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NON-HOMOGENEOUS KÄHLER-EINSTEIN METRICS ON COMPACT COMPLEX MANIFOLDS II

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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In the previous paper K-S [12] we have considered $P^1(\boldsymbol{C})$-bundles over compact Kähler-Einstein manifolds to obtain non-homogeneous Kähler-Einstein manifolds with positive Ricci tensor. The purpose of this paper is to give more examples of non-homogeneous compact Kähler-Einstein manifolds, more precisely, compact almost homogeneous Kähler-Einstein manifolds with disconnected exceptional set. By [1] and [8], the structure of orbits of almost homogeneous projective algebraic manifolds with disconnected exceptional set have been investigated, but no explicit examples were given in [1] and [8] except complex projective spaces. To construct these examples, we start again with $P^1(\boldsymbol{C})$-bundles over Kähler C-spaces and consider compact complex manifolds obtained from these $P^1(\boldsymbol{C})$-bundles by blowing down. Note that compact complex manifolds obtained from projective algebraic manifolds by blowing down are not Kähler in general as an example of Moisezon [14] Chap. 3, section 3 shows. We construct our compact complex manifolds in section 3 and prove that our compact almost homogeneous complex manifolds are Kähler and have positive first Chern class (Theorem 4.1). But in general these almost homogeneous manifolds may be homogeneous. We give a sufficient condition for these Kähler manifolds being non-homogeneous (Theorem 5.1). In section 6 we show that for each positive integer d there are compact Kähler-Einstein manifolds which have cohomogeneity d. We follow the notation in Kobayashi-Nomizu [11] which is slightly different from the one in [12].

1 Kähler C-spaces and Dynkin diagrams

We recall known facts on compact simply connected homogeneous Kähler manifolds, called Kähler C-spaces (cf. Takeuchi [18]).

Let Π be a Dynkin diagram and Π_0 a subdiagram of Π. The pair (Π, Π_0) is said to be *effective* if Π_0 does not contain any irreducible component of Π. Let Σ be the root system with the fundamental root system Π. Choose a lexicographic order $>$ on Σ such that the set of simple roots with respect to $>$ coincides with Π. Take a compact semi-simple Lie algebra $\mathfrak{g}_u$ with the root system Σ and let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}_u$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the complexification of $\mathfrak{g}_u$ and $\mathfrak{t}$ respectively. We identify a weight of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$ with an element of $\sqrt{-1}\mathfrak{t}$ by the duality defined by the Killing form (,) of $\mathfrak{g}$. In particular, the root system Σ of $\mathfrak{g}$ relative to $\mathfrak{h}$ is a subset of $\sqrt{-1}\mathfrak{t}$. Let $\{\Lambda_\alpha\}_{\alpha\in\Pi}\subset\sqrt{-1}\mathfrak{t}$ be the fundamental weights of $\mathfrak{g}$ corresponding to Π:

$$(1.1) \qquad \frac{2(\Lambda_\alpha, \beta)}{(\beta, \beta)} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \,. \end{cases}$$

Let Σ^+ be the set of all positive roots and $\{\Pi_0\}_Z$ the subgroup of $\sqrt{-1}\mathfrak{t}$ generated by Π_0. Put $\Sigma_0=\Sigma\cap\{\Pi_0\}_Z$. We define a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ by

$$(1.2) \qquad \mathfrak{u} = \mathfrak{h} + \sum_{\alpha\in\Sigma_0\cup\Sigma^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ for $\alpha\in\Sigma$. Let G be a simply connected complex Lie group whose Lie algebra is $\mathfrak{g}$, and let U be the connected (closed) complex subgroup of G generated by $\mathfrak{u}$. Put $M=G/U$. Then it is known that the complex manifold $M=G/U$ is compact, simply connected and admits a homogeneous Kähler metric. Let G_u be the compact connected subgroup of G generated by $\mathfrak{g}_u$. Put $K=G_u\cap U$. Then K is connected, G_u acts on M transitively and $M=G/U=G_u/K$ as a smooth manifold. This homogeneous complex manifold M is s said to be *associated to the pair* (Π, Π_0) *of Dynkin diagrams.*

We define a subspace $\mathfrak{c}$ of $\sqrt{-1}\mathfrak{t}$ by

$$(1.3) \qquad \mathfrak{c} = \sum_{\alpha\in\Pi-\Pi_0} \boldsymbol{R}\Lambda_\alpha \,.$$

Then $\sqrt{-1}\mathfrak{c}$ coincides with the center of the Lie algebra $\mathfrak{k}$ of K. We also define lattices Z of $\sqrt{-1}\mathfrak{t}$ and $Z_\mathfrak{c}$ of $\mathfrak{c}$ by

$$(1.4) \qquad Z = \{\lambda\in\sqrt{-1}\mathfrak{t} \,|\, 2(\lambda, \alpha)/(\alpha, \alpha) \text{ is an integer for each } \alpha\in\Sigma\}$$

and

$$(1.5) \qquad Z_\mathfrak{c} = Z\cap\mathfrak{c} \,.$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}_u$ with respect to the Killing form (,); $\mathfrak{g}_u=\mathfrak{k}+\mathfrak{m}$. The subspace $\mathfrak{m}$ is K-invariant under the adjoint action and identified with the tangent space $T_o(M)$ of M at the origin $o\in M$. Put

$$(1.6) \qquad \Sigma_\mathfrak{m}^+ = \Sigma^+ - \Sigma_0 \,, \quad \Sigma_\mathfrak{m}^- = -\Sigma_\mathfrak{m}^+ \,.$$

We define K-invariant subspaces $\mathfrak{m}^\pm$ of $\mathfrak{g}$ by

$$(1.7) \qquad \mathfrak{m}^\pm = \sum_{\alpha\in\Sigma_\mathfrak{m}^\pm} \mathfrak{g}_{-\alpha} \,.$$

Then the complexification $\mathfrak{m}^C$ of $\mathfrak{m}$ is the direct sum:

$$\mathfrak{m}^C = \mathfrak{m}^+ + \mathfrak{m}^- .$$

We denote by $X \to \bar{X}$ the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_u$. Then $\mathfrak{m}^{\mp} = \overline{\mathfrak{m}^{\pm}}$. We choose $E_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Sigma$ with the following properties and fix them from now on:

$$(1.8) \qquad [E_\alpha, E_{-\alpha}] = -\alpha\,, \quad (E_\alpha, E_{-\alpha}) = -1\,, \quad \bar{E}_\alpha = E_{-\alpha} \qquad \text{for} \quad \alpha \in \Sigma\,.$$

Let $\{\omega^\alpha\}_{\alpha\in\Sigma}$ be the linear forms on $\mathfrak{g}$ dual to $\{E_\alpha\}_{\alpha\in\Sigma}$, that is, linear forms defined by

$$(1.9) \qquad \begin{cases} \omega^\alpha(\mathfrak{h}) = \{0\} \\ \omega^\alpha(E_\beta) = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta\,. \end{cases} \end{cases}$$

Let T be the toral subgroup of G_u generated by $\mathfrak{t}$. The tangent space $T_e(T)$ of T at the identity element e is identified with $\mathfrak{t}$. Let $\mathcal{I}^1(T)$ denote the space of T-invariant real 1-forms on T. Then we have natural linear isomorphisms:

$$(1.10) \qquad \mathfrak{t} \to \mathfrak{t}^* = T_e^*(T) \to \mathcal{I}^1(T) \to H^1(T, \boldsymbol{R})\,.$$

We identify $\mathfrak{t}$ with $H^1(T, \boldsymbol{R})$. Then we have

$$(1.11) \qquad \frac{1}{2\pi\sqrt{-1}} Z = H^1(T, \boldsymbol{Z})\,.$$

It is known that the inclusion $\iota\colon T \to K$ induces an injective linear map $\iota^*\colon H^1(K, \boldsymbol{R}) \to H^1(T, \boldsymbol{R})$ with $\iota^* H^1(K, \boldsymbol{R}) = 1/(2\pi\sqrt{-1})\mathfrak{c}$ and $\iota^* H^1(K, \boldsymbol{Z}) = 1/(2\pi\sqrt{-1}) Z_{\mathfrak{c}}$, and that the transgression for the principal bundle $K \to G \to M$ defines a linear isomorphism $\tau\colon H^1(K, \boldsymbol{R}) \to H^2(M, \boldsymbol{R})$ with $\tau(H^1(K, \boldsymbol{Z})) = H^2(M, \boldsymbol{Z})$. We define a linear map $\underline{\tau}\colon \mathfrak{c} \to H^2(M, \boldsymbol{R})$ by

$$\underline{\tau}(\lambda) = -\tau(\lambda/(2\pi\sqrt{-1})) \text{ for } \lambda \in \mathfrak{c}\,,$$

where $H^1(K, \boldsymbol{R})$ is identified with $1/(2\pi\sqrt{-1})\mathfrak{c}$ through ι^*. Then $\underline{\tau}(Z_{\mathfrak{c}})$ coincides with $H^2(M, \boldsymbol{Z})$ (cf. Borel-Hirzebruch [4]).

We define a cone $\mathfrak{c}^+$ in $\mathfrak{c}$ by

$$(1.12) \qquad \mathfrak{c}^+ = \{\lambda \in \mathfrak{c} \mid (\lambda, \alpha) > 0 \text{ for each } \alpha \in \Pi - \Pi_0\}$$

and put $Z_{\mathfrak{c}}^+ = Z \cap \mathfrak{c}^+$. Then we have

$$(1.13) \qquad \mathfrak{c}^+ = \sum_{\alpha \in \Pi - \Pi_0} \boldsymbol{R}^+ \Lambda_\alpha ,$$

$$(1.14) \qquad Z_\mathfrak{c}^+ = \sum_{\alpha \in \Pi - \Pi_0} \boldsymbol{Z}^+ \Lambda_\alpha .$$

Moreover, the cone $\mathfrak{c}^+$ is characterized by

$$\mathfrak{c}^+ = \{\lambda \in \mathfrak{c} \,|\, (\lambda, \alpha) > 0 \text{ for each } \alpha \in \Sigma_\mathfrak{m}^+\} .$$

Lemma 1.1 (Takeuchi [18]). *Let $\mathcal{I}^2_{G_u}(M)$ be the space of closed G_u-invariant real 2-forms on M and $\mathcal{H}^2(M, g)$ the space of real harmonic 2-forms on M with respect to a G_u-invariant Riemannian metric g on M. Then $\mathcal{I}^2_{G_u}(M) = \mathcal{H}^2(M, g)$.*

Let $\lambda \in \mathfrak{c}$. Regarding each ω^α a G_u-invariant $\boldsymbol{C}$-valued 1-form on G_u, we define a G_u-invariant $\boldsymbol{C}$-valued 2-form $\eta(\lambda)$ on G_u by

$$(1.15) \qquad \eta(\lambda) = \frac{1}{2\pi\sqrt{-1}} \sum_{\alpha \in \Sigma_\mathfrak{m}^+} (\lambda, \alpha) \omega^{-\alpha} \wedge \overline{\omega^{-\alpha}} .$$

We define a complex linear form $\tilde{\lambda}$ on $\mathfrak{g}_u$ by

$$\tilde{\lambda}(X) = (\lambda, X) \qquad \text{for} \quad X \in \mathfrak{g}_u ,$$

and regard $\tilde{\lambda}$ as a G_u-invariant $\boldsymbol{C}$-valued 1-form on G_u. Thus $1/(2\pi\sqrt{-1})\tilde{\lambda}$ is regarded as a G_u-invariant $\boldsymbol{R}$-valued 1-form on G_u. Then we have

$$\eta(\lambda) = -d(1/(2\pi\sqrt{-1})\tilde{\lambda}) ,$$

and $\eta(\lambda)$ can be pulled down to a unique form in $\mathcal{I}^2_{G_u}(M)$. Thus the correspondence $\lambda \to \eta(\lambda)$ defines a linear map $\eta \colon \mathfrak{c} \to \mathcal{I}^2_{G_u}(M)$.

Lemma 1.2 (Takeuchi [18]). *Let ψ be the natural map assigning $\omega \in \mathcal{I}^2_{G_u}(M)$ to the de Rham class $[\omega]$ in $H^2(M, \boldsymbol{R})$. Then we have the following commutative diagram consisting of linear isomorphisms:*

$$(1.16) \qquad \begin{tikzcd} \mathfrak{c} \arrow[rr, "\tau"] \arrow[dr, "\eta"'] & & H^2(M, \boldsymbol{R}) \\ & \mathcal{I}^2_{G_u}(M) . \arrow[ur, "\psi"'] & \end{tikzcd}$$

We define elements $\delta_\mathfrak{m}$, δ of $\sqrt{-1}\mathfrak{t}$ by

$$(1.17) \qquad \delta_\mathfrak{m} = \frac{1}{2} \sum_{\alpha \in \Sigma_\mathfrak{m}^+} \alpha , \quad \delta = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$$

respectively. It is known that $2\delta_\mathfrak{m} \in Z_\mathfrak{c}^+$ and $\delta = \sum_{\alpha \in \Pi} \Lambda_\alpha$.

Now we recall the following facts.

FACT 1 (cf. Borel-Hirzebruch [4], Takeuchi [18]). Let $M=G/U=G_u/K$ be the compact homogeneous complex manifold associated to an effective pair (Π, Π_0) of Dynkin diagrams. Then we have the followings.

1) *For* $\lambda \in \mathfrak{c}$,

$$g(\lambda)=\frac{1}{2\pi}\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}}(\lambda, \alpha)\omega^{-\alpha}\cdot\overline{\omega^{-\alpha}} \tag{1.18}$$

defines a G_u*-invariant real covariant symmetric tensor field of degree 2 on* M, *and the correspondence* $\lambda \to g(\lambda)$ *gives a bijection from* $\mathfrak{c}^{+}$ *to the set of* G_u*-invariant Kähler metrics on* M.

2) *The first Chern class* $c_1(M)$ *of* M *is given by* $c_1(M)=\underline{\tau}(-2\delta_{\mathfrak{m}})$. *For the Kähler metric* g *corresponding to* $\lambda\in\mathfrak{c}^{+}$, *the Kähler form* ω *(defined by* $\omega(X, Y)=g(X, JY)$, *where* J *is the almost complex structure of* M*), the Ricci tensor* r *and the Ricci form* ρ *are given by*

$$\omega=\eta(\lambda)=-(\sqrt{-1}/(2\pi))\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}}(\lambda, \alpha)\omega^{-\alpha}\wedge\overline{\omega^{-\alpha}}, \tag{1.19}$$

$$r=4\pi g(2\delta_{\mathfrak{m}})=2\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}}(2\delta_{\mathfrak{m}}, \alpha)\omega^{-\alpha}\cdot\overline{\omega^{-\alpha}}, \tag{1.20}$$

$$\rho=4\pi\eta(2\delta_{\mathfrak{m}})=-2\sqrt{-1}\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}}(2\delta, \alpha)\omega^{-\alpha}\wedge\overline{\omega^{-\alpha}}. \tag{1.21}$$

FACT 2 (cf. Ise [9]). *For each* $\Lambda\in \boldsymbol{Z}_{\mathfrak{c}}$, *there is a unique holomorphic character* χ_Λ *of* U *such that*

$$\chi_\Lambda(\exp H)=\exp(\Lambda, H) \qquad \textit{for each} \quad H\in\mathfrak{h}.$$

Let L_Λ *denote the holomorphic line bundle on* M *associated to the principal bundle* $U\to G\to M$ *by the character* χ_Λ. *The correspondence* $\Lambda\to L_\Lambda$ *induces an isomorphism from* $Z_{\mathfrak{c}}$ *onto the group* $H^1(M, \theta^*)$ *of all holomorphic line bundles on* M. *Moreover, under this isomorphism the subset* $-Z_{\mathfrak{c}}^{+}$ *corresponds to the set of all very ample holomorphic line bundles on* M. *The first Chern class* $c_1(L_\Lambda)$ *of* L_Λ *contains a unique* G_u*-invariant 2-form*

$$\eta(\Lambda)=-\frac{\sqrt{-1}}{2\pi}\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}}(\Lambda, \alpha)\omega^{-\alpha}\wedge\overline{\omega^{-\alpha}} \tag{1.22}$$

on M.

2 Kähler C-spaces as projective bundles

Let E be a holomorphic vector bundle of rank r over a complex manifold N. The complex projective bundle $P(E)$ associated to E is defined as follows. Let $\boldsymbol{C}^*$ act freely on $E-$(0-section) by scalar multiplication. Then $P(E)$ is the quotient complex manifold

$$P(E)=E-(0\text{-section})/\boldsymbol{C}^*\,.$$

Thus a point of $P(E)$ over $x\in N$ represents a complex line in the fiber E_x of E at x. We organize various spaces and maps by the following diagram:

$$\begin{tikzcd} P(E) \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & E-(0\text{-section}) \subset E \arrow[d, "p"] \\ & N \end{tikzcd} \tag{2.1}$$

Using the projection $\varphi\colon P(E)\to N$, we pull back the bundle E to obtain the vector bundle φ^*E of rank r over $P(E)$. We define the tautological line bundle $L(E)$ over $P(E)$ as a subbundle of φ^*E as follows. The fiber $L(E)_\xi$ at $\xi\in P(E)$ is the complex line in $E_{\varphi(\xi)}$ represented by ξ. Note also that if L is a holomorphic line bundle over N, then $P(E)$ is canonically identified with $P(E\otimes L)$ as complex manifolds and $L(E\otimes L)=L(E)\otimes\varphi^*L$ as holomorphic line bundles.

Let Π be a Dynkin diagram and Π_0 a subdiagram of Π such that Π_0 is of type A_{l-1} ($\Pi_0=\emptyset$ if $l=1$). Consider also a subdiagram Π_1 such that Π_1 contains Π_0 as a subdiagram and Π_1 is of type A_l. Put $\Sigma_1=\Sigma\cap\{\Pi_1\}_{\boldsymbol{Z}}$. We define a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by

$$\mathfrak{p}=\mathfrak{h}+\sum_{\alpha\in\Sigma_1\cup\Sigma^+}\mathfrak{g}_\alpha \tag{2.2}$$

as in section 1. We denote by $G/U,G/P$ the Kähler C-spaces associated to the pairs (Π,Π_0), (Π,Π_1) of Dynkin diagrams respectively. Put $\{\alpha_0\}=\Pi_1-\Pi_0$ and $\Lambda_0=\Lambda_{\alpha_0}\in Z$. We define a subalgebra $\mathfrak{g}(1)$ of $\mathfrak{p}$ by

$$\mathfrak{g}(1)=\mathfrak{h}+\sum_{\alpha\in\Sigma_1}\mathfrak{g}_\alpha\,, \tag{2.3}$$

and let $G(1)$ be the complex subgroup of G generated by $\mathfrak{g}(1)$. Then there is an irreducible representation $\rho_{\Lambda_0}\colon G(1)\to GL(V_{\Lambda_0})$ of $G(1)$ with the highest weight Λ_0. The representation ρ_{Λ_0} can be uniquely extended to an irreducible representation of P, which is also denoted by $\rho_{\Lambda_0}\colon P\to GL(V_{\Lambda_0})$. Note also that $\dim_{\boldsymbol{C}}V_{\Lambda_0}=l+1$.

We denote by E_{Λ_0} the homogeneous vector bundle over G/P defined by the representation $\rho_{\Lambda_0}\colon P\to GL(V_{\Lambda_0})$ and by $P(E_{\Lambda_0})$ the complex projective bundle over G/P associated to the vector bundle E_{Λ_0}. Then G acts on E_{Λ_0} and $P(E_{\Lambda_0})$ in natural ways. We denote by $[g,v]$ the element of E_{Λ_0} defined by $(g,v)\in G\times V_{\Lambda_0}$ and let $p\colon E_{\Lambda_0}-(0\text{-section})\to P(E_{\Lambda_0})$ be the projection. Take a highest weight vector v_{Λ_0} of $\rho_{\Lambda_0}\colon P\to GL(V_{\Lambda_0})$, that is, v_{Λ_0} is a non-zero vector of V_{Λ_0} such that

$$(2.4)\qquad \begin{cases} \rho_{\Lambda_0}(H)v_{\Lambda_0} = (\Lambda_0, H)v_{\Lambda_0} & \text{for } H\in\mathfrak{h} \\ \rho_{\Lambda_0}(E_\alpha)v_{\Lambda_0} = 0 & \text{for } E_\alpha\in\mathfrak{g}_\alpha,\ \alpha\in\Sigma^+ \end{cases}$$

and fix it.

Lemma 2.1. *We have an identification*: $P(E_{\Lambda_0})=G/U$.

Proof. At first note that G acts on $P(E_{\Lambda_0})$ transitively, since $G(1)$ acts on $P(V_{\Lambda_0})$ transitively. Put $o=p([e, v_{\Lambda_0}])$. Consider the isotropy subgroup G_o of G at $o\in P(E_{\Lambda_0})$. Then we have $G_o=\{g\in G\mid g\in P,\ \rho_{\Lambda_0}(g)v_{\Lambda_0}=\lambda(g)v_{\Lambda_0}$ for some $\lambda(g)\in \boldsymbol{C}-(0)\}$. Thus the Lie algebra $\mathfrak{g}_o$ of G_o is given by

$$(2.5)\qquad \begin{aligned}\mathfrak{g}_o &= \{X\in\mathfrak{p}\mid \rho_{\Lambda_0}(X)v_{\Lambda_0}\in \boldsymbol{C}v_{\Lambda_0}\} \\ &= \mathfrak{h}+\sum_{(\Lambda_0,\alpha)\geq 0,\,\alpha\in\Sigma_1\cup\Sigma^+}\mathfrak{g}_\alpha = \mathfrak{h}+\sum_{\alpha\in\Sigma_0\cup\Sigma^+}\mathfrak{g}_\alpha\,.\end{aligned}$$

Hence $\mathfrak{g}_o=\mathfrak{u}$. Since the normalizer of the parabolic subgroup U coincides with U, we see that $U=G_o$ and $P(E_{\Lambda_0})=G/G_o=G/U$. q.e.d.

Now we consider the homogeneous vector bundle E_{Λ_0} over G/P. Then $E_{\Lambda_0}-$(0-section) is a $\boldsymbol{C}^*$-bundle over $P(E_{\Lambda_0})$. Let $L(E_{\Lambda_0})$ be the tautological line bundle over $P(E_{\Lambda_0})$ associated to the vector bundle E_{Λ_0} over G/P. Then we have an identification: $E_{\Lambda_0}-$(0-section)$=L(E_{\Lambda_0})-$(0-section).

Lemma 2.2. *The tautological line bundle $L(E_{\Lambda_0})$ is the holomorphic line bundle L_{Λ_0} over $P(E_{\Lambda_0})=G/U$ associated to the principal bundle $U\to G\to G/U$ by the character χ_{Λ_0} of U.*

Proof. Since $(\Lambda_0, \alpha)=0$ for each $\alpha\in\Sigma_0$, $\rho_{\Lambda_0}(E_\alpha)v_{\Lambda_0}=0$ for each $\alpha\in\Sigma_0$. Thus ρ_{Λ_0} induces a representation $\rho_{\Lambda_0}\colon U\to GL(\boldsymbol{C}v_{\Lambda_0})$, which is identified with the character χ_{Λ_0} of U, since $\rho_{\Lambda_0}(\exp H)v_{\Lambda_0}=\exp(\Lambda_0, H)v_{\Lambda_0}$ for $H\in\mathfrak{h}$. Note that by Lemma 2.1 each element of $L(E_{\Lambda_0})$ can be written as $[g, \lambda v_{\Lambda_0}]$ $(g\in G, \lambda\in\boldsymbol{C})$. Now let $[g, \lambda v_{\Lambda_0}]$, $[g', \mu v_{\Lambda_0}]$ be elements of $L(E_{\Lambda_0})$. Then $[g, \lambda v_{\Lambda_0}]=[g', \mu v_{\Lambda_0}]$ in $L(E_{\Lambda_0})$ if and only if $g'=gu$ $(u\in U)$ and $\rho_{\Lambda_0}(u)\mu v_{\Lambda_0}=\lambda v_{\Lambda_0}$. Thus we get our claim. q.e.d.

Now we recall the following general formula for the canonical line bundle of a projective bundle. Let $\varphi\colon E\to N$ be a holomorphic vector bundle of rank r over a complex manifold N and let $K_{P(E)}$, K_N denote the canonical line bundle on $P(E)$, N respectively. Then

$$(2.7)\qquad K_{P(E)} = \varphi^*(K_N\otimes\det E^*)\otimes L(E)^r$$

where $\det E^*$ denotes the holomorphic line bundle $\overset{r}{\wedge} E^*$.

We apply this formula to compute the first Chern class of $P(E_{\Lambda_0})=G/U$.

Lemma 2.3. *The element* $-2\delta_{\mathfrak{m}}\in Z_{\mathfrak{c}}^{+}$ *corresponding to the first Chern class* $c_1(P(E_{\Lambda_0}))$ *of* $P(E_{\Lambda_0})=G/U$ *is given by*

$$(2.8)\qquad -2\delta_{\mathfrak{m}}=-(l+1)\Lambda_0+\sum_{\alpha\in\Pi-\Pi_1}-n_\alpha\Lambda_\alpha \qquad \textit{for some}\quad n_\alpha\in \boldsymbol{N}\,.$$

Proof. Since $2\delta_{\mathfrak{m}}\in Z_{\mathfrak{c}}^{+}$, it is of the form

$$2\delta_{\mathfrak{m}}=\sum_{\alpha\in\Pi-\Pi_0} n_\alpha\Lambda_\alpha \qquad (n_\alpha\in\boldsymbol{N})\,.$$

Since $K_{G/P}\otimes\det E_{\Lambda_0}^{*}$ is a holomorphic line bundle over G/P, the Chern class of $\varphi^{*}(K_{G/P}\otimes\det E_{\Lambda_0}^{*})$ contains a unique G_u-invariant 2-form $\eta(\Lambda_1)$ with

$$\Lambda_1=\sum_{\alpha\in\Pi-\Pi_1} m_\alpha\Lambda_\alpha \qquad (m_\alpha\in\boldsymbol{Z})\,.$$

By Lemma 2.2 and Fact 2, the first Chern class $c_1(L(E_{\Lambda_0}))$ of $L(E_{\Lambda_0})$ contains a unique G_u-invariant 2-form $\eta(\Lambda_0)$. Since E_{Λ_0} is a holomorphic vector bundle of rank $l+1$, we see that

$$2\delta_{\mathfrak{m}}=\Lambda_1+(l+1)\Lambda_0$$

by the formula (2.7) and Fact 2, and hence we get our claim. q.e.d.

REMARK. We may prove Lemma 2.3 by a computation on root systems as follows. Put $\Pi_0=\{\alpha_{i_1},\cdots,\alpha_{i_{l-1}}\}$. Since Π_0 is of type A_{l-1}, we have

$$\sum_{\alpha\in\Sigma_0^+}\alpha=(l-1)\alpha_{i_1}+\cdots+j(l-j)\alpha_{i_j}+\cdots+(l-1)\alpha_{i_{l-1}}\,,$$

where $\Sigma_0^+=\Sigma^+\cap\Sigma_0$.
Since Π_1 is of type A_l and $\Pi_1-\Pi_0=\{\alpha_0\}$, we may assume that

$$\frac{2(\alpha_0,\alpha_{i_1})}{(\alpha_0,\alpha_0)}=-1,\qquad \frac{2(\alpha_0,\alpha_{i_j})}{(\alpha_0,\alpha_0)}=0\qquad \text{for}\quad 2\leq j\leq l-1\,.$$

Thus we see that

$$\sum_{\alpha\in\Sigma_0^+}\alpha=-(l-1)\Lambda_0+\sum_{\alpha\in\Pi-\{\alpha_0\}} m_\alpha\Lambda_\alpha \qquad (m_\alpha\in\boldsymbol{Z})$$

Hence we have

$$\begin{aligned}2\delta_{\mathfrak{m}}&=2\delta-\sum_{\alpha\in\Sigma_0^+}\alpha=2(\sum_{\alpha\in\Pi}\Lambda_\alpha)+(l-1)\Lambda_0-\sum_{\alpha\in\Pi-\{\alpha_0\}} m_\alpha\Lambda_\alpha\\ &=(l+1)\Lambda_0+\sum_{\alpha\in\Pi-\{\alpha_0\}} n_\alpha\Lambda_\alpha=(l+1)\Lambda_0+\sum_{\alpha\in\Pi-\Pi_1} n_\alpha\Lambda_\alpha\,,\end{aligned}$$

where $n_\alpha\in\boldsymbol{N}$ for each $\delta\in\Pi-\Pi_1$, since $2\delta_{\mathfrak{m}}\in Z_{\mathfrak{c}}^{+}$.

3 $P^1(C)$-bundles over Kähler C-spaces and blowing down

Let N_1, N_2 be compact complex manifolds and consider holomorphic vector bundles E_1 of rank $l+1\geq 2$ over N_1, E_2 of rank $k+1\geq 2$ over N_2. We also assume that the total spaces $P(E_1)$ and $P(E_2)$ of projective bundles coincide as complex manifolds, which is denoted by M, and that there are holomorphic line bundles L_1' over N_1 and L_2' over N_2 such that the tautological line bundles $L(E_1\otimes L_1'^{-1})$ over $P(E_1\otimes L_1'^{-1})$ and $L(E_2\otimes L_2'^{-1})$ over $P(E_2\otimes L_2'^{-2})$ satisfy $L(E_1\otimes L_1'^{-1})^{-1}=L(E_2\otimes L_2'^{-1})$, more precisely, there is a holomorphic bundle isomorphism $L(E_1\otimes L_1'^{-1})^{-1}\to L(E_2\otimes L_2'^{-1})$ compatible with the identification: $P(E_1\otimes L_1'^{-1})\cong P(E_1)\cong P(E_2)\cong P(E_2\otimes L_2'^{-1})$. We also consider the $P^1(\boldsymbol{C})$-bundle $P(1\oplus L(E_1\otimes L_1'^{-1}))=P(L(E_2\otimes L_2'^{-1})\oplus 1)$ over $M=P(E_1\otimes L_1'^{-1})=P(E_2\otimes L_2'^{-1})$, whose total space is denoted by X. Note that complex submanifolds M_1, M_2 of X defined by the 0-section of $L(E_1\otimes L_1'^{-1})$ and 0-section of $L(E_2\otimes L_2'^{-1})$ are identified with $M=P(E_1\otimes L_1'^{-1})$ and $M=P(E_2\otimes L_2'^{-1})$ respectively.

We organize various spaces and maps by the following diagram:

$$
\begin{array}{ccccc}
X= & P(1\oplus L(E_1\otimes L_1'^{-1})) & = & P(L(E_2\otimes L_2'^{-1})\oplus 1) & \\
 & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \\
M= & P(E_1\otimes L_1'^{-1}) & = & P(E_2\otimes L_2'^{-1}) & \\
 & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & \\
 & N_1 & & N_2 &
\end{array}
\tag{3.1}
$$

Now the following lemma is a special case of Nakano [16], Fujiki-Nakano [6] (cf. Moisezon [14]).

Lemma 3.1. *There exists a complex manifiod Y containing N_1, N_2 as complex submanifolds and a holomorphic map Φ: $X\to Y$ in such a way that (X, Φ) is a composition of monoidal transforms from Y with centers N_1, N_2 and $M_1=\Phi^{-1}(N_1)$, $M_2=\Phi^{-1}(N_2)$, that is, Y is a complex manifold obtained from X by blowing down $M_1=P(E_1\otimes L_1'^{-1})$ to N_1 and $M_2=P(E_2\otimes L_2'^{-1})$ to N_2.*

Proof. Note that the normal bundle of $P(E_1\otimes L_1'^{-1})$ is the line bundle $L(E_1\otimes L_1'^{-1})$. Thus the condition in Fujiki-Nakano [6] is satisfied. q.e.d.

REMARK. Note that the tautological line bundle $L(E)$ over a projective bundle $P(E)$ is obtained from E by blowing up the 0-section of E to $P(E)$. Note also that $P(1\oplus L(E_1\otimes L_1'^{-1}))$ is a union of complex submanifolds $L(E_1\otimes L_1'^{-1})$ and $L(E_2\otimes L_2'^{-1})$ with the intersection $L(E_1\otimes L_1'^{-1})\cap L(E_2\otimes L_2'^{-1})=L(E_1\otimes L_1'^{-1})-$(0-section)$=L(E_2\otimes L_2'^{-1})-$(0-section). Thus Y is a union of the canonically imbedded complex submanifolds $E_1\otimes L_1'^{-1}$ and $E_2\otimes L_2'^{-1}$ with the intersection $E_1\otimes L_1^{-1}\cap E_2\otimes L_2'^{-1}=E_1\otimes L_1'^{-1}-$(0-section)$=E_2\otimes L_2'^{-1}-$(0-section), which is also $L(E_1\otimes L_1'^{-1})-$(0-section)$=L(E_2\otimes L_2'^{-1})-$(0-section).

Now we consider the triples (Π, Π_1^1, Π_0), (Π, Π_1^2, Π_0) of Dynkin diagrams which are one of the followings.

(a) The Dynkin diagram Π is connected, Π_0 is a subdiagram of Π and of type A_{l-1}, and subdiagrams Π_1^1, Π_1^2 of Π are of type A_l and contain Π_0 as a subdiagram.

(b) The Dynkin diagram Π has two connected components $\Pi(1)$ and $\Pi(2)$, and Π_0 is a subdiagram of Π which has also two connected components $\Pi_0(1)$ of type A_{l-1} and $\Pi_0(2)$ of type A_{k-1}. Subdiagrams Π_1^1, Π_1^2 of Π have also two connected components $\Pi_1^1(1)$ and $\Pi_1^1(2)$, $\Pi_1^2(1)$ and $\Pi_1^2(2)$ respectively, and we assume they satisfy the following conditions:

(1) $\Pi_1^1(1)$ is a subdiagram of $\Pi(1)$, of type A_l and contains $\Pi_0(1)$ as a subdiagram, and $\Pi_1^1(2)$ conicides with $\Pi_0(2)$.

(2) $\Pi_1^2(2)$ is a subdiagram of $\Pi(2)$, of type A_k and contains $\Pi_0(2)$ as a subdiagram, and $\Pi_1^2(1)$ conicides with $\Pi_0(1)$.

EXAMPLES 3.1. The vertices contained in Π_0, $\Pi_1^i-\Pi_0$, $\Pi-\Pi_1^i$ of a Dynkin diagram Π are denoted by ○, □, × for $i=1, 2$ respectively.

(a)	(Π, Π_1^1, Π_0)	×—□—○—○—○—○—×—×	
	(Π, Π_1^2, Π_0)	×—×—○—○—○—○—□—×	
(b)	(Π, Π_1^1, Π_0)	×—□—○—○—×	×—○—○—○—×—×
	(Π, Π_1^2, Π_0)	×—×—○—○—×	×—○—○—○—□—×

Put $\{\alpha_0(i)\}=\Pi_1^i-\Pi_0$ and $\Lambda_0(i)=\Lambda_{\alpha_0(i)}$ for $i=1, 2$.

We consider Kähler C-spaces associated to pairs of Dynkin diagrams and $P^1(\boldsymbol{C})$-bundles over Kähler C-spaces.

Case (a). We denote by G/U, G/P_1, G/P_2 the Kähler C-spaces associated to the pairs (Π, Π_0), (Π, Π_1^1), (Π, Π_1^2) respectively, and by E_1, E_2 the homogeneous vector bundles $E_{\Lambda_0(1)}$, $E_{\Lambda_0(2)}$ over G/P_1, G/P_2 respectively. By Lemma 2.1, we have $M=P(E_1)=P(E_2)=GU$, and $L(E_1)=L_{\Lambda_0(1)}$, $L(E_2)=L_{\Lambda_0(2)}$ by Lemma 2.2. Put $L_1H=L_{\Lambda_0(1)}$ and $L_2=L_{\Lambda_0(2)}$. Note that there is a holomorphic line bundle L_1' (resp. L_2') over $N_1=G/P_1$ (resp. over $N_2=G/P_2$) such that $\varphi_1^*L_1'=L_2$ (resp. $\varphi_2^*L_2'=L_1$), where φ_1: $M=G/U\to N_1=G/P_1$ (resp. φ_2: $M=G/U\to N_2=G/P_2$) is the projection. We thus have $L(E_1\otimes L_1'^{-1})=L_1\otimes L_2^{-1}=L(E_2\otimes L_2'^{-1})^{-1}$. Note also that the $P^1(\boldsymbol{C})$-bundle X is given by $P(1\oplus L_1\otimes L_2^{-1})$.

Case (b). We denote by G_1/U_1, G_1/P_1, G_2/U_2, G_2/P_2 the Kähler C-spaces associated to the pairs $(\Pi(1), \Pi_0(1))$, $(\Pi(1), \Pi_1^1(1))$, $(\Pi(2), \Pi_0(2))$, $(\Pi(2), \Pi_1^2(2))$ respectively and by E_1, E_2 the homogeneous vector bundles $E_{\Lambda_0(1)}$, $E_{\Lambda_0(2)}$ over G_1/P_1, G_2/P_2 respectively. We regard the vector bundle E_1 over G_1/P_1 (resp. E_2 over G_2/P_2) as a vector bundle over $N_1=G_1/P_1\times G_2/U_2$ (resp. $N_2=G_1/U_1\times G_2/P_2$), which is also denoted by E_1 (resp. E_2). By Lemma 2.1, we have $M=P(E_1)=P(E_2)=G_1/U_1\times G_2/U_2$, and $L(E_1)=L_{\Lambda_0(1)}$ and $L(E_2)=L_{\Lambda_0(2)}$ by Lemma 2.2. Put $L_1=L_{\Lambda_0(1)}$ and $L_2=L_{\Lambda_0(2)}$. Note that there is a holomorphic line bundle L_1' (resp. L_2') over N_1 (resp. over N_2) such that $\varphi_1^*L_1'=L_2$ (resp. $\varphi_2^*L_2'=L_1$), where

$\varphi_1\colon M\to N_1$ (resp. $\varphi_2\colon M\to N_2$) is the natural projection. We thus have $L(E_1\otimes L_1'^{-1})=L_1\otimes L_2^{-1}=L(E_2\otimes L_2'^{-1})^{-1}$. Note also that the $P^1(\boldsymbol{C})$-bundle X is given by $P(1\oplus L_1\otimes L_2^{-1})$. Put $G=G_1\times G_2$ and $U=U_1\times U_2$.

In case (a) and (b), we call X the *$P^1(C)$-bundle associated to the triples* (Π, Π_1^1, Π_0), (Π, Π_1^2, Π_0) *of Dynkin diagrams.* We also call Y obtained as in Lemma 3.1 the *compact complex manifold obtained from X by blowing down associated to the triples* (Π, Π_1^1, Π_0), (Π, Π_1^2, Π_0) *of Dynkin diagrams.* Note that in this case Y is almost homogeneous with respect to the complex Lie group G, since $E_1\otimes L_1'^{-1}-(0\text{-section})=L_1\otimes L_2^{-1}-(0\text{-section})$ is an open G-orbit in Y, and Y has a disconnected exceptional set which consists of two G-orbits N_1, N_2. Note also that N_1, N_2 are Kähler C-spaces associated to the pairs (Π, Π_1^1), (Π, Π_1^2) respectively.

4 Almost homogeneous Fano manifolds

A compact complex manifold is called *Fano* if its first Chern class is positive. In this section we prove the following.

Theorem 4.1. *Let* (Π, Π_1^1, Π_0) *and* (Π, Π_1^2, Π_0) *be triples of Dynkin diagrams, as in section* 3, *X the $P^1(\boldsymbol{C})$-bundle associated to these triples of Dynkin diagrams and Y the compact complex manifold obtained from X by blowing down associated these triples of Dynkin diagrams. Then Y is a Kähler manifold with positive first Chern class.*

First we recall the notation of K-S [12]. Let $\pi\colon L\to M$ be a holomorphic line bundle over a compact Kähler manifold M with a hermitian metric h. Denote by $\mathring{L}$ the open set $L-(0\text{-section})$ of L. Let t be a function on $\mathring{L}$ which depends only on the norm s of h and increases for the norm. Then the horizontal lift $\tilde{X}$ of a vector field X of M to $\mathring{L}$ with respect to the canonical hermitian connection of L is characterized by

$$(4.1)\qquad \pi_*\tilde{X}=X\,,\quad \tilde{X}[t]=(\tilde{J}\tilde{X})[t]=0$$

where $\tilde{J}$ is the almost complex structure of the total space of L. We decompose the group $\boldsymbol{C}^*$ into $S^1\times\boldsymbol{R}^+$ and define holomorphic vector fields S, H on $\mathring{L}$ generated by S^1-action, $\boldsymbol{R}^+$-action respectively so that

$$(4.2)\qquad \exp 2\pi S=id\,,\quad H=-\tilde{J}S\,,\quad H[t]>0\,.$$

If we denote by ρ_L the Ricci form of L, then we have

$$(4.3)\qquad [\tilde{X}, \tilde{Y}]-\widetilde{[X, Y]}=-\rho_L(X, Y)S\,.$$

Define a hermitian 2-form B on M, the Ricci tensor of L, by

$$B(X, Y) = \rho_L(X, JY)\,, \tag{4.4}$$

where J is the almost complex structure of M.

We also cinsider a riemannian metric $\tilde{g}$ on $\mathring{L}$ of the form

$$\tilde{g} = dt^2 + (dt\circ \tilde{J})^2 + \pi^* g_t \tag{4.5}$$

where $\{g_t\}$ is a one-parameter family of riemannian metrics on M. Define a positive function u on $\mathring{L}$ depending only on t by

$$u(t)^2 = \tilde{g}(H, H)\,. \tag{4.6}$$

Then, by Lemma 1.1 of K-S [12], the metric $\tilde{g}$ on $\mathring{L}$ is a Kähler metric if and only if each g_t is a Kähler metric on M and $\frac{d}{dt} g_t = -u(t)B$. We also assume that the range of t contains 0. Put

$$U(t) = \int_0^t u(t)dt\,, \tag{4.7}$$

then we have

$$g_t = g_0 - U(t)B\,. \tag{4.8}$$

We put

$$u(t) = a\cos\frac{t}{a} \quad \text{with} \quad t \in \left(-\frac{\pi}{2}a, \frac{\pi}{2}a\right) \qquad \text{for} \quad a>0\,, \tag{4.9}$$

and define $U(t)$ by (4.7). Take a Kähler metric g_0 on M and assume that each g_t defined by (4.8) is positive definite. We condider the Kähler metric $\tilde{g}$ on $\mathring{L}$ of the form (4.5) satisfying (4.6). Then we have

$$U(t) = a^2 \sin\frac{t}{a} + b\,. \tag{4.10}$$

We may assume that the range of U is $(-(l+1), k+1)$ for given positive integers k and l, by changing the origin of U and $a>0$ if necessary. Thus we have

$$a^2 = \frac{1}{2}(l+k+2)\,, \quad b = \frac{1}{2}(k-l)\,. \tag{4.11}$$

Lemma 4.2. *Let s be the norm of the hermitian line bundle $\pi\colon L \to M$. Then on $\mathring{L}$*

$$U(t) = \frac{(a^2+b)s^2 + (b-a^2)}{s^2+1}\,, \tag{4.12}$$

by replacing $t(s)$ by $t(cs)$ for a positive constant c if necessary.

Proof. Note that, in terms of polar coordinates (r, θ) on C^*, the natural complex structure $\tilde{J}$ on C^* is given by

$$\tilde{J}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \tilde{J}\frac{\partial}{\partial\theta} = -r\frac{\partial}{\partial r}$$

and that if $s=\tilde{c}r$ for a constant $\tilde{c}>0$, $\tilde{J}\frac{\partial}{\partial\theta}=-s\frac{\partial}{\partial s}$. Note also that the restriction to a fiber C^* of the C^*-action on $\mathring{L}$ coincides with the group action of C^*. Thus the vector field H restricted to a fiber C^* satisfies

$$H = -\tilde{J}S = s\frac{\partial}{\partial s} = s\frac{dt}{ds}\frac{\partial}{\partial t}$$

and thus

$$s\frac{dt}{ds} = u(t) = a\cos\frac{t}{a}.$$

Since $\int \sec x\,dx=\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$, we see that

$$cs = \tan\left(\frac{\pi}{4}+\frac{t}{2a}\right) = \frac{1+\tan(t/2a)}{1-\tan(t/2a)} \qquad \left(-\frac{\pi a}{2}<t<\frac{\pi a}{2}\right)$$

for some positive constant c, and $\tan(t/2a)=(cs-1)/(cs+1)$. Thus we have

$$U(t) = a^2\sin\frac{t}{a}+b = a^2\frac{2\tan(t/2a)}{1+\tan^2(t/2a)}+b = a^2\frac{(cs)^2-1}{(cs)^2+1}+b. \qquad \text{q.e.d.}$$

In general, let $p: E\to N$ be a holomorphic vector bundle over a compact complex manifold N, $\varphi: P(E)\to N$ the associated projective bundle over N and $\pi: L(E)\to P(E)$ the tautological line bundle over $P(E)$. Denote by $\mathring{E}$ the open set $E-$(0-section) of E. Let h_1 be a hermitian metric on E. Since $\mathring{E}=\mathring{L}(E)$ $=L(E)-$(0-section), a metric h_1 on E defines a hermitian metric h on $L(E)$: for $x\in P(E)$ and $v, w\in\mathring{L}(E)=\mathring{E}$ with $\pi(v)=\pi(w)=x$, $h_x(v, w)=(h_1)_{\varphi(x)}(v, w)$.

REMARK. In general a fiber metric on $L(E)$ does not define a hermitian metric on E. There is a natural one-to-one correspondence between complex Finsler structures in E and hermitian structures in $L(E)$. See Kobayashi [10].

Corollary 4.3. *Let N_1, N_2, E_1, E_2, L_1' and L_2' be as in* (3.1) *with M Kähler. Assume that there are hermitian metrics h_1 on $E_1\otimes L_1'^{-1}$ and h_2 on $E_2\otimes L_2'^{-1}$ with the following properety: If we denote the hermitian metric on $L=L(E_1\otimes L_1'^{-1})$ induced from h_1 by h and the norm of h by s, the norm s_2 of the hermitian metric on $L(E_2\otimes L_2'^{-1})$ induced from h_2 depends only on s, under the identification: $\mathring{L}(E_1\otimes L_1'^{-1})=\mathring{L}(E_2\otimes L_2'^{-1})$. Assume further that we can construct a Kähler metric*

$\tilde{g}$ on $\mathring{L}$ in the above way, that is, each g_t in (4.8) *is positive definite. We choose the function t in such a way that the range $(-(l+1), k+1)$ of U is $l+1=\operatorname{rank} E_1=$ codimension N_1 in Y and $k+1=\operatorname{rank} E_2=$ codimension N_2 in Y. Then the function U on the open set $\mathring{L}$ of the compact complex manifold Y is extended to a smooth function U on Y such that the range of U on the complex submanifold $E_1\otimes L_1'^{-1}$ is $[-(l+1), k+1)$ and the range of U on $E_2\otimes L_2'^{-1}$ is $(-(l+1), k+1]$.*

In general, for a Kähler metric g the corresponding Kähler form is denoted by ω_g. We now seek the condition that the metric $\tilde{g}$ on $\mathring{L}=\mathring{L}(E_1\otimes L_1'^{-1})=\mathring{L}(E_2\otimes L_2'^{-1})$ can be written as

$$\omega_{\tilde{g}} = (\varphi_1\circ\pi)^*\underline{\omega}_1-2\sqrt{-1}d'd''f_0 = (\varphi_2\circ\pi)^*\underline{\omega}_2-2\sqrt{-1}d'd''f_\infty$$

where $\underline{\omega}_i$ is a Kähler form of a Kähler metric $\underline{g}_i$ on N_i for $i=1, 2$ and f_0, f_∞ are smooth functions on $\mathring{L}$ depending only on t.

Lemma 4.4. *Under the assumptions in Corollary* 4.3, *if the Kähler metric g_0 on $M=P(E_1\otimes L_1'^{-1})=P(E_2\otimes L_2'^{-1})$ and the hermitian form B on M satisfy that $g_0+(l+1)B=\varphi_1^*\underline{g}_1$ where $\underline{g}_1$ is a Kähler metric on N_1 and $g_0-(k+1)B=\varphi_2^*\underline{g}_2$ where $\underline{g}_2$ is a Kähler metric on N_2, then there are smooth functions f_0: $E_1\otimes L_1'^{-1}\to \boldsymbol{R}$ and f_∞: $E_2\otimes L_2'^{-1}\to\boldsymbol{R}$ such that on $\mathring{L}$*

$$\omega_{\tilde{g}} = (\varphi_1\circ\pi)^*\underline{\omega}_1-2\sqrt{-1}d'd''f_0 = (\varphi_2\circ\pi)^*\underline{\omega}_2-2\sqrt{-1}d'd''f_\infty\,. \tag{4.13}$$

Proof. We use the notation $\hat{\partial}_\alpha$, $\hat{\partial}_{\bar{\alpha}}$ $(0\le\alpha\le n)$ used in K-S [12]. We may assume that $\hat{\partial}_\alpha t=\hat{\partial}_{\bar{\alpha}}t=0$ $(1\le\alpha\le n)$ on a fiber. First we consider a function f on $\mathring{L}$ satisfying $\omega_{\tilde{g}}=(\varphi_1\circ\pi)^*\underline{\omega}_1-2\sqrt{-1}d'd''f$. Since $\tilde{g}_{0\bar{0}}=\hat{\partial}_{\bar{0}}\hat{\partial}_0 f$, we have

$$2u^2 = u\frac{d}{dt}\left(u\frac{df}{dt}\right) \tag{4.14}$$

by Lemmas 1.2 and 1.3 of K-S [12]. As (2.15) in K-S [12], we put $\varphi(U)=u^2$. Then the equation (4.14) is given by

$$2 = \frac{d}{dU}\left(\varphi(U)\frac{df}{dU}\right), \quad \text{since} \quad \frac{d}{dt} = u\frac{d}{dU}\,. \tag{4.15}$$

By solving this equation, we have

$$\frac{df}{dU} = \frac{2U+C}{\varphi(U)} \qquad \text{for some constant} \quad C\in\boldsymbol{R}\,. \tag{4.16}$$

Now $\varphi(U)=u^2=a^2\cos^2\dfrac{t}{a}=a^2\left(1-\sin^2\left(\dfrac{t}{a}\right)\right)$. By (4.10) and (4.11), we see that

$$(4.17)\qquad \varphi(U)=\frac{1}{a^2}(a^2+b-U)(a^2-b+U)=\frac{1}{a^2}(k+1-U)(l+1+U)\,.$$

Let f_0 denote a solution of (4.16) with $C=2(l+1)$. Then the equation (4.16) is given by

$$(4.18)\qquad \frac{df_0}{dU}=\frac{2a^2}{k+1-U}$$

and hence $f_0=-2a^2\log(k+1-U)+C'$ ($C'\in\boldsymbol{R}$) and f_0 is extended to a smooth function on $E_1\otimes L_1'^{-1}$. Similarly we have a solution

$$f_\infty=-2a^2\log(l+1+U)+C''\qquad (C''\in\boldsymbol{R})$$

of (4.16) with $C=-2(k+1)$, which is a smooth function on $E_2\otimes L_2'^{-1}$. By K-S [12] Lemma 1.3, we have

$$(4.19)\qquad \begin{cases}\hat{\partial}_{\bar{0}}\hat{\partial}_\beta f=0\\ \hat{\partial}_{\bar{\alpha}}\hat{\partial}_\beta f=-\dfrac{1}{2}u\dfrac{df}{dt}B_{\bar{\alpha}\beta}=-\dfrac{1}{2}\varphi(U)\dfrac{df}{dU}B_{\bar{\alpha}\beta}\,.\end{cases}$$

Since $\dfrac{1}{2}\varphi(U)\dfrac{df_0}{dU}=U+l+1$ by (4.17) and (4.18), we have

$$(g_0-UB)+\frac{1}{2}\varphi(U)\frac{df_0}{dU}B=g_0+(l+1)B=\varphi_1^*\underline{g}_1\,.$$

Thus $\omega_{\tilde{g}}=(\varphi_1\circ\pi)^*\omega_1-2\sqrt{-1}\,d'd''f_0$ on $\mathring{L}$. Similarly, $\omega_{\tilde{g}}=(\varphi_2\circ\pi)^*\underline{\omega}_2-2\sqrt{-1}d'd''f_\infty$ on $\mathring{L}$. q.e.d.

Corollary 4.5. *Under the same assumption of Corollary* 4.3 *and Lemma* 4.4, *the Kähler metric $\tilde{g}$ on $\mathring{L}$ can be extended to a Kähler metric on the complex manifold Y.*

Proof. Note that by (4.11) and (4.12) we have $k+1-U=\dfrac{l+k+2}{s^2+1}$ where s^2 is the square of the norm of the hermitian metric h_1 on $E_1\otimes L_1'^{-1}$. Thus we have

$$(4.20)\qquad f_0=2a^2\log(1+s^2)-2a^2\log(k+l+2)+C'\,.$$

Let $p_1\colon E_1\otimes L_1'^{-1}\to N_1$ be the projection. It is easy to see that

$$(4.21)\qquad p_1^*\underline{\omega}_1-4a^2\sqrt{-1}d'd''\log(1+s^2)$$

is the Kähler form of a Kähler metric on a neighborhood of 0-section of $p_1\colon E_1\otimes L_1'^{-1}\to N_1$. Since $p_1=\varphi_1\circ\pi$ on $E_1\otimes L_1'^{-1}-$(0-section)$=\mathring{L}$, the metric $\tilde{g}$ on $\mathring{L}$ can be extended to a Kähler metric

$$p_1^*\underline{\omega}_1-4a^2\sqrt{-1}d'd''\log(1+s^2)$$

on $E_1\otimes L_1'^{-1}$. Similarly the metric $\tilde{g}$ on $\mathring{L}$ can be extended to a Kähler metric on $E_2\otimes L_2'^{-1}$ and hence to a Kähler metric on Y. q.e.d.

Corollary 4.6. *Under the same assumption of Theorem* 4.1, *the compact complex manifold Y is Kähler. More precisely a Kähler metric $\tilde{g}$ on $\mathring{L}=L_1\otimes L_2^{-1}-$* (0-section) *can be extended tc a Kähler metric on Y, which is also denoted by $\tilde{g}$.*

Proof. Let g_0 be the G_u-invariant Kähler metric on $M=G/U=P(E_1\otimes L_1'^{-1})=P(E_2\otimes L_2'^{-1})$ corresponding to $8\pi\delta_{\mathfrak{m}}$ as in Fact 1 in section 1 and h a G_u-invariant hermitian metric on the homogeneous line bundle $L=L(E_1\otimes L_1'^{-1})=L(E_2\otimes L_2'^{-1})^{-1}$ over M. Since we are in G_u-invariant situation, the first assumption in Corollary 4.3 is satisfied. And the hermitian form B on M is G_u-invariant and corresponds to $4\pi(-\Lambda_0(1)+\Lambda_0(2))\in\mathfrak{c}$ by Fact 2 in section 2. Thus g_t is G_u-invariant and corresponds to $4\pi\{2\delta_{\mathfrak{m}}+U(t)(\Lambda_0(1)-\Lambda_0(2))\}$, which belongs to $\mathfrak{c}^+$ by Lemma 2.3. So the second assumption in Corollary 4.3 is satisfied. In the same way we see that $g_0+(l+1)B=\varphi_1^*\underline{g}_1$ where $\underline{g}_1$ is a G_u-invariant Kähler metric on the Kähler C-space N_1 associated to the pair (Π, Π_1^1) and $g_0-(k+1)B=\varphi_2^*\underline{g}_2$ where $\underline{g}_2$ is a G_u-invariant Kähler metric on the Kähler C-space N_2 associated to the pair (Π, Π_1^2). Thus the Kähler metric $\tilde{g}$ on $\mathring{L}$ can be extended to a Kähler metric on Y. q.e.d.

From now on we assume further that the eigenvalues of B, regarded as a hermitian form on a holomorphic tangent space of M, with respect to g_0 are constant on M. Note that the assumption in Lemma 4.4 implies that $\frac{-1}{l+1}$ $\left(\text{resp. } \frac{1}{k+1}\right)$ is an eigenvalue of B with respect to g_0 with multiplicity l (resp. k) because $\varphi_1^*\underline{g}_1$ (resp. $\varphi_2^*\underline{g}_2$) is a positive semi-definite hermitian form of nullity l (resp. k). Thus the function $\det(g_0^{-1}g_t)=Q(U)$ on $\mathring{L}$ is given by

$$Q(U)=\det(1-Ug_0^{-1}B)=\left(1+\frac{U}{l+1}\right)^l\left(1-\frac{U}{k+1}\right)^k Q_1(U) \tag{4.22}$$

where $Q_1(U)$ is a polynomial of U such that $Q_1(U)\neq 0$ on $[-(l+1), k+1]$. Here also $g_0^{-1}g_t$ and $g_0^{-1}B$ are regarded as endomorphisms on homolorphic tangent spaces of M.

Theorem 4.7. *Under the assumption above, together with assumptions in Corollary* 4.3 *and Lemma* 4.4, *if the Ricci tensor r_0 of the Kähler metric g_0 on M is equal to g_0, then the first Chern class $c_1(Y)$ of Y is positive. More precisely, let $\tilde{\rho}$ be the Ricci form of the Kähler metric $\tilde{g}$ on Y, then there is a C^∞ function $F(U)$ of U on $[-(l+1), k+1]$ such that*

$$\tilde{\rho}-\omega_{\tilde{g}} = -2\sqrt{-1}d'd''F. \tag{4.23}$$

Proof. By Lemmas 1.2, 1.3 and 1.4 in K-S [12], we see that the equation (4.23) is equiavlent to the equation

$$\varphi\frac{d}{dU}\log(\varphi Q)+2U+\varphi\frac{dF}{dU} = 0. \tag{4.24}$$

By solving this equation,

$$F = -\log(\varphi Q)-2\int\frac{U}{\varphi}dU. \tag{4.25}$$

By (4.17) and (4.22),

$$\log(\varphi Q) = (l+1)\log(l+1+U)+(k+1)\log(k+1-U)+\log Q_1+C_1 \tag{4.26}$$

where $C_1 \in \boldsymbol{R}$.

By (4.11) and (4.17),

$$2\frac{U}{\varphi} = 2a^2\frac{U}{(k+1-U)(l+1+U)} = \frac{k+1}{k+1-U}-\frac{l+1}{l+1+U}$$

and hence

$$2\int\frac{U}{\varphi}dU = -(k+1)\log(k+1-U)-(l+1)\log(l+1+U). \tag{4.27}$$

Thus $F=-\log Q_1+C_2$ $(C_2 \in \boldsymbol{R})$.
Since $Q_1(U)\neq 0$ on $[-(l+1), k+1]$, F is a smooth function on $[-(l+1), k+1]$ and hence, it is smooth on Y. q.e.d.

Proof of Theorem 4.1. Since g_0 and B in Corollary 4.6 are G_u-invariant, the eigenvalues of B with respect to g_0 are constant. By (1.20) we have $r_0=g_0$. Note that the assumptions in Corollary 4.3 and Lemma 4.4 are astisfied as in the proof of Corollary 4.6. Thus our theorem follows from Theorem 4.7. q.e.d.

REMARK. Note that, under the assumption in Theorem 4.1, by taking $L=L_1\otimes L_2^{-1}$, $\hat{L}=Y$, $M=P(E_1)=P(E_2)=G/U$ and the metric $\tilde{g}$ on Y as in Corollray 4.6, the following assumptions A) and B) in K-S [12] are satisfied for a Kähler metric $\tilde{g}$ on $\mathring{L}$ of the form (4.5).

Assumption A). Let $(\min t, \max t)$ be the range of t. The function t extends to a continuous function on $\hat{L}$ with range $[\min t, \max t]$, and the subset $M_{\min}$ (resp. $M_{\max}$) of $\hat{L}$ defined by $t=\min t$ (resp. $t=\max t$) is a complex submanifold of $\hat{L}$ with codimension $D_{\min}$ (resp. $D_{\max}$). Moreover the Kähler metric $\tilde{g}$ extends to a Kähler metric on $\hat{L}$, which is also denoted by $\tilde{g}$.

Assumption B). (1) The Kähler form of the metric $\tilde{g}$ on $\hat{L}$ is cohomologous to the Ricci form $\tilde{\rho}$ of $\tilde{g}$. (2) The eigenvalues of the Ricci tensor r_0 of g_0 with respect to g_0 are constant on M.

5 Non-homogeneous Kähler-Einstein metrics

Let $\pi: L \to M$ be a hermitian holomorphic line bundle over a compact Kahler manifold M. As above we consider a Kähler metric $\tilde{g}$ on $\mathring{L}$ of the form (4.5). We also assume that the eigenvalues of B with respect to a Kähler metric g_0 on M are constant and a compactification $\hat{L}$ of $\mathring{L}$ satisfies the assumptions A) and B). By Lemma 2.2 of K-S [12], we may assume that the range of U is $[-D_{\min}, D_{\max}]$.

Now we give a necessary condition for a Kähler-Einstein metric on $\hat{L}$ of the form (4.5) being homogeneous.

Theorem 5.1. *Under the above situation, assume further that the Ricci tensor $\tilde{r}$ of the Kähler metric $\tilde{g}$ of $\hat{L}$ of the form* (4.5) *is equal to $\tilde{g}$. If $\tilde{g}$ is riemannian homogeneous, the followings hold.*

(1) *If the codimensions $D_{\min}=D_{\max}=1$, then $B=0$.*

(2) *If one of the codimensions $D_{\min}$, $D_{\max}$ is equal to* 1 *and the other* >1, *then the non-zero eigenvalues of $g_0^{-1}B$ are all equal.*

(3) *If both codimensions $D_{\min}$, $D_{\max}>1$, then the number of distinct non-zero eigenvalues of $g_0^{-1}B$ are* 2.

First we recall the following.

Lemma 5.2. *Every complete totally geodesic submanifold of a homogeneous riemannian manifold is homogeneous.*

Proof. See K-N [11] Chap. 7, Corollary 8.10.

Proof of Theorem 5.1. Since the closure S^2 of each fiber $\boldsymbol{C}^*$ is a totally geodesic submanifold of $(\hat{L}, \tilde{g})$ and $\tilde{g}$ is homogeneous, it is a riemannian homogeneous manifold by Lemma 5.2. We use the notations in K-S [12]. Note that the induced metric $\tilde{g}_{\bar{0}0}=2u^2$ is an Einstein metric on S^2, since S^2 is 2-dimensional. Thus we have

$$(5.1)\qquad -\hat{\partial}_{\bar{0}}\hat{\partial}_0(\log(2u^2)) = c\cdot 2u^2 \qquad \text{where } c \text{ is a constant.}$$

Note that $u^2=\varphi$, $u\dfrac{d}{dt}=\varphi\dfrac{d}{dU}$. By Lemma 1.3 of K-S [12], we see that the equation (5.1) is given by

$$(5.2)\qquad -\varphi\frac{d}{dU}\left(\varphi\frac{d}{dU}(\log\varphi)\right) = c\cdot\varphi$$

and hence

$$\frac{d\varphi}{dU} = -cU + \text{constant}. \tag{5.3}$$

Thus φ is a quadric polynomial of U. On the other hand φ vanishes at $U = -D_{\min}, D_{\max}$. Therefore φ is of the form

$$\varphi = c'(U + D_{\min})(U - D_{\max}) \qquad \text{for some } c' \in \boldsymbol{R}\,.$$

By (4.1.5) in K-S [12], the first term of Taylor expansion of $\varphi(U)$ at $U = -D_{\min}$ is given by $2(U + D_{\min})$. Thus φ is given by

$$\varphi = \frac{-2}{D_{\min} + D_{\max}}(U + D_{\min})(U - D_{\max})\,. \tag{5.4}$$

Since $\tilde{r} = \tilde{g}$, the polynomial Q of U satisfies the equation

$$\frac{d}{dU}\varphi + 2U + \frac{\varphi}{Q}\cdot\frac{dQ}{dU} = 0 \tag{5.5}$$

by Lemma 2.2 in K-S [12]. By (5.4) and (5.5), we have

$$\begin{aligned}\frac{d}{dU}\log Q &= -\frac{(2U + D_{\min} - D_{\max}) - U(D_{\max} + D_{\min})}{(U + D_{\min})(U - D_{\max})} \\ &= -\frac{1 - D_{\min}}{U + D_{\min}} - \frac{1 - D_{\max}}{U - D_{\max}}\,.\end{aligned}$$

Thus we have

$$\log Q = -(1 - D_{\min})\log(U + D_{\min}) - (1 - D_{\max})\log|U - D_{\max}| + c''$$

and thus we have

$$Q = C(U + D_{\min})^{D_{\min}-1}(D_{\max} - U)^{D_{\max}-1}\,. \tag{5.6}$$

Since $Q = \det(1 - Ug_0^{-1}B)$, we get our claim. q.e.d.

Now we recall the following theorem in K-S [12].

Theorem 5.3 (Theoerm 4.2 in K-S [12]). *Let M be a compact Kähler-Einstein manifold whose Kähler form represents the first Chern class $c_1(M)$ and L a hermitian holomorphic line bundle over M. Assume that there is a Kähler metric $\tilde{g}$ on a compactification $\hat{L}$ of $\mathring{L}$ of the form* (4.5) *with g_0 Kähler-Einstein, whose Kähler form is cohomologous to the Ricci form of $\hat{L}$ and that the eigenvalues of the Ricci form B of L with respect to g_0 are constant. Then the complex manifold $\hat{L}$ admits a Kähler-Einstein metric if and only if the integral*

$$(5.7) \qquad F(\hat{L}) = \int_{-D_{\min}}^{D_{\max}} UQ(U)dU$$

vanishes.

Now let (Π, Π_0) be an effective pair of Dynkin diagrams as in section 1 and $M=G/U$ the Kähler C-space associated to (Π, Π_0). Consider the Kähler-Einstein metric g_0 on G/U corresponding to $8\pi\delta_{\mathfrak{m}}\in\mathfrak{c}^+$ with $r_0=g_0$ and a holomorphic line bundle L_Λ on G/U for $\Lambda\in Z_{\mathfrak{c}}$ with a G_u-invariant hermitian metric. Note that a unique G_u-invariant form in the first Chern class $c_1(L_\Lambda)$ is given by $\eta(\Lambda)$ of (1.22). Let B be the Ricci tensor of L_Λ which is the G_u-invariant hermitian form on M corresponding to $-4\pi\Lambda\in\mathfrak{c}$.

Lemma 5.4. *Under the assumption above, we have*

$$(5.8) \qquad Q(x) = \det(1-xg_0^{-1}B) = \prod_{\alpha\in\Sigma_{\mathfrak{m}}^+}\left(1+\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}x\right).$$

Proof. Straightforwards by (1.18).

Let ρ be an automorphism of Dynkin diagram Π such that $\rho^2=id$ and $\rho\neq id$. It is known that if Π is irreducible and it admits such an automorphism ρ, then Π is of type A_n $(n\geq 2)$, D_n $(n\geq 4)$ or E_6 (cf. [5]). Note also that if Π has two connected components $\Pi(1)=\{\alpha_1, \cdots, \alpha_n\}$, $\Pi(2)=\{\beta_1, \cdots, \beta_n\}$ and $\Pi(1)$, $\Pi(2)$ are isomorphic by the map $\alpha_i\to\beta_i$, then the map $\rho\colon\Pi\to\Pi$ defined by $\rho(\alpha_i)=\beta_i$, $\rho(\beta_i)=\alpha_i$ (for each i) is such an automorphism of Π, and from now on we consider this automorphism ρ exclusively in the case when a Dynkin diagram Π is reducible. A pair (Π, Π_0) of Dynkin diagram is said to be *admissible* for ρ if $\rho(\Pi_0)=\Pi_0$.

Lemma 5.5. *Let* (Π, Π_0) *be an admissible pair of Dynkin diagrams for an automorphism* ρ *and assume that* $\Lambda\in Z_{\mathfrak{c}}$ *satisfies* $\rho(\Lambda)=-\Lambda$. *Then*

$$Q(x) = \prod_{\alpha\in\Sigma_{\mathfrak{m}}^+}\left(1+\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}x\right)$$

is an even function of x.

Proof. We use notation in section 1. Since ρ induces the bijections $\rho\colon\Sigma^+\to\Sigma^+$ and $\rho\colon\Sigma_0^+\to\Sigma_0^+$, it also induces the bijection $\rho\colon\Sigma_{\mathfrak{m}}^+\to\Sigma_{\mathfrak{m}}^+$. Since $2\delta_{\mathfrak{m}}=\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+}\alpha$, we have $\rho(2\delta_{\mathfrak{m}})=2\delta_{\mathfrak{m}}$. Note that $(\Lambda, \rho(\alpha))=(\rho(\Lambda), \alpha)=-(\Lambda, \alpha)$. Thus if $\rho(\alpha)=\alpha$, $(\Lambda, \alpha)=0$. For $\alpha\in\Sigma_{\mathfrak{m}}^+$, $\alpha\neq\rho(\alpha)$,

$$\left(1+\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}x\right)\left(1+\frac{(\Lambda, \rho(\alpha))}{(2\delta_{\mathfrak{m}}, \rho(\alpha))}x\right) = 1-\left(\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}\right)^2\cdot x^2.$$

Thus we get our claim. q.e.d.

Corollary 5.6. *Let G/U be a Kähler C-space associated to an admissible pair (Π, Π_0) for an automorphism ρ. Put $2\delta_{\mathfrak{m}}=\sum_{\alpha\in\Pi-\Pi_0} a_\alpha\Lambda_\alpha$. Let L_Λ be a holomorphic line bundle over G/U such that $\rho(\Lambda)=-\Lambda$ and $\Lambda=\sum_{\alpha\in\Pi-\Pi_0} b_\alpha\Lambda_\alpha$ with $|b_\alpha|<a_\alpha$ for each $\alpha\in\Pi-\Pi_0$. Then the $P^1(\boldsymbol{C})$-bundle $P(1\otimes L_\Lambda)$ over G/U admits an Kähler-Einstein metric.*

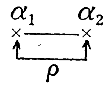
Proof. Note that by the assumption for Λ the absolute values of eigenvalues of B are less than 1. By Theorem 5.4 in K-S [12], it is sufficient to see that the integral

$$\int_{-1}^{1} UQ(U)dU=\int_{-1}^{1} U\cdot\det(1-Ug_0^{-1}B)dU$$

vanishes. Since $\det(1-Ug_0^{-1}B)$ is an even function of U by Lemma 5.5, we get our claim. q.e.d.

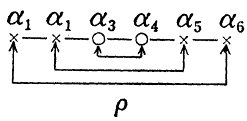
EXAMPLES 5.1. In the following cases the $P^1(\boldsymbol{C})$-bundle $P(1\oplus L_\Lambda)$ over a Kähler C-space G/U admits an Kähler-Einstein metric. The vertices contained in Π_0, $\Pi-\Pi_0$ of a Dynkin diagram Π are denoted by $\bigcirc$, $\times$ respectively.

(1) (Π, Π_0)



$2\delta_{\mathfrak{m}}=2(\Lambda_{\alpha_1}+\Lambda_{\alpha_2})$. Put $\Lambda=\Lambda_{\alpha_1}-\Lambda_{\alpha_2}$. Then $\rho(\Lambda)=-\Lambda$. In this case the associated $P^1(\boldsymbol{C})$-bundle $P(1\oplus L_\Lambda)$ is the Example 5.10 in K-S [12].

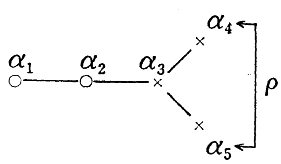
(2) (Π, Π_0)



$2\delta_{\mathfrak{m}}=2\Lambda_{\alpha_1}+4\Lambda_{\alpha_2}+4\Lambda_{\alpha_5}+2\Lambda_{\alpha_6}$. Put $\Lambda=\Lambda_{\alpha_1}+\Lambda_{\alpha_2}-\Lambda_{\alpha_5}-\Lambda_{\alpha_6}$. Then $\rho(\Lambda)=-\Lambda$. In this case $G=SL(7, \boldsymbol{C})$ and U is given by

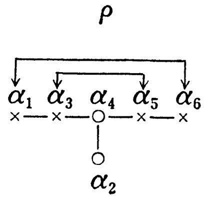
$$U=\left\{\begin{pmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}\right\}.$$

(3) (Π, Π_0)



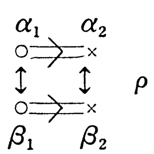
$2\delta_{\mathfrak{m}}=4\Lambda_{\alpha_3}+2\Lambda_{\alpha_4}+2\Lambda_{\alpha_5}$. Put $\Lambda=\Lambda_{\alpha_4}-\Lambda_{\alpha_5}$. Then $\rho(\Lambda)=-\Lambda$.

(4) (Π, Π_0)



$2\delta_{\mathfrak{m}}=2\Lambda_{\alpha_1}+4\Lambda_{\alpha_3}+4\Lambda_{\alpha_5}+2\Lambda_{\alpha_6}$. Put $\Lambda=\Lambda_{\alpha_3}-\Lambda_{\alpha_5}$. Then $\rho(\Lambda)=-\Lambda$.

(5) (Π, Π_0)



$2\delta_{\mathfrak{m}}=4\Lambda_{\alpha_2}+4\Lambda_{\beta_2}$. Put $\Lambda=\Lambda_{\alpha_2}-\Lambda_{\beta_2}$. Then $\rho(\Lambda)=-\Lambda$.

Now we consider triples (Π, Π_1^1, Π_0), (Π, Π_1^2, Π_0) of Dynkin diagrams as in section 3. These triples are said to be *admissible for an automorphism* ρ of Dynkin diagram Π if $\rho(\Pi_0)=\Pi_0$, and $\rho(\alpha_0(1))=\alpha_0(2)$. Note that the holomorphic line bundle $L_1\otimes L_2^{-1}=L(E_1\otimes L_1'^{-1})=L(E_2\otimes L_2'^{-1})^{-1}$ over $M=G/U$ is given by L_Λ, where $\Lambda=\Lambda_0(1)-\Lambda_0(2)$ and thus $\rho(\Lambda)=-\Lambda$. By Lemma 2.4 Λ satisfies the assumption in Corollary 5.6. Recall that the $P^1(\boldsymbol{C})$-bundle X is given by $X=P(1\oplus L_\Lambda)$.

Corollary 5.7. *Let* (Π, Π_1^1, Π_0), (Π, Π_1^2, Π_0) *be admissible triples of Dynkin diagrams for* ρ. *Then both the* $P^1(\boldsymbol{C})$*-bundle* X *associated to these triples of Dynkin diagrams and the compact complex manifold* Y *obtained from* X *by blowing down associated to these triples of Dynkin diagrams admit Kähler-Einstein metrics.*

Proof. By the last Remark in section 4 and Theorem 5.3, it is enough to see that the integral (5.7) vanishes. Since $D_{\max}=D_{\min}$ and $\underset{\sim}{Q}(U)$ is an even function of U by Lemma 5.5, we get our claim. q.e.d.

REMARK. As in K-S [12], X and Y admit a Kähler-Einstein metric if and only if Futaki's integral $F(H)$ of the holomorphic vector field H vanishes. We can explain Corollaries 5.6 and 5.7 as follows. The automorphism ρ of the Dynkin diagram induces automorphisms γ_X and γ_Y of the complex manifolds X and Y respectively, such that $\gamma_* H=-H$. Thus Futaki's integral $F(H)$ vanishes, because it is invariant under complex automorphisms (cf. Futaki [7] Theorem 2.1). However the existence of such an automorphism γ is not necessary to the existence of a Kähler-Einstein metric. See Example 5.3 (2) and Example 6.4 in [17].

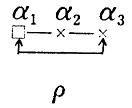
Corollary 5.8. *Under the same notation as in Corollary 5.7, if the number of elements in* $\Pi-\Pi_0 \geq 3$, *then the Kähler-Einstein metric on* Y *is non-homogeneous.*

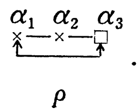
Proof. Since $\Pi-(\Pi_1^1 \cup \Pi_1^2) \neq \emptyset$ by our assumption, we can take an element $\alpha \in \Pi-(\Pi_1^1 \cup \Pi_1^2)$. Note that $\Pi_1^1=\Pi_0 \cup \{\alpha_0(1)\}$ and $\Pi_1^2=\Pi_0 \cup \{\alpha_0(2)\}$. We may assume that there is a connected subdiagram Π' of Π such that α and $\alpha_0(1)$ are terminal vertices of Π' and $\alpha_0(2)$ is not a vertex of Π', taking $\rho(\alpha)$ instead of α if necessary. Note that $\gamma=\sum_{\beta \in \Pi'} \beta$ is a positive root (cf. Bourbaki [5] Chap. 6, Prop. 19 Cor. 3 b)) and hence $\gamma \in \Sigma_{\mathrm{m}}^+$. Put $\Lambda=\Lambda_{\alpha_0(1)}-\Lambda_{\alpha_0(2)}$. Since $(\Lambda, \gamma)=(\Lambda, \alpha_0(1))$ and $(2\delta_{\mathrm{m}}, \gamma)>(2\delta_{\mathrm{m}}, \alpha_0(1))$, we see that

$$\frac{(\Lambda, \alpha_0(1))}{(2\delta_{\mathrm{m}}, \alpha_0(1))}>\frac{(\Lambda, \gamma)}{(2\delta_{\mathrm{m}}, \gamma)}>0>\frac{(\Lambda, \alpha_0(2))}{(2\delta_{\mathrm{m}}, \alpha_0(2))},$$

and hence the number of distinct non-zero eigenvalues of $g_0^{-1}B$ are greater than or equal to 3, by Lemma 5.4. Thus we get our claim by Theorem 5.1. q.e.d.

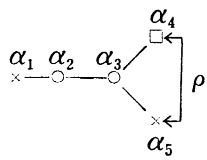
EXAMPLES 5.2. In the following cases the blowing down Y admits a non-homogeneous Kähler-Einstein metric. The vertices contained in Π_0, $\Pi_1^i-\Pi_0$, $\Pi-\Pi_1^i$ of a Dynkin diagram Π are denoted by $\bigcirc$, $\square$, $\times$ for $i=1, 2$ respectively as in section 3.

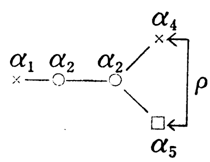
(1) (Π, Π_1^1, Π_0) 

(Π, Π_1^2, Π_0) 

.

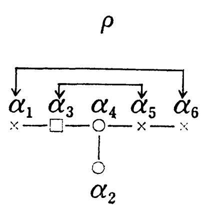
Note that $\Lambda=\Lambda_{\alpha_1}-\Lambda_{\alpha_3}$.

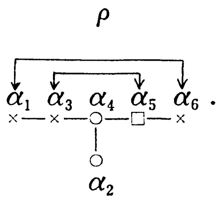
(2) (Π, Π_1^1, Π_0) 

(Π, Π_1^2, Π_0) 

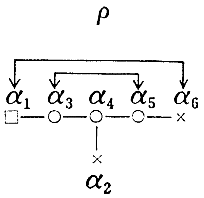
.

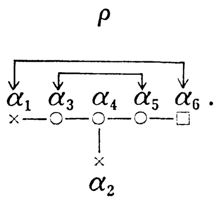
Note that $\Lambda=\Lambda_{\alpha_4}-\Lambda_{\alpha_5}$.

(3) (Π, Π_1^1, Π_0) 

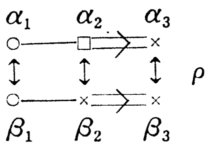
(Π, Π_1^2, Π_0) 

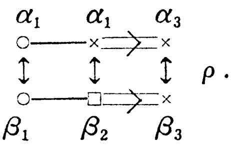
Note that $\Lambda = \Lambda_{\alpha_3} - \Lambda_{\alpha_5}$.

(4) (Π, Π_1^1, Π_0) 

(Π, Π_1^2, Π_0) 

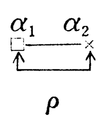
Note that $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\alpha_6}$.

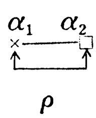
(5) (Π, Π_1^1, Π_0) 

(Π, Π_2^1, Π_0) 

Note that $\Lambda = \Lambda_{\alpha_2} - \Lambda_{\beta_2}$.

Now we give examples of Y being homogeneous.

EXAMPLES 5.3. (1) (Π, Π_1^1, Π_0) 

(Π, Π_1^2, Π_0) 

In this case $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\alpha_2}$, compact Kähler manifolds N_1, N_2 are $P^2(\boldsymbol{C})$, M is the flag manifold $SL(3, \boldsymbol{C})/B$ where B is a Borel subgroup of $SL(3, \boldsymbol{C})$, X is the $P^1(\boldsymbol{C})$-bundle $P(1 \oplus L_\Lambda)$ over M and Y is the complex quadric $Q^4(\boldsymbol{C})$.

(2) (Π, Π_1^1, Π_0) 

(Π, Π_1^2, Π_0) 

In this case $\Lambda=\Lambda_{\alpha_1}-\Lambda_{\beta_1}$, compact Kähler manifolds N_1, N_2 are $P^m(\boldsymbol{C})$, $P^n(\boldsymbol{C})$ respectively, M is $P^n(\boldsymbol{C})\times P^m(\boldsymbol{C})$, X is the $P^1(\boldsymbol{C})$-bundle $P(1\oplus L_\Lambda)$ over M and Y is the complex projective space $P^{n+m+1}(\boldsymbol{C})$.

6. Remarks

A riemannian manifold N is said to have *cohomogeneity d* if the codimension of the principal orbits for the action of the isometry group is d, and d is denoted by cohomg(N). For a given positive integer d we give examples of Kähler-Einstein manifolds which have cohomogeneity d.

Lemma 6.1. *Let M_1, M_2 be Fano manifolds of n_1-dimension and n_2-dimension $(n_1, n_2\geq 2)$ and let F_1, F_2 be holomorphic line bundles on M_1, M_2 respectively such that $c_1(F_1)>0$ and $c_1(F_2)>0$. Then $H^1(M_1\times M_2, \mathrm{End}(1\oplus F_1\otimes F_2^{-1}))=(0)$.*

Proof. By Künneth formula, $H^1(M_1\times M_2, F_1\otimes F_2^{-1})=\sum_{i=0,1} H^i(M_1, F_1)\otimes H^{1-i}(M_2, F_2^{-1})$. Since $c_1(F_2^{-1})<0, H^j(M_2, F_2^{-1})=(0)$ for $j<n_2$. Thus $H^1(M_1\times M_2, F_1\otimes F_2^{-1})=(0)$. Also we get $H^1(M_1\times M_2, F_1^{-1}\otimes F_2)=(0)$ by the same way. Since $\mathrm{End}(1\oplus F_1\otimes F_2^{-1})=1\oplus(F_1\otimes F_2^{-1})\oplus(F_1^{-1}\otimes F_2)\oplus 1$ and M_1, M_2 are simply conected, we get our claim. q.e.d.

In general, for a compact complex manifold X let $\mathrm{Aut}_0(X)$ denote the identity component of the group of all holomorphic automorphisms of X. Let E be a holomorphic vector bundle of rank r over a compact complex manifold M and $P(E)$ the associated projective bundle over M. By a theorem of Blanchard [3], we see that $\mathrm{Aut}_0(P(E))$ coincides with the identity component of all fiber preserving automorphisms of $P(E)$. Thus the projection $\pi: P(E)\to M$ induces a homomoıphism $\pi: \mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$. Note also that the group of all fiber preserving holomorphic automorphisms of $P(E)$ is naturally isomorphic to the group of all fiber preserving holomorphic automorphisms of the principal fiber bundle $P(M, PGL(r, \boldsymbol{C}), \pi)$ associated to the bundle $\pi: P(E)\to M$.

Lemma 6.2. *Under the assumption as in Lemma 6.1, the homomorphism $\pi: \mathrm{Aut}_0(P(1\oplus F_1\otimes F_2^{-1}))\to\mathrm{Aut}_0(M_1\times M_2)$ is surjective.*

Proof. By Proposition 2 in [15] and Proposition 9 in [2], it is enough to show that $H^1(M_1\times M_2, \mathrm{End}(1\oplus F_1\otimes F_2^{-1}))=(0)$. Thus we get our claim by Lemma 6.1. q.e.d.

We consider a holomorphic line bundle L over a compact complex manifold M and the $P^1(\boldsymbol{C})$-bundle $P(1\oplus L)$ over M. We assume that M has a Kähler-Einstein metric g_0 with $r_0=g_0$ and that L has a hermitian fiber metric such that the eigenvalues of the Ricci tensor B are constant on M and their absolute values are less than 1. We also assume that

$$\int_{-1}^{1} U \cdot \det(1-Ug_0^{-1}B)dU = 0 .$$

Thus $P(1\oplus L)$ admits a Kähler-Einstein metric by Theorem 5.4 in K-S [12].
Now we recall the following.

Proposition 6.3. *In the above situation, if the homomorphism* $\pi: \mathrm{Aut}_0(P(1\oplus L))\to\mathrm{Aut}_0(M)$ *is surjective and* B *is non-trivial on each irreducible factor of the Kähler manifold* M*, then the Kähler-Einstein manifold* $P(1\otimes L)$ *is irreducible and* $\mathrm{cohomg}(P(1\oplus L))=\mathrm{cohomg}(M)+1$.

Proof. See K-S [12] Proposition 5.6. Note that the homomoprhism π: $\mathrm{Isom}_0(P(1\oplus L))\to\mathrm{Isom}_0(M)$ is surjective by a theorem of Matsushima [13]. q.e.d.

Let $N_0=P^n(\boldsymbol{C})$, H the holomorphic line bundle over $P^n(\boldsymbol{C})$ corresponding to a hyperplane and $L_0=H^m$ for $1\leq m\leq n$. Then we have $c_1(L_0)=(m/(n+1))c_1(N_0)$ and we get an almost homogeneous Kähler-Einstein manifold $P(L_0\oplus L_0)$ of cohomogeneity one. Let $N'=P(L_0\oplus L_0)$, $\pi: N'\to N_0\times N_0$ the projection and $\xi=L(L_0\oplus L_0)$ the tautological line bundle ovcr N'. Then we have

$$c_1(N') = (n+1-m)\pi^*(c_1(H)\oplus c_1(H))+2c_1(\xi) .$$

Thus, if $n+1-m$ is even, there exists a holomorphic line bundle L' over N' such that $c_1(L')=(1/2)c_1(N')$.

Now we construct a Kähler-Eintein manifold of cohomogeneity d for each given positive integer d. If d is even, put $d=2k$, and if d is odd, put $d=2k+1$ (we may assume $d\geq 2$). Consider the product $M_1=N'\times\cdots\times N'$ of $d-1$ copies of N' and the product $F_1=L'\otimes\cdots\otimes L'$ of $d-1$ holomorphic line bundles on M_1 induced from L' on N'. Then $c_1(F_1)=(1/2)c_1(M_1)$. If d is even, consider the complex projective space M_2 of $(2n+1)(d-1)$ dimension. Then $c_1(M_2)=((2n+1)(d-1)+1)c_1(H)$. Put $F_2=H^l$ where $l=((2n+1)(d-1)+1)/2$. If d is odd, consider the complex quadric M_2 of $(2n+1)(d-1)$ dimension. Then $c_1(M_2)=(2n+1)(d-1)c_1(H)$ for a holomorphic line bundle H over M_2. Put $F_2=H^l$ where $l=(2n+1)(d-1)/2$. Consider the $P^1(\boldsymbol{C})$-bundle $P(1\oplus F_1\otimes F_2^{-1})$ over $M_1\times M_2$. Then, by Lemma 6.2 and Proposition 6.3, we see that $P(1\oplus F_1\otimes F_2^{-1})$ has cohomogeneity d.

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