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Osaka J. Math.
29 (1992), 63-70

REMARKS ON OPEN SURFACES

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(Received February 8, 1991)

0. Introduction

Let X be a smooth complex analytic open surface: that is, X is biholomorphically equivalent to $M \setminus D$, where M is a compact complex variety of dimension 2 and D is a closed analytic subvariety of M. We can assume that M is also smooth.

We shall be interested in the case where X is strongly pseudoconvex (see [6] for definitions). Such an X contains a distinguished compact analytic subset Z, which is the union of all closed analytic subspaces of X of positive dimension. Z is empty if and only if X is Stein.

A famous remark of Serre, in [8], points out that M is not determined up to bimeromorphic equivalence by X. If $X = \boldsymbol{C}^* \times \boldsymbol{C}^*$ then M can be rational or elliptic ruled, or (an observation due to Igusa, see [1]) a non-elliptic Hopf surface. Naturally one asks: for what other such X, if any, is M not unique up to bimeromorphic equivalence?

This question and some related ones have been considered by (among others) Tan, in a series of papers ([11], [12], [13], [14]). There it is always assumed that M is minimal. This, however, imposes a further restriction on X. The purpose of this note is to see what happens for general M.

1. A non-minimal example

We give an easy example of a Stein open surface X for which only non-minimal compactifications exist. Let M' be a surface whose universal cover is a bounded domain, say a ball in $\boldsymbol{C}^2$. Then M' is a strongly minimal surface of general type and is hyperbolic in the sense of [4]. Let $\pi: M \to M'$ be the blow-up of M' in a point p, and let $C = \pi^{-1}(p)$ (so that C is a (-1)-curve in M). Fix some projective embedding of M and let D be a general hyperplane section (so $C \not\subseteq D$). Put $X = M \setminus D$: thus X is affine, and therefore Stein. As we shall see below, the fact that M is of general type implies that it is determined by X up to bimeromorphic equivalence. By the uniqueness of minimal models, M' is the only minimal surface which can possibly contain an open subset biholomorphically equivalent to X. Suppose it does: let $\varphi: X \hookrightarrow M'$ be a holomorphic embedding. By Theorem VI.4.1 of [4], φ extends to a holomorphic map $\tilde{\varphi}: M \to M'$. By strong minimality, the bimeromorphic map $\alpha = \pi\tilde{\varphi}^{-1}$: $M' \dashrightarrow M'$ is biholomorphic. But $\pi = \alpha\tilde{\varphi}$ contracts C: therefore $\tilde{\varphi}$ contracts C, but $\tilde{\varphi}$ is an embedding on X so this is impossible.

A similar argument can be used to give an example of an X which does have a minimal embedding, but only if the boundary D is allowed to have worse than normal crossings singularities. This, too, will force us to consider non-minimal surfaces.

2. Non-algebraic compactifications

We shall say that a surface X is compactifiable if an M and a D such that X is biholomorphically equivalent to $M \setminus D$ exist.

Theorem 2.1. *Let X be a compactifiable Stein surface. Then X has an algebraic compactification.*

Proof. (This is proved in [14] under the assumption that X has a minimal compactification.) Let M be a compactification. By [1] either M is algebraic, in which case there is nothing to prove, or $b_1(M)=1$ and M has no non-constant meromorphic functions. Assume the latter. Then M contains only finitely many irreducible curves, $C_1, \cdots, C_k$ say. Put $C = C_1 \cup \cdots \cup C_k$. The boundary $D = M \setminus X$ is a non-empty connected compact curve: since D meets all the curves C_j and there are no others, C is connected.

The intersection form on C must be negative semidefinite. For if there is a divisor C_0 such that $C_0^2 > 0$ then, by Riemann-Roch,

$$h^0(\mathcal{O}_M(nC_0)) + h^0(\mathcal{O}_M(K_M - nC_0)) = n^2 C^2/2 + o(n).$$

So for $n \gg 0$ one of $\mathcal{O}_M(nC_0)$ and $\mathcal{O}_M(K_M - nC_0)$ has two independent sections, and their ratio gives a meromorphic function on M. Furthermore, if the intersection form on C is negative definite then D can be contracted to a (possibly singular) point and Hartogs' Theorem contradicts the holomorphic convexity of X. Therefore there is a divisor D_0, supported on D, such that $D_0^2 = 0$.

By [15], Lemma 2, we can assume that D_0 is effective and $\operatorname{Supp} D_0 = C$. So $D = C$, so all curves in M, and in particular all (-1)-curves, are contained in the boundary. Hence we can contract the (-1)-curves and assume that M is minimal, which reduces us to the situation of Theorem 3 of [14]. ■

3. Uniqueness questions

Suppose M is algebraic: we ask whether M is unique up to bimeromorphic (or birational) equivalence. In general, of course, it is not. In [13] it is shown that there is usually at most one minimal compactification. We shall show that there is usually just one compactification M.

We denote by $\kappa_{\mathrm{an}}(X)$ the analytic Kodaira dimension of Sakai ([7]). $\kappa_{\mathrm{an}}(X)$ is a biholomorphic invariant of X. We denote by $\bar{\kappa}(X)$ the logarithmic Kodaira dimension of X (see [2]): this may depend on the choice of compactification. If L is a divisor on M then $\kappa(M, L)$ will be the L-dimension and $\kappa(M)$ the usual Kodaira dimension.

Theorem 3.1. *Let X be a compactifiable strongly pseudoconvex surface and suppose that $\kappa(M)\geq 0$ for some algebraic compactification. Then M is determined by X up to birational equivalence.*

We shall prove this theorem via a series of lemmas, some of which we shall refer to later.

Lemma 3.2. *If $\kappa_{\mathrm{an}}(X)=2$ then M is unique up to bimeromorphic equivalence.*

Proof. If M_1 and M_2 are compactifications of X then the map $\varphi\colon X\times X\to X\times X$ given by $\varphi(x_1, x_2)=(x_2, x_1)$ extends to a bimeromorphic map $\bar{\varphi}\colon M_1\times M_2 \dashrightarrow M_1\times M_2$. This follows from [7], Proposition 4.3, using the fact that $\kappa_{\mathrm{an}}(M_1\times M_2)=4$. For a general point $\xi\in M_2$, the map

$$M_1 \xrightarrow{\sim} M_1\times\{\xi\} \hookrightarrow M_1\times M_2 \overset{\bar{\varphi}}{\dashrightarrow} M_1\times M_2 \xrightarrow{pr_2} M_2$$

is bimeromorphic. In particular, if $\bar{\kappa}(M)=2$ then M is unique, as was stated in §1 above. ■

Lemma 3.3. (Tan) *If $\kappa(M)\geq 0$ and M is minimal then $\bar{\kappa}(X)=2$.*
Note that this implies that $\kappa_{\mathrm{an}}(X)=2$ and therefore that M is unique.

Proof. Suppose $\kappa(M)=\bar{\kappa}(X)=1$. Since M is minimal, the canonical divisor K is nef. Suppose that $K.D_j>0$ for some irreducible component D_j of D. Then $(D+rK)^2>0$ for $r\gg 0$. If r is such that the rth plurigenus $P_r\geq 2$, then rK is effective, so by Lemma 8.5 of [2] we have $\kappa(M, D+rK)=2$ and therefore, by Lemma 10.5 of [2], $\bar{\kappa}(X)=2$. So we can assume that $K.D_j=0$ for all D_j. Moreover, M is elliptic. D must be contained in the fibres (because of $K.D_j=0$); but then X contains infinitely many complete curves.

If $K\equiv 0$ we can use Lemma 10.3 of [2] to reduce to the case $K=0$. If $\bar{\kappa}(X)=1$ then Theorem 8.6 of [2] shows that X contains infinitely many complete curves; but if M is an abelian surface $h^0(\mathcal{O}_M(2D))\geq 2$, so $\bar{\kappa}(X)\neq 0$. On the other hand if M is a $K3$ then the boundary is contractible (as in [2], § 10.4). ■

REMARK. Tan proves essentially the same lemma in [13] by using the structure theorem in [5] (II.2.3.1). However, there are some details to be checked and the above argument is more elementary.

To conclude the proof of Theorem 3.1 we need to deal with $\kappa(M)=0$ or 1 and M non-minimal. We may assume that Z contains no (-1)-curves (if it does we simply contract them). If f: $M \to M'$ is the contraction of a (-1)-curve E, we put $D'=f(D)$ and $X'=M' \setminus D'$. Since f is biholomorphic away from E and $X'=f(X \setminus E)$, X' is biholomorphically equivalent to $X \setminus E$. By [9], Lemma 2, $X \setminus E$ is strongly pseudoconvex, so X' is also. If g: $M \to M''$ is a birational morphism to a minimal surface, g can be factored into finitely many contractions of (-1)-curves so, by induction, $X''=M'' \setminus g(D)$ is strongly pseudoconvex. By Lemma 3.3, $\bar{\kappa}(X')=2$. By [2], Theorem 11.4 (iii), $\bar{\kappa}(X)=\bar{\kappa}(X')$, so we are done by Lemma 3.2. ■

We can sharpen Theorem 3.1 further. Suppose now that M_i is an algebraic compactification of X_i with boundary $D_i=M_i \setminus X_i$, that X is strongly pseudoconvex and that X is biholomorphically equivalent to X_i for $i=1,2$. After blowing up one point of D_i if necessary (in case it happens that $M_i \cong \boldsymbol{P}^2$), we can assume that M_i are equipped with morphisms π_i: $M_i \to B_i$, where B_i is a smooth complete curve of genus g_i and the general fibre F_i is a $\boldsymbol{P}^1$. We put $\alpha_i=D_i.F_i$ and let β_i be the number of fibers of π_i contained in D_i. Let θ: $X_1 \to X_2$ be biholomorphic.

Theorem 3.4. *Suppose M_1 and M_2 are not bimeromorphically equivalent. Then $g_i \leq 1$, and if $g_i=1$ then $\beta_i=0$ and $\alpha_i=1$ or 2.*

Proof. Suppose first that, say, $g_2>1$ or that $g_2=1$ and $\beta_2>0$. Then $\pi_2(X_2)$ is $B_2 \setminus (\beta_2$ points), which is hyperbolic. By [4], Ch. IV, Corollary 2.6, the map $\pi_2\, \theta$: $F_1 \cap X_1 \to \pi_2(X_2)$ extends to a map $F_1 \to B_2$, which is constant by Lüroth's theorem. This determines a map $\hat{\theta}$: $\hat{B}_1 \to B_2$, given by $\hat{\theta}(p)=\pi_2\, \theta\, (\pi_1^{-1}(p) \cap X_1)$ and defined on the Zariski-open set $\hat{B}_1 \subseteq B_1$ in which the fibres of π_1 are smooth and not contained in D_1. It is clear that $\hat{\theta}$ extends to an isomorphism between B_1 and B_2, so in this case M_1 and M_2 are bimeromorphically equivalent.

Now suppose that for one of the compactifications, M, we have $g=1$ and $\beta=0$, so that B is a complete curve of genus 1, and that $\alpha \geq 3$. I claim that $\bar{\kappa}(X)=2$, so that M is unique by Lemma 3.2. By the open version of $C_{2,1}$ proved in [3] we have

$$\bar{\kappa}(X) \geq \bar{\kappa}(B)+\bar{\kappa}(\boldsymbol{P}^1 \setminus (\alpha \text{ points})) = 1\,.$$

If $\bar{\kappa}(X)=1$, we apply the structure theorem in [5], II.2.3.1 already mentioned. Suppose first we are in the quasi-elliptic case, so that there is a birational morphism f: $M \to M_0$ and a divisor D_0 on M_0 such that $f^*(D_0+K_{M_0})=(D+K_M)^+$ (the arithmetically effective part of $D+K_M$). M_0 is equipped with a morphism π_0: $M_0 \to B_0=B$, and $D_0.F_0=2$ where F_0 is the general fibre of π_0. If C is a component of D such that $C.F \neq 0$, then C is not rational so f_*C is a component of D_0 with multiplicity 1 ([5], p. 153). But $f_*F=F_0$, so $2=D_0.F_0=D.F=\alpha\geq 3$. This leaves only $\bar{\kappa}(X)=2$.

If we are in the elliptic case of [5], II.2.3.1, then there is a birational morphism f: $M\to M_0$ as above, and M_0 is equipped with a morphism σ: $M_0\to B'$ whose general fibre is elliptic and which contracts no (-1)-curves. Since M_0 is birationally ruled over B there is also a morphism π: $M_0\to B$, and $B'\cong \boldsymbol{P}^1$. Then σ factors through $B\times B'$, so M_0 must be minimal (otherwise σ does contract some (-1)-curves).

According to [5], D_0 is a sum of fibres of σ (with multiplicity). So therefore, is $D_0+K_{M_0}$. The general fibre E of σf: $M\to\boldsymbol{P}^1$ must meet D. Let C be a component of D such that $E.C>0$. Since Supp $(D_0+K_{M_0})^+$ lies in the fibres of σ, Supp $(D+K_M)^+$ lies in the fibres of σf. Hence $C\subseteq$ Supp $(D+K_M)^-$, and in particular it is rational ([5], p. 153). It is therefore contained in a fibre F_0 of $\pi_0 f$: $M_0\to B$. It is not contracted by f, since $C.E>0$. All other components of F_0 are contracted by f, so they occur in Supp K_M. So the whole of F_0 occurs in Supp $(D+K_M)$, and indeed in the arithmetically negative part. But $F_0^2=0$, and the intersection form on the arithmetically negative part of a divisor is negative definite (see, for instance, [5], p. 37). ■

REMARK. Note that θ itself need not extend to a bimeromorphic map M_1 $-\to M_2$. In general, as is shown in [10], it does not.

This result allows us to give another, partial, extension of a result of Tan to arbitrary smooth surfaces.

Theorem 3.5. (cf. [13], Theorem 4.) *Suppose X is strongly pseudoconvex but not Stein, and an algebraic compactification M exists. Then M is unique up to bimeromorphic equivalence except possibly in the following cases:*

a) Z contains a unique smooth elliptic curve E, and there are exactly two algebraic compactifications, M_1 rational and M_2 birationally equivalent to $E\times\boldsymbol{P}^1$ with $Z_2.F_2=D_2.F_2=1$;

b) Z can be contracted to give finitely many singularities belonging to a restricted class (possibly smooth points) any nonrational compactification M is birationally elliptic ruled, $D.F=2$ and Z is contained in the singular fibres of M. If in this case D is irreducible, the possible singularities are A_k and D_k.

Proof. Put $X^*=X\backslash Z$. Both M_1 and M_2 are compactifications of X^*, so by Lemma 3.2 we must have $\bar{\kappa}(X^*)<2$. By Theorem 3.1, if there is more than one compactification then one of them must be birationally elliptic ruled. Let us abuse notation and call this M, so Z is now a curve in M. The boundary of X^* in M is $D+Z$ so, by the argument used in proving Lemma 3.4, $(D+Z).F\leq 2$. Clearly $D.F>0$, so we have two cases, $Z.F=1$ and $Z.F=0$. In the case $Z.F=1$ a section E of $M\to B$ occurs as a component of Z and $D.F=1$. This is case (a).

Suppose, then, that $Z.F=0$: thus Z is contained in finitely many fibres, obviously singular ones. A singular fibre F_0 is a tree of rational curves. We contract all (-1)-curves which are contained in either D or Z. If all of Z is then contracted, we are in case (*b*) with smooth points only.

It is well known that the intersection form on F_0 is negative semidefinite. F_0 certainly contains at least one (-1)-curve: say

$$F_0 = \sum \lambda_i C_i' + \sum \mu_j C_j$$

where the C_i' are (-1)-curves and the C_j are not. Every curve in F_0 either meets D or is contained in Z, because Z is maximal, and the C_i' are not contained in Z nor in D. Hence $D.C_i \geq 1$. If in addition D is irreducible then

$$2 = D.F_0 = \sum \lambda_i D.C_i' + \sum \mu_j D.C_j \geq \sum \lambda_i ,$$

where the inequality comes from the fact that $C_j \not\subset \operatorname{Supp} D$, so that $C_j.D \geq 0$. Furthermore, $p_a(F_0)=0$ so $F_0.K=-2$. Hence

$$\begin{aligned} -2 &= \sum \lambda_i C_i'.K + \sum \mu_j C_j.K \\ &= -\sum \lambda_i + \sum \mu_j C_j.K \\ &\geq -2 + \sum \mu_j C_j.K . \end{aligned}$$

We have $C_j^2 < -1$, since the intersection form is negative semidefinite and $C_j^2 \neq -1$. (It is easy to see that $C_j^2 \neq 0$.) Also C_j is rational, so $C_j.K > -1$. But $\mu_j \geq 0$ for all j, and $\sum \mu_j C_j.K \leq 0$: hence $C_j.K=0$ for all j, and the C_j are (-2)-curves. Thus $Z \cap F_0 = \sum C_j$ is a union of (-2)-curves and the intersection form is negative definite. This means that each connected component of Z has a configuration of type A_k, D_k, E_6, E_7 or E_8. However, if the configuration is E_k then one can easily check that the intersection foım on F_0 is not negative semidefinite. ■

Fibres as described in (*b*) can occur (that is, we cannot eliminate A_k or D_k as we eliminated E_k) but I do not know whether case (*b*) actually occurs, apart from the smooth case wihch certainly does.

The restricted class of singularities referred to in (*b*) is those whose weighted resolution graph can be embedded in the weighted graph of a singular fibre of genus 0 in such a way that its removal leaves at most two connected components. If we know more about D we can say more about Z, by doing calculations similar to those used above for the case D irreducible, but we shall not give any further examples here.

4. Other remarks

The argument in §2 also shows that strongly pseudoconvex surfaces with a non-algebraic, not necessarily minimal compactification are either as described in [11] or modifications of smooth Stein surfaces.

I do not know whether it is true in general that for smooth compactifiable Stein surfaces $\kappa_{an}(X)=2$ or $-\infty$(cf. [14]). However, the only smooth curves C with $\kappa_{an}(C)=0$ are complete elliptic curves, so it follows from the fibration theorem (Theorem 2.3 of [7]) that if $\kappa_{an}(X)=1$ then X contains infinitely many complete curves. The corresponding result for affines is true: indeed we have the following stronger result.

Corollary 4.1. *Let X be a smooth affine n-dimensional variety. Then $\kappa_{an}(X)=-\infty$ or n.*

Proof. By Theorem 2.3 of [7] it is enough to show that $\kappa_{an}(X)\neq 0$ (and proceed by induction on n). By the argument in [14], Proposition 4, it is enough to show that $\kappa(M, D')\neq 0$ for a suitable ample divisor D' supported on D, i.e., that $h^0(\mathcal{O}_M(D'))>1$. This is obvious. ■

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