gamma\mathfrak{m}'_{l-1} = \mathfrak{m}'_l$, $\gamma\mathfrak{m}'_l = \mathfrak{m}'_1$. Thus γ^l preserves each $\mathfrak{m}'_i$. We consider two cases.

Case (1) $l=1$.

In this case, $\mathfrak{m}' \cong \mathfrak{sl}_n$ and $M' \cong SL_n$. In the following we identify $\mathfrak{m}'$ with $\mathfrak{sl}_n$ and M' with SL_n. There is a standard decomposition $\mathfrak{sl}_n = \mathfrak{t}_n \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$ where $\mathfrak{t}_n$ is the maximal toral subalgebra of $\mathfrak{sl}_n$ consisting of diagonal matrices with trace zero and $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) is the nilpotent subalgebra of $\mathfrak{sl}_n$ consisting of upper (resp. lower) triangular matrices with zero diagonal entries. By Lemma 4.3 we may assume that the induced action of $\gamma \in \Gamma$ on $\mathfrak{m}'$ is conjugation by a diagonal matrix. Hence the Γ-action on $\mathfrak{t}_n$ is trivial and $\mathfrak{u}^\pm$ are Γ-invariant.

Given $A \in \mathfrak{sl}_n$ such that $s^r A$ is Γ-invariant, we decompose $A = A_0 + A_+ + A_-$ where $A_0 \in \mathfrak{t}_n$, $A_+ \in \mathfrak{u}^+$, $A_- \in \mathfrak{u}^-$. Since

$$\exp(s^r A_+)\exp(s^r A_-) = I + s^r(A_+ + A_-) + O(s^{r+1}) \in SL_n(\boldsymbol{B})_1^\Gamma,$$

we may reduce to the case where $A \in \mathfrak{t}_n$. Furthermore, we may reduce to the case where $\mathfrak{m}' = \mathfrak{sl}_2$ and $A \in \mathfrak{t}_2$ since $\mathfrak{t}_n$ is isomorphic to a direct sum of $\mathfrak{t}_2 \subset \mathfrak{sl}_2$.

Let $A = \alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{t}_2$, where α is a scalar. By Lemma 4.3 we may assume that the action of γ on $\mathfrak{sl}_2$ is the conjugation by a diagonal 2×2 matrix with diagonal entries λ_1 and λ_2 where λ_i are d-th roots of unity. From the Γ-invariance of $s^r A$, we have $d \mid r$. Set

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then σ_1 and σ_2 are nilpotent, $[\sigma_1, \sigma_2] = \sigma_3$ and the γ-actions on them are:

$$\gamma\sigma_1 = \lambda_1\lambda_2^{-1}\sigma_1, \quad \gamma\sigma_2 = \lambda_1^{-1}\lambda_2\sigma_2, \quad \gamma\sigma_3 = \sigma_3.$$

If $r \neq d$ or the γ-actions on σ_1 and σ_2 are not trivial (i.e. $\lambda_1 \neq \lambda_2$), then there are positive integers a, b such that $s^a\sigma_1$, $s^b\sigma_2$ are Γ-invariant and $[s^a\sigma_1, s^b\sigma_2] = s^r\sigma_3$ since $\lambda_i^d = 1$ and $d \mid r$. Hence

$$g(s) := \exp(\alpha s^a\sigma_1)\exp(s^b\sigma_2)\exp(-\alpha s^a\sigma_1)\exp(-s^b\sigma_2)$$

is in $SL_2(\boldsymbol{B})_1^\Gamma$ and $g(s) = I + \alpha s^r\sigma_3 + O(s^{r+1})$, so it is the desired element. If $r = d$ and the γ-actions on σ_1 and σ_2 are trivial, then one can easily check that

$$g(s) := \exp(-\alpha s^r\sigma_2)\exp(\alpha s^r\sigma_1)\exp(\sigma_2)\exp(-\alpha s^r\sigma_1)\exp(-\sigma_2)$$

is the desired element.

Case (2) $l \geq 2$.

In this case, $d = kl$ for some positive integer k and each $\mathfrak{m}_i' \cong \mathfrak{sl}_n$ is stable under the action of $\Gamma^l := \{\gamma^{jl} \mid j = 0, 1, \cdots, k-1\}$. Let $A = A_1 \oplus \cdots \oplus A_l$ where $A_i \in \mathfrak{m}_i' \cong \mathfrak{sl}_n$. Since s^rA is Γ-invariant, s^rA_1 is Γ^l-invariant. It follows from Case (1) that we can find $g_1(s) \in SL_n(\boldsymbol{B})_1^{\Gamma^l}$ such that $g_1(s) = I + s^rA_1 + O(s^{r+1})$. Then $g(s) = \Pi_{i=1}^l g_i(s)$, where $g_{i+1}(s) = (\gamma g_i)(s)$ for $1 \leq i \leq l-1$, is the desired element. $\square$

Denote the canonical map $M \to M/M'$ by τ. Since $M \cong \Pi_{i=1}^q GL_{n_i}$ and $M' \cong \Pi_{i=1}^q SL_{n_i}$, $M/M' \cong (\mathbf{C}^*)^q$ and τ is viewed as the determinant map on each factor GL_{n_i}. Let Z be the center of M. Then Z is isomorphic to $(\mathbf{C}^*)^q$ and the map τ restricted to Z induces an isomorphism of the Lie algebras. Note that Z is invariant under the Γ-action. We define $\mathfrak{m}(\hat{\boldsymbol{B}})_r^\Gamma := \mathfrak{m}(\hat{\boldsymbol{B}})^\Gamma \cap \mathfrak{m}(\hat{\boldsymbol{B}})_r$ where $\mathfrak{m}(\hat{\boldsymbol{B}})_r = \{\mu \in \mathfrak{m}(\hat{\boldsymbol{B}}) \mid \mu = O(s^r)\}$. Similar definition applies to $\mathfrak{m}'$ and $\mathfrak{m}/\mathfrak{m}'$.

Proposition 5.3. *For $r \geq 1$, there is a commutative diagram of split exact sequences:*

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{m}'(\hat{\boldsymbol{B}})_r^\Gamma & \to & \mathfrak{m}(\hat{\boldsymbol{B}})_r^\Gamma & \xrightarrow{\tau_\#} & (\mathfrak{m}/\mathfrak{m}')(\hat{\boldsymbol{B}})_r^\Gamma & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & M'(\hat{\boldsymbol{B}})_r^\Gamma & \to & M(\hat{\boldsymbol{B}})_r^\Gamma & \xrightarrow{\tau_*} & (M/M')(\hat{\boldsymbol{B}})_r^\Gamma & \to & 1
\end{array}$$

where τ induces $\tau_\#$ and τ_*, and the vertical maps are isomorphisms induced from exponential maps. Moreover $M(\hat{\boldsymbol{B}})_r^\Gamma = M'(\hat{\boldsymbol{B}})_r^\Gamma Z(\hat{\boldsymbol{B}})_r^\Gamma$.

Proof. Exactness of the upper sequence, commutativity of the diagram and isomorphisms of exponential maps are clear. The existence of a splitting map of $\tau_\#$ follows from the fact that the canonical map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}' \cong \mathbf{C}^q$ is an isomorphism on the Lie algebra $\mathfrak{z}$ of the center Z. This implies that τ_* also has a splitting and $\tau_\#$ is an isomorphism on $\mathfrak{z}(\hat{\boldsymbol{B}})_r^\Gamma$. Since $\mathfrak{z}(\hat{\boldsymbol{B}})_r^\Gamma \cong Z(\hat{\boldsymbol{B}})_r^\Gamma$ via the exponential map, it follows that τ_* is an isomorphism on $Z(\hat{\boldsymbol{B}})_r^\Gamma$. Thus exactness of the lower sequence and the last statement follow. $\square$

Lemma 5.4. *$M(\hat{B})_1^\Gamma = M(\tilde{B})_1^\Gamma M(\hat{B})_r^\Gamma$ for all $r \geq 1$.*

Proof. From Proposition 5.3, $M(\hat{B})_r^\Gamma = M'(\hat{B})_r^\Gamma Z(\hat{B})_r^\Gamma$ for $r \geq 1$. Since M' has the approximation property, we reduce to the case where $M = Z \cong (\mathbf{C}^*)^q$.

Let $\hat{z} = (z_1(s), \cdots, z_q(s)) \in Z(\hat{B})_1^\Gamma$ where $z_i(s) = 1 + \Sigma_{j=1}^{r-1} a_{ij} s^j + O(s^r)$ $(a_{ij} \in \mathbf{C})$ for $1 \leq i \leq q$. Define $\tilde{z} = (\tilde{z}_1(s), \cdots, \tilde{z}_q(s))$ by $\tilde{z}_i(s) = 1 + \Sigma_{j=1}^{r-1} a_{ij} s^j$ for $1 \leq i \leq q$. Since the action of $\gamma \in \Gamma$ on $M \cong \mathrm{GL}(Q)^H$ is a conjugation by an element of $\mathrm{GL}Q$, the Γ-action preserves the grading of $Z(\hat{B}) \subset M(\hat{B})$. Hence $\tilde{z} \in Z(\tilde{B})_1^\Gamma$ and $\tilde{z}^{-1}\hat{z} \in Z(\hat{B})_r^\Gamma$. $\square$

Lemma 5.5. *For $r \geq 1$, $M(\hat{B})_r^\Gamma$ and $M(B)_1^\Gamma M(\hat{B})_r^\Gamma$ are both normal subgroups of $M(\hat{B})_1^\Gamma$.*

Proof. It is easy to see that $M(\hat{B})_r^\Gamma$ is normal, so we prove that $M(B)_1^\Gamma M(\hat{B})_r^\Gamma$ is normal. From Proposition 5.3, $M(\hat{B})_1^\Gamma = M'(\hat{B})_1^\Gamma Z(\hat{B})_1^\Gamma$. On the other hand $M(B)_1^\Gamma M(\hat{B})_r^\Gamma \supset M'(B)_1^\Gamma M'(\hat{B})_r^\Gamma = M'(\hat{B})_1^\Gamma$ by Theorem 5.1. Since $Z(\hat{B})_1^\Gamma$ is the center of $M(\hat{B})_1^\Gamma$ and $M'(\hat{B})_1^\Gamma$ is contained in $M(B)_1^\Gamma M(\hat{B})_r^\Gamma$, it follows that $M(B)_1^\Gamma M(\hat{B})_r^\Gamma$ is a normal subgroup of $M(\hat{B})_1^\Gamma$. $\square$

6. Moduli of vector bundles

In this section we analyze the set $VEC_G(X, Q) \cong VEC_G(X_{cl}, Q)$ using the results in the previous sections, in particular we prove the Theorem (1) in the introduction.

Let $\mathfrak{E}(A) := \mathrm{Mor}(X_{cl}, \mathrm{End}Q)^G$. It is a free $\mathcal{O}(A)$-module of rank $\dim \mathrm{End}(Q)^H = \dim \mathfrak{m}$ by Lemma 2.2 (2). Note that the map $\varphi: B *^\Gamma F \to X_{cl}$ induces an $\mathcal{O}(A)$-module homomorphism $\varphi_\#: \mathfrak{E}(A) \to \mathfrak{m}(B)^\Gamma$.

Proposition 6.1. *Let $\{A_i\}$ $(1 \leq i \leq \dim \mathfrak{m})$ be a homogeneous basis of $\mathfrak{E}(A)$ over $\mathcal{O}(A)$ and let $A_i' := A_i|_F \in \mathfrak{m} = \mathrm{Mor}(F, \mathrm{End}Q)^G$. Write $\deg A_i = k_i d + a_i$ where $0 \leq a_i < d$. Then*

(1) *$\{s^{a_i} A_i'\}$ is an $\mathcal{O}(A)$-module basis of $\mathfrak{m}(B)^\Gamma$.*

(2) $\varphi_\# A_i = t^{k_i}(s^{a_i} A_i')$.

Hence $\varphi_\#: \mathfrak{E}(A) \to \mathfrak{m}(B)^\Gamma$ is an injection of free $\mathcal{O}(A)$-modules and is of full rank.

Proof. (1) The set $\{A_i'\}$ is a basis of $\mathfrak{m}$ over $\mathbf{C}$ by Lemma 2.2 (2). Since $s^r A_i' \in \mathfrak{m}(B)^\Gamma$ if and only if $r \equiv a_i \bmod d$, any element of $\mathfrak{m}(B)^\Gamma$ is a linear combination of $s^{a_i} A_i'$ over $\mathcal{O}(A)$. Suppose that $\Sigma_i f_i(t) s^{a_i} A_i' = 0$ for $f_i(t) \in \mathcal{O}(A)$, where we may assume $f_i(t)$ are homogeneous. Then $\Sigma_i f_i(1) A_i' = 0$ by evaluating the identity at $s = 1$. Since the set $\{A_i'\}$ is a basis of $\mathfrak{m}$, $f_i(1) = 0$ for all i and hence $f_i(t)$ are identically 0 since they are homogeneous.

(2) For $b \in \dot{\boldsymbol{B}}$, $f \in F$, we have

$$(\varphi_\# A_i)(b)(f) = A_i(bf) = b^{\deg A_i} A_i(f) = b^{dk_i + a_i} A_i'(f) = (t^{k_i} s^{a_i} A_i')(b)(f).$$

This proves that $\varphi_\# A_i = t^{k_i}(s^{a_i} A_i')$ on $\dot{\boldsymbol{B}}$ and bence on $\boldsymbol{B}$ by continuity. $\square$

Let $\mathfrak{E}(\hat{A}) := \mathrm{Mor}(\hat{A} \times_A X_{cl}, \mathrm{End} Q)^G$. Since

$$\mathfrak{E}(\hat{A}) \cong \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} (\mathcal{O}(X_{cl}) \otimes \mathrm{End} Q)^G (\cong \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} \mathfrak{E}(A)),$$

it inherits a grading from $\mathcal{O}(X_{cl})$. Let $\mathfrak{E}(\hat{A})_r$ be the ideal of $\mathfrak{E}(\hat{A})$ generated by homogeneous elements of degree $\geq r$. Note that $\mathfrak{P}(\hat{A}) = \mathrm{Mor}(\hat{A} \times_A X_{cl}, \mathrm{GL} Q)^G$ is a subset of $\mathfrak{E}(\hat{A})$ and set $\mathfrak{P}(\hat{A})_r := \{A \in \mathfrak{P}(\hat{A}) \mid A - I \in \mathfrak{E}(\hat{A})_r\}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{P}(\hat{A})_r & \xrightarrow{\varphi_*} & M(\hat{\boldsymbol{B}})_r^\Gamma \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{E}(\hat{A})_r & \xrightarrow[\varphi_\#]{} & \mathfrak{m}(\hat{\boldsymbol{B}})_r^\Gamma \end{array}$$

where the vertical maps are isomorphisms incuced from the exponential map $\mathrm{End} Q \to \mathrm{GL} Q$.

Lemma 6.2.

(1) *For any sufficiently larger r we have $\varphi_* \mathfrak{P}(\hat{A})_r = M(\hat{\boldsymbol{B}})_r^\Gamma$, in particular $\varphi_* \mathfrak{P}(\hat{A})_1 \supset M(\hat{\boldsymbol{B}})_r^\Gamma$.*

(2) $M(\boldsymbol{B})_1^\Gamma \varphi_* \mathfrak{P}(\hat{A})_1 \supset M'(\hat{\boldsymbol{B}})_1^\Gamma$.

Proof. (1) Since $\mathfrak{E}(\hat{A}) = \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} \mathfrak{E}(A)$ and $\mathfrak{m}(\hat{\boldsymbol{B}})^\Gamma = \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} \otimes \mathfrak{m}(\boldsymbol{B})^\Gamma$, it follows from Proposition 6.1 that $\varphi_\# \mathfrak{E}(\hat{A})_r = \mathfrak{m}(\hat{\boldsymbol{B}})_r^\Gamma$ for any sufficiently large r. This together with the above diagram proves (1).

(2) It follows from (1) that $M(\boldsymbol{B})_1^\Gamma \varphi_* \mathfrak{P}(\hat{A})_1 \supset M(\boldsymbol{B})_1^\Gamma M(\hat{\boldsymbol{B}})_r^\Gamma$ for a sufficiently large r. On the other hand $M'(\hat{\boldsymbol{B}})_1^\Gamma = M'(\boldsymbol{B})_1^\Gamma M'(\hat{\boldsymbol{B}})_r^\Gamma$ for any $r \geq 1$ by Theorem 5.1. Hence (2) follows. $\square$

Remember that

$$VEC_G(X_{cl}, Q) \cong \tilde{D}\mathfrak{P} = \mathfrak{P}(\dot{A}) \backslash \mathfrak{P}(\widetilde{\hat{A}}) / \mathfrak{P}(\tilde{A}).$$

Proposition 6.3. *The canonical map*

$$\tilde{D}\mathfrak{P} = \mathfrak{P}(\dot{A}) \backslash \mathfrak{P}(\widetilde{\hat{A}}) / \mathfrak{P}(\tilde{A}) \to \mathfrak{P}(\dot{A}) \backslash \mathfrak{P}(\hat{A}) / \mathfrak{P}(\hat{A}) = D\mathfrak{P}$$

is a bijection.

Proof. The injectivity is easy. We show the surjectivity. Remember that

$$\tilde{D}\mathfrak{P} = \mathfrak{P}(\dot{A})\backslash\mathfrak{P}(\tilde{\hat{A}})/\mathfrak{P}(\tilde{A}) \underset{\sim}{\overset{\varphi_*}{\rightarrow}} M(\dot{B})^\Gamma\backslash M(\tilde{\hat{B}})^\Gamma/\varphi_*\mathfrak{P}(\tilde{A}).$$

Since M has the decomposition property (Theorem 4.1), the latter is isomorphic to

$$M(\dot{B})^\Gamma \cap M(\tilde{B})_1^\Gamma\backslash M(\tilde{B})_1^\Gamma/\varphi_*\mathfrak{P}(\tilde{A}) \cap M(\tilde{B})_1^\Gamma \cong M(B)_1^\Gamma\backslash M(\tilde{B})_1^\Gamma/\varphi_*\mathfrak{P}(\tilde{A})_1.$$

Similarly, $D\mathfrak{P} \cong M(B)_1^\Gamma\backslash M(\hat{B})_1^\Gamma/\varphi_*\mathfrak{P}(\hat{A})_1$. Thus the canonical inclusion $\tilde{D}\mathfrak{P} \subsetneq D\mathfrak{P}$ reduces to an inclusion

$$M(B)_1^\Gamma\backslash M(\tilde{B})_1^\Gamma/\varphi_*\mathfrak{P}(\tilde{A})_1 \subsetneq M(B)_1^\Gamma\backslash M(\hat{B})_1^\Gamma/\varphi_*\mathfrak{P}(\hat{A})_1.$$

Here $M(\hat{B})_1^\Gamma = M(\tilde{B})_1^\Gamma M(\hat{B})_r^\Gamma$ for all $r \geq 1$ by Lemma 5.4 and $\varphi_*\mathfrak{P}(\hat{A})_1 \supset M(\hat{B})_r^\Gamma$ for a sufficiently large r by Lemma 6.2 (1). This implies the surjectivity. □

Proposition 6.4.

$$\begin{aligned} D\mathfrak{P} &\cong (M/M')(\hat{B})_1^\Gamma/\tau_*\varphi_*\mathfrak{P}(\hat{A})_1 \\ &\cong (\mathfrak{m}/\mathfrak{m}')(\hat{B})_1^\Gamma/\tau_\#\varphi_\#\mathfrak{E}(\hat{A})_1 \\ &\cong (\mathbf{C}^p, +) \ (\mathbf{C}^p \textit{ as a vector group under addition}). \end{aligned}$$

Proof. It follows from the proof of Proposition 6.3 that

$$D\mathfrak{P} \cong M(B)_1^\Gamma\backslash M(\hat{B})_1^\Gamma/\varphi_*\mathfrak{P}(\hat{A})_1.$$

For a sufficiently large r, this double coset is isomorphic to

$$\begin{aligned} & M(B)_1^\Gamma\backslash M(\hat{B})_1^\Gamma/M(\hat{B})_r^\Gamma\varphi_*\mathfrak{P}(\hat{A})_1 && \text{(by 6.2 (1))} \\ \cong\ & M(B)_1^\Gamma M(\hat{B})_r^\Gamma\backslash M(\hat{B})_1^\Gamma/\varphi_*\mathfrak{P}(\hat{A})_1 && \text{(by 5.5)} \\ \cong\ & M(\hat{B})_1^\Gamma/M(B)_1^\Gamma M(\hat{B})_r^\Gamma\varphi_*\mathfrak{P}(\hat{A})_1 && \text{(by 5.5)} \\ \cong\ & M(\hat{B})_1^\Gamma/M(B)_1^\Gamma\varphi_*\mathfrak{P}(\hat{A})_1 && \text{(by 6.2 (1))} \\ \cong\ & (M(\hat{B})_1^\Gamma/M'(\hat{B})_1^\Gamma)/[(M(B)_1^\Gamma\varphi_*\mathfrak{P}(\hat{A})_1)/M'(\hat{B})_1^\Gamma] && \text{(by 6.2 (2))} \\ \cong\ & (M/M')(\hat{B})_1^\Gamma/\tau_*\varphi_*\mathfrak{P}(\hat{A})_1. \end{aligned}$$

In the last isomorphism we use Proposition 5.3 and the fact that τ_* is nothing but the determinant map on each factor so that τ_* is trivial on $M(B)_1^\Gamma$.

Since the exponential map induces an isomorphism $(\mathfrak{m}/\mathfrak{m}')(\hat{B})_1^\Gamma \tilde{\rightarrow} (M/M')(\hat{B})_1^\Gamma$ and τ and φ commute with the exponential maps, we have

$$(M/M')(\hat{B})_1^\Gamma/\tau_*\varphi_*\mathfrak{P}(\hat{A})_1 \cong (\mathfrak{m}/\mathfrak{m}')(\hat{B})_1^\Gamma/\tau_\#\varphi_\#\mathfrak{E}(\hat{A})_1.$$

The latter set has a natural vector group structure induced from $(\mathfrak{m}/\mathfrak{m}')(\hat{\boldsymbol{B}})_1^\Gamma$ and it is finite dimensional by Proposition 6.1, thus it is isomorphic to $\mathbf{C}^p$ for some p. □

The reader will find that the former part of the Theorem (1) in the introduction will follow from Theorems 3.3 (2), 3.4 and Propositions 6.3, 6.4.

Here is a formula to compute the dimension p in Proposition 6.4.

Theorem 6.5. *Let B_i $(1\le i\le q)$ be homogeneous elements of $\mathfrak{E}(A)_1$ which project to a basis of $\mathfrak{m}/\mathfrak{m}'$ and have minimal degrees possible. Then $p=\Sigma_{i=1}^q[(\deg B_i-1)/d]$.*

REMARK. We note that $n_i=1$ for all i (see §4 for n_i) if and only if $\mathfrak{m}'=0$, and in this case $q=\dim\mathfrak{m}$. The condition that $n_i=1$ for all i is called in [7] that Q is multiplicity free with respect to H.

Proof. Write $\deg B_i=k_id+b_i$ where $0<b_i\le d$. Then $s^{b_i}B_i'$ (where $B_i'=B_i|_F\in\mathfrak{m}$) are elements of $\mathfrak{m}(\boldsymbol{B})_1^\Gamma$ which project to a basis of $(\mathfrak{m}/\mathfrak{m}')(\boldsymbol{B})_1^\Gamma$ over $\mathcal{O}(A)$. Since the B_i have minimal degrees possible, the set

$$\{t^{j-1}(s^{b_i}B_i')\in\mathfrak{m}(\hat{\boldsymbol{B}})_1^\Gamma \mid i=1,\cdots,q,\ j=1,\cdots,k_i\}$$

projects to a $\mathbf{C}$-basis of $(\mathfrak{m}/\mathfrak{m}')(\hat{\boldsymbol{B}})_1^\Gamma/\tau_\sharp\varphi_\sharp\mathfrak{E}(\hat{A})_1$. This shows that $p=\Sigma_{i=1}^q k_i=\Sigma_i[(\deg B_i-1)/d]$. □

Finally we complete the proof of the Theorem (1) in the introduction, i.e. we prove

Theorem 6.6. *There is a G-vector bundle $\mu:\mathfrak{B}\to X\times VEC_G(X,Q)$ such that for every $E\in Vec_G(X,Q)$ the G-vector bundle $\mu^{-1}(X\times[E])$ is an element of $Vec_G(X,Q)$ isomorphic to E.*

Proof. Remember that $\mathfrak{z}(\hat{\boldsymbol{B}})_1^\Gamma\cong(\mathfrak{m}/\mathfrak{m}')(\hat{\boldsymbol{B}})_1^\Gamma$ via $\tau_\sharp$ (see Proposition 5.3 and its proof). Let C_i $(1\le i\le p)$ be elements of $\mathfrak{z}(\boldsymbol{B})_1^\Gamma$ which project to a $\mathbf{C}$-basis of $(\mathfrak{m}/\mathfrak{m}')(\hat{\boldsymbol{B}})_1^\Gamma/\tau_\sharp\varphi_\sharp\mathfrak{E}(\hat{A})_1$. We identify $VEC_G(X_{cl},Q)$ with $\mathbf{C}^p$ by these generators. By Lemma 6.2 (1) there is a positive integer r such that $\varphi_*\mathfrak{P}(\hat{A})_r=M(\hat{\boldsymbol{B}})_r^\Gamma$. We fix such an r and define

$$\exp_r z:=1+z+z^2/2+\cdots+z^{r-1}/(r-1)!.$$

Let $c=(c_1,\cdots,c_p)\in\mathbf{C}^p$ and $C_c:=\Sigma_{i=1}^p c_iC_i$. Then

$$\exp_r C_c\in Z(\tilde{\boldsymbol{B}})_1^\Gamma\subset M(\tilde{\boldsymbol{B}})_1^\Gamma\subset M(\tilde{\hat{\boldsymbol{B}}})^\Gamma.$$

We consider the element $\varphi_*^{-1}\exp_r C_c\in\operatorname{Mor}(\widetilde{\hat{A}}\times_A X_{cl},\mathrm{GL}Q)^G$. As observed in the proof of Theorem 3.3 (2) the element extends to an element of $\mathrm{Mor}(\widetilde{A}\times_A X,\mathrm{GL}Q)^G$ which we denote by ξ_c.

Let $B_c:=\{b\in B|\det(\exp_r C_c)(b)\neq 0\}$ and $A_c:=B_c/\Gamma$. For all $c\in \mathbf{C}^p$, A_c is an open subset of A containing the origin. We set $X_c:=A_c\times_A X$. We glue 2 trivial G-vector bundles (with fibre Q) over $\dot{\mathfrak{U}}:=\dot{X}\times\mathbf{C}^p$ and $\mathfrak{U}':=\{(x,c)\in X\times\mathbf{C}^p\,|\,x\in X_c\}$ via the following transition function

$$\Psi:\dot{\mathfrak{U}}\cap\mathfrak{U}'\to \mathrm{GL}Q$$
$$(x,c)\mapsto \xi_c(x).$$

It is easy to see that the G-vector bundle over $X\times\mathbf{C}^p$ defined by Ψ has the required property. □

7. The structure of $VEC_G(X,Q)$

In this section, we complete the proof of the theorem in the introduction, i.e. we prove

Theorem 7.1. *Let Q, Q_1 and Q_2 be G-modules.*

(1) *Whitney sum induces an epimorphism of vector groups*:

$$WS\colon VEC_G(X,Q_1)\times VEC_G(X,Q_2)\to VEC_G(X,Q_1\oplus Q_2).$$

If $\mathrm{Hom}(Q_1,Q_2)^H=\{0\}$, *then WS is an isomorphism.*

(2) *Let $E_1,E_2\in Vec_G(X,Q)$. Then $E_1\oplus E_2\cong E_3\oplus\Theta_Q$ where $[E_3]:=[E_1]+[E_2]$.*

(3) *The stabilization map*

$$Stab\colon VEC_G(X,Q)\to VEC_G(X,Q\oplus Q)$$
$$[E]\mapsto[E\oplus\Theta_Q]$$

is an isomorphism.

Proof. (1) Let $\mathfrak{m}_i=\mathrm{Mor}(F,\mathrm{End}Q_i)^G$ for $i=1,2$ and $\tilde{\mathfrak{m}}=\mathrm{Mor}(F,\mathrm{End}(Q_1\oplus Q_2))^G$. The additive structures of $(\mathfrak{m}_i/\mathfrak{m}_i')(\hat{B})_1^\Gamma$ and $(\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}')(\hat{B})_1^\Gamma$, which induce the vector group structures on $VEC_G(X,Q_i)$ and $VEC_G(X,Q_1\oplus Q_2)$, come from the ones of $\mathrm{End}Q_i$ and $\mathrm{End}(Q_1\oplus Q_2)$, respectively. While, the Whitney-sum map WS comes from the natural homomorphism $\mathrm{End}Q_1\times\mathrm{End}Q_2\to\mathrm{End}(Q_1\oplus Q_2)$. Hence, WS is a homomorphism of vector groups.

Since the natural map $\mathfrak{m}_1/\mathfrak{m}_1'\times\mathfrak{m}_2/\mathfrak{m}_2'\to\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}'$ is surjective and $\mathfrak{m}_i/\mathfrak{m}_i'\cong\mathfrak{z}_i$ etc., the induced map $(\mathfrak{m}_1/\mathfrak{m}_1')(\hat{B})_1^\Gamma\times(\mathfrak{m}_2/\mathfrak{m}_2')(\hat{B})_1^\Gamma\to(\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}')(\hat{B})_1^\Gamma$ is also surjecitve. Thus, WS is an epimorphism.

If $\mathrm{Hom}(Q_1,Q_2)^H=\{0\}$, then the natural map $\mathfrak{m}_1\times\mathfrak{m}_2\to\tilde{\mathfrak{m}}$ is an isomorphism, which implies that WS is an isomorphism.

(2) Let $\mathfrak{m} = \mathrm{Mor}(F, \mathrm{End}Q)^G$ and $\tilde{\mathfrak{m}} = \mathrm{Mor}(F, \mathrm{End}(Q \oplus Q))^G$. Let $A_1, A_2 \in \mathfrak{m}(\hat{\boldsymbol{B}})_1^{\Gamma}$ be elements which represent $[E_1], [E_2] \in VEC_G(X, Q)$ respectively. Every element $A \in \mathrm{Mor}(\hat{\boldsymbol{B}}, \mathrm{End}(Q_1 \oplus Q_2)^H)$ can be expressed as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{where} \quad A_{ij} \in \mathrm{Mor}(\hat{\boldsymbol{B}}, \mathrm{Hom}(Q_j, Q_i)^H).$$

Using this expression for $Q_1 = Q_2 = Q$, one sees that

$$A_3 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad A_3' = \begin{pmatrix} A_1 + A_2 & 0 \\ 0 & 0 \end{pmatrix}$$

represent $[E_1] \oplus [E_2]$ and $([E_1] + [E_2]) \oplus \Theta_Q$ respectively. Since $\tilde{\tau} : \tilde{\mathfrak{m}} \to \tilde{\mathfrak{m}} / \tilde{\mathfrak{m}}'$ is the trace map on each factor of $\tilde{\mathfrak{m}} \cong \mathrm{End}(Q \oplus Q)^H \cong \oplus_i \mathfrak{gl}_{n_i + n_i}$, we have $\tilde{\tau}_\#(A_3) = \tilde{\tau}_\#(A_3')$. This means that $[E_1] \oplus [E_2] = ([E_1] + [E_2]) \oplus \Theta_Q$.

(3) The stabilization map is induced from

$$\mathrm{End}(Q) \to \mathrm{End}(Q \oplus Q)$$
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

This induces an isomorphism $\mathfrak{m} / \mathfrak{m}' \xrightarrow{\sim} \tilde{\mathfrak{m}} / \tilde{\mathfrak{m}}'$. In fact, the inverse is induced from

$$\mathrm{End}(Q \oplus Q) \to \mathrm{End}(Q)$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto A + D.$$

This implies that the map Stab is an isomorphism. □

REMARK. Besides Whitney sum there are some bundle operations such as tensor product and exterior power. One can see that tensor product induces a (not necessarily surjective) homomorphism of vector groups:

$$VEC_G(X, Q_1) \times VEC_G(X, Q_2) \to VEC_G(X, Q_1 \otimes Q_2).$$

If $Q_2 = \mathbf{C}^m$ (the trivial G-module of dimension m), then $VEC_G(X, Q_2) = \{*\}$ and the above map is nothing but m-fold Whitney sum; so it is an isomorphism in this case. One can also see that i-fold tensor or exterior product induces a (not necessarily surjective) homomorphism:

$$\otimes^i : VEC_G(X, Q) \to VEC_G(X, \otimes^i Q), \qquad \wedge^i : VEC_G(X, Q) \to VEC_G(X, \wedge^i Q).$$

8. Example

We give an example of a non-tirvial moduli of G-vector bundles over a weighted G-cone with smooth one dimensional quotient. This example was first treated by [8] (see also [9]).

Let G be a dihedral group $D_n = \boldsymbol{Z}/2\boldsymbol{Z} \ltimes \boldsymbol{Z}/n\boldsymbol{Z}$. Let τ and λ be generators of $\boldsymbol{Z}/2\boldsymbol{Z}$ and $\boldsymbol{Z}/n\boldsymbol{Z}$, respectively. For a positive integer m we denote by V_m the 2-dimensional G-module defined by

$$\tau(a,b) = (b,a) \qquad \lambda(a,b) = (\lambda^m a, \lambda^{-m} b)$$

where $(a,b) \in V_m (= \mathbf{C}^2)$ and λ is identified with $\exp(2\pi\sqrt{-1}/n)$. Note that $V_m = V_{m+n}$ and if $m \leq n$, $V_m \cong V_{n-m}$; so we may assume $2m \leq n$.

Let X be a G-invariant affine cone defined by

$$X := \begin{cases} \{(c\zeta, c\zeta^{-1}) \in V_1 \mid c \in \mathbf{C},\ \zeta^n = 1\} & n:\text{even} \\ \{(a,b,c) \in V_1 \times \mathbf{C} \mid ab = c^2,\ a^n = b^n = c^n\} & n:\text{odd.} \end{cases}$$

Then

$$\mathcal{O}(X) = \begin{cases} \mathbf{C}[x,y]/(x^{n/2} - y^{n/2}) & n:\text{even} \\ \mathbf{C}[x,y,z]/(xy - z^2, x^n - y^n, y^n - z^n) & n:\text{odd} \end{cases}$$

and $\mathcal{O}(X)^G = \mathbf{C}[t]$ where

$$t = \begin{cases} xy & n:\text{even} \\ z & n:\text{odd.} \end{cases}$$

Henece X is a weighted G-cone with smooth one dimensional quotient.

Theorem 8.1. *Let $G = D_n$, $2m \leq n$ and X be as above. Then $VEC_G(X, V_m) \cong \mathbf{C}^p$ where*

$$p = \begin{cases} \min\{m-1, n/2 - m - 1\} & n:\textit{even} \neq 2m \\ 0 & n = 2m \\ \min\{2m-1, n-2m-1\} & n:\textit{odd.} \end{cases}$$

Proof. We apply Theorem 6.5. The principal isotropy group H of X is $\boldsymbol{Z}/2\boldsymbol{Z}$ (the second factor of $G = \boldsymbol{Z}/n\boldsymbol{Z} \rtimes \boldsymbol{Z}/2\boldsymbol{Z}$) and V_m is multiplicity free with respect to H. Hence it suffices to see the homogeneous generators of $\mathrm{Mor}(X, \mathrm{End}\, V_m)^G$ as an $\mathcal{O}(X)^G$-module as remarked after Theorem 6.5. Since $\dim \mathrm{End}(V_m)^H = 2$, the module $\mathrm{Mor}(X, \mathrm{End}\, V_m)^G$ is of rank 2 (hence $q = 2$ in Theorem 6.5). It is not hard to see that the generators are given by

$$A_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad A_2=\begin{cases}\begin{pmatrix}0 & y^{n-2m}\\ x^{n-2m} & 0\end{pmatrix} & n<4m\\ \begin{pmatrix}0 & x^{2m}\\ y^{2m} & 0\end{pmatrix} & n\geq 4m.\end{cases}$$

Since $\deg A_1=0$ and $\deg A_2=\min\{2m,n-2m\}$, B_1 and B_2 in Theorem 6.5 are tA_1 and A_2 if $n\neq 2m$, and tA_1 and tA_2 if $n=2m$. Noting that $\deg t=2$ or 1 according as n is even or odd, one sees that the theorem follows from Theorem 6.5. □

REMARK. Let $VEC_G(X,V_m;\mathbf{C}):=\{[E]\in VEC_G(X,V_m)\mid[E\oplus\Theta_{\mathbf{C}}]$ is trivial$\}$. It is isomorphic to a $\mathbf{C}$-vector group and its dimension is computed in [8], which agrees with that of $VEC_G(X,V_m)$. Thus, $VEC_G(X,V_m)=VEC_G(X,V_m;\mathbf{C})$, i.e. $E\oplus\Theta_{\mathbf{C}}$ is isomorphic to a trivial bundle for any $E\in Vec_G(X,V_m)$.

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