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A "NON-STANDARD" APPROACH TO THE FIELD EQUATIONS IN THE FUNCTIONAL FORM

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1. Nonlinear functional equations

Schwinger [1] gave a couple of equations for the Green functions of the nucleon and meson fields with interaction, in the form of functional differential equations in terms of an ordinary external field. For neutral pseudoscalar mesons, they are

$$(1.1)\quad [-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\phi(x)+ig\gamma_5\int d\xi\Delta(x, \xi; \phi)\frac{\delta}{\delta\phi(\xi)}]G(x, x'; \phi)=\delta(x-x'),$$

$$(1.2)\quad (-\frac{\partial^2}{\partial\xi^2}+k^2)\Delta(\xi, \xi'; \phi) = \delta(\xi-\xi')+igT_r\gamma_5\int d\eta\Delta(\xi, \eta; \phi)\frac{\delta G(\xi', \xi'; \phi)}{\delta\phi(\eta)} -gT_r\gamma_5\phi(\xi)G(\xi', \xi'; \phi).$$

Equations (1.1) and (1.2) form a system from which the Green functions G and Δ may be determined as functionals of the external field $\phi(x)$. By using these equations it is possible to obtain closed expressions for the complete Green functions. In electrodynamics these equations are usually treated by expanding in powers of the coupling constant. But in meson theory, this procedure is inapplicable. Edwards and Peierls [2] simplified equation (1.2) to a linear one by assuming Δ to be a given function of x and x' independent of ϕ. Starting from these equations, they obtained the Green function in an explicit form by using Fourier transform of functionals. They used a sort of functional integrals but they did not give mathematical definitions of integrations or measures with respect to which integrations are performed.

Paul Lévy [3] has developed a potential theory on an infinite dimensional space. He used "la méthode du passage du fini à l'infini" in which an infinite dimensional Laplacian and harmonic functions are defined by a limiting procedure of finite dimensional potential theory.

Recently, Hasegawa [4], [5] introduced the following space of sequences

$$(1.3)\quad E=\{x=(x_1, x_2, \cdots, x_N, \cdots)\in R^\infty, \sup_N \frac{1}{N}\sum_{n=1}^N x_n^2<\infty\}$$

with semi-norms

$$||x||_N = (\frac{1}{N}\sum_{n=1}^{N} x_n^2)^{1/2} \text{ and } ||x||_\infty = \overline{\lim_{N\uparrow\infty}}\, ||x||_N ,$$

and reformulated the results of Lévy in terms of an infinite dimensional Brownian motion B on the space E with the infinitesimal generator $\frac{1}{2}\Delta_\infty$. In the derivation of Schwinger equations, for example in [6], a product of distribution is used and it does not seem to be treated very conveniently in the space (1.3).

This paper is an attempt to give mathematical foundations of the calculations which have been done by Edwards and Peierls in solving Schwinger equations. The method of non-standard analysis developed by A. Robinson [7] is applied to give a mathematical definition of functional integrals. A passage to limit similar to that of Lévy is used in a much bigger space than (1.3). Usually, the functional (continuous) integral is defined as the appropriate limit of certain finite dimensional integrals with respect to $\prod_{j=0}^{n} dx_j$ which is the volume element of integrations in the discrete function space. But the limiting transition to the continuous function space has been carried out formally. One must justify the transition to the limit for $n\to\infty$. Non-standard analysis offers a new approach to definition of functional integrals.

2. Ultra numbers and ultra Euclidean spaces

In this section, we give definitions and notations which will be used throughout this paper.

i) Let N be the set of all non-negative integers. Take and fix an ultra filter $\mathcal{F}_0$ on N which does not contain any finite subset of N. Let R be the set of all real numbers, and let a and b be elements of R^N. We use notations $a=(a_0, a_1, \cdots)=(a_n)_{n\in N}$ and $b=(b_0, b_1, \cdots)=(b_n)_{n\in N}$, where a_n and b_n are real numbers and called the n-th coordinate of a and b.

The relations and operations $a\equiv b$, $a<b$, $a\pm b$ and $a\cdot b$ are defined to be $\{i;\ a_i=b_i\}\in\mathcal{F}_0$, $\{i;\ a_i<b_i\}\in\mathcal{F}_0$, $(a_n\pm b_n)_{n\in N}$ and $(a_nb_n)_{n\in N}$, respectively. The relation $\equiv$ is an equivalence relation compatible with $<$, $\pm$, and $\cdot$. $^*R$ is defined to be $R^N/\equiv$ which is also written as $R^N/\mathcal{F}_0$ ($^*R$ is called ultra real numbers and its element is written as $[a_n]$). An imbedding isomorphism j from R to $^*R$ is defined by $j(a_0)=(a_0, a_0, \cdots)$. We consider R as a sub-field of $^*R$ by the isomorphisms j.[1] Ultra complex numbers $^*\boldsymbol{C}$ is similarly defined.

Similarly, we define the ultra-power of R^{n+1}, $E=\prod_{n\in N} R^{n+1}/\mathcal{F}_0$, where R^{n+1}

1) An element in R is called a "standard" number.

is the real $n+1$ dimensional space. For $a, b \in \prod_{n\in N} R^{n+1}$, and $\alpha=[\alpha_n]\in {}^*R$, we define $a\equiv b$, αa, (a, b), and $a+b$ as $\{n;\ a^{(n)}=b^{(n)} \text{ in } R^{n+1}\}\in\mathcal{F}_0$, $[\alpha_n a^{(n)}]$, $[(a^{(n)}, b^{(n)})]$, $[a^{(n)}+b^{(n)}]$, respectively, where $a^{(n)}$ is the $n+1$ dimensional vector in R^{n+1}, i.e.; $a^{(n)}=(a_0^{(n)}, a_1^{(n)}, \cdots, a_n^{(n)})$. The space of real sequence $a=(a_p)_{p\in N}$, or the product space R^N can be considered as a subspace of E, R^N being imbedded into E by the injection $a=[(a_0, a_1, \cdots, a_n)]$. Now by a similar way we can consider the space of slowly growing sequences[2], s', as a subspace of E.

ii) the space E_h

Now let $\{h_p(x)\}_{p\in N}$ be the Hermite complete orthonormal system;

$$h_p(x) = \frac{e^{-x^2/2}}{\sqrt{2^p p!\sqrt{\pi}}} H_p(x)$$

$$H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2}.$$

To any element of the space of E defined in (i), there corresponds a ${}^*R$-valued function $\psi(x)$ such that $\psi(x)=[\psi_n(x_n)]=[\sum_{p\leq n} a_p^{(n)} h_p(x_n)]$ for $a=[a^{(n)}]\in E$ and $x=[x_n]\in {}^*R$. The totality of these functions forms a linear space isomorphic to E. We denote this space by E_h or $E_h({}^*R)$. Clearly, for mappings $a\to\phi(x)$, $b\to\psi(x)$, we have

$$(a, b) = [\int_R \phi_n(x_n)\psi_n(x_n)dx_n] = (\phi, \psi).$$

We note the Hermite expansion $u\to(u_p)_{p\in N}$ is an isomorphism of the space of slowly increasing distribution in R, $\mathcal{S}'(R)$, onto the space of slowly growing sequences. Hence we can imbed $\mathcal{S}'(R)$ into $E_h({}^*R)$. The algebraic operations are defined naturally in this space.

Set $E^c=\prod_{n\in N} \boldsymbol{C}^{n+1}/\mathcal{F}_0$ and define the space $E_h^c({}^*R)$ of ${}^*\boldsymbol{C}$-valued functions isomorphic to E^c similar to above.

Here we have restricted ourselves to case of the one variable, but our formulation can be easily generalized to the case of several variables.

DEFINITILN 1. Let $a, b\in {}^*R$ and, let $\phi(x)$, $\psi(x)\in E_h$. Then the sum $\phi(x)+\psi(x)$ and the integral $\int_a^b \phi(x)dx$ are defined as follows:

$$\phi(x)+\psi(x) = [\phi_n(x_n)+\psi_n(x_n)], \tag{2.1}$$

$$\int_a^b \phi(x)dx = [\int_{a_n}^{b_n} \phi_n(x_n)dx_n]. \tag{2.2}$$

Let $\Delta(x, y)$ be the standard Green's function. The non-standard version

2) A sequence $a=(a_p)_{p\in N}$ is said to be slowly growing if there is a constant $k\geqq 0$ such that the sequence $((1+p)^{-k}a_p)_{p\in N}$ is bounded.

of the Green's function defined by $\Delta(x,y)=[\Delta_n(x_n,y_n)]$, where $\Delta_n(x,y)$ is the Cesaro mean of the first order of Hermite expansion of the Green's function,

$$\Delta_n(x,y)=\sum_{\max(p,q)\leqq n}\frac{n-(p-1)}{n+1}\frac{n-(q-1)}{n+1}h_p(x)h_q(y)\iint\Delta(u,v)h_p(u)h_q(v)dudv\,.$$

DEFINITILN 2 (a nonstandard version of the Green's function and its inverse). We introduce *the inverse of the Green's function* defined by the equation

$$\int\Delta(x,x')\Delta^{-1}(x',y)dx'=\int\Delta^{-1}(x,x')\Delta(x',y)dx'=\delta(x,y)\,, \tag{2.3}$$

where $\delta(x,y)$ is *the nonstandard version of the standard Dirac function* such that

$$\delta(x,y)=[\sum_{p\leqq n}h_p(x_n)h_p(y_n)]\in E_h(^*(R_x\times R_y))\,. \tag{2.4}$$

REMARK 1. A number $a\in{}^*R$ will be called infinitesimal if $|a|<\varepsilon$ for all positive numbers ε in R. For any $r\in R$ the set of all numbers $a\in{}^*R$ which differ from r only by an infinitesimal amount is said to be monad of r, $\mu(r)$. If $a\in\mu(r)$, then we write $r={}^\circ a$ and we call r the standard part of a. We say that a property holds $\mathcal{F}_0$ a.s. (almost surely), if the set of elements of N for which the property holds is contained in $\mathcal{F}_0$. We define a subspace $\tilde{E}_h$ of E_h as follows: For each $\psi\in\mathcal{S}(R)$, $\tilde{E}_h=\{\phi;\ \int\phi_n(x)\psi(x)dx$ is bounded $\mathcal{F}_0$ a.s.$\}$. For any $\phi\in\tilde{E}_h$, we can define a linear functional T_ϕ over $\mathcal{S}$ by putting $\lim_{\mathcal{F}_0}\langle\phi,\psi\rangle={}^\circ[\langle\phi_n,\psi\rangle]=T_\phi\psi$. If the map $\psi\to T_\phi\psi$ is continuous on $\mathcal{S}(R)$, then T_ϕ defines a distribution.

REMARK 2. Instead of the Hermite complete orthonormal system, we can choose a complete system of normalized eigenfunctions of the symmetric integral operator $A\psi(x)=\int_R\Delta(x,y)\psi(y)dy$, where $\Delta(x,y)=\Delta(y,x)$.

3. A formulation of functional integral

If $F_n(x,y;\phi;\Delta)$ $(n=0,1,\cdots)$ is a sequence of $\boldsymbol{C}$-valued standard functions of $(x,y;\phi;\Delta)$ in the domain

$$R\times R,\ R^R,\ R^{R\times R}\,,$$

then we define a $^*\boldsymbol{C}$-valued functional by

$$F(x,y;\phi;\Delta)=[F_n(x_n,y_n;\phi_n;\Delta_n)]\,.$$

$F^c(^*(R\times R),\ E_h(^*R),\ E_h(^*(R\times R)))$ or, in short, $F^c(E_h)$ will be used to denote the set of such $^*\boldsymbol{C}$-valued functionals. From the definition, $F_n(x_n,y_n;\phi_n;\Delta_n)$ can be considered as a function of $a^{(n)}$. Hence we can identify a functional defined on E_h with a function defined on E.

The nonstandard analogue of the space $L^p(R)$ is the space $L^p(E_h)$ ($\subset F^c(E_h)$), consisting of all functions whose p-th powers are Lebesgue integrable $\mathscr{F}_0$ a.s. Similarly, we shall use the notations $\mathcal{S}'(E_h)$, $C^m(E_h)$ and so on.

Our way of introducing integration on E_h is clear and simple. We shall define an integral over E_h and a functional derivative by the following way:

DEFINITILN 3. If $F(\phi)\in F^c(E_h)$ is integrable in the sense of the improper Riemann integrable $\mathscr{F}_0$ a.s., then we define

$$\int_{E_h} F(\phi)\delta\phi = [\int_{R^{n+1}} F(a^{(n)})da_0^{(n)} \cdots da_n^{(n)}]\in {}^*C\,. \tag{3.1}$$

DEFINITION 4. If $F(\phi)\in F^c(E_h)$ is differentiable $\mathscr{F}_0$ a.s., then we define

$$\frac{\delta F(\phi(\cdot))}{\delta\phi(t)} = [\sum_{p\leqslant n} \frac{\partial F}{\partial a_p^{(n)}} h_p(t_n)]\in E_h({}^*R)\,, \tag{3.2}$$

where $\phi=[\sum_{p\leqslant n} a_p^{(n)}h_p]$, $t=[t_n]\in {}^*R$.

DEFINITION 5. Let $F(\phi)\in F^c(E_k)$ be rapidly decreasing $\mathscr{F}_0$ a.s. We define Fourier transform $(\mathscr{F}F)(\lambda)$ by

$$(\mathscr{F}F)(\lambda) = \int F(\phi)e^{i\int\phi(s)\lambda(s)ds}\delta\phi\,. \tag{3.3}$$

We also the Fourier transform of a tempered distribution functional $\mathscr{F}_0$ a.s. as a tempered distribution functional $\mathscr{F}_0$ a.s.

We note that the integral (3.1) by definition reduces to an ordinary finite dimensional integral and its evaluation does not involve any difficulties in principle. By using these definitions, we can easily show the following properties.

Theorem 1 (properties of the integral).
(i) *If $F(\phi)$ and $G(\phi)$ are integrable $\mathscr{F}_0$ a.s., then $\alpha F(\phi)$ and $\beta G(\phi)$ is integrable $\mathscr{F}_0$ a.s. and*

$$\int(\alpha F(\phi)+\beta G(\phi))\delta\phi = \alpha\int F(\phi)\delta\phi+\beta\int G(\phi)\delta\phi\,. \tag{3.4}$$

(ii) *If $F(\phi)$ is integrable $\mathscr{F}_0$ a.s., then under the following translation of functions in E_h*

$$\phi(\cdot)\rightarrow\Omega(\cdot) = \phi(\cdot)+\phi_0(\cdot)$$

($\phi_0(\cdot)$ *is a fixed function) the integral is invariant, i.e.,*

$$\int F(\phi+\phi_0)\delta\phi = \int F(\Omega)\delta\Omega\,. \tag{3.5}$$

(iii) *If we introduce a change of variable*

$$\phi(t) \to \lambda(t) = \int K(t, s)\phi(s)ds \equiv K\phi ,$$

then we have

(3.6) $$\int F(K\phi)\delta\phi = |\det K^{-1}| \int F(\lambda)\delta\lambda = \int F(\lambda)\frac{\partial(\phi)}{\partial(\lambda)}\delta\lambda \quad (symbolically),$$

where $K(t, s) = [\sum_{\max(p,q)\leqq n} K^{(n)}_{p,q} h_p(t_n) h_p(s_n)]$, *$K(t, s)$ is invertible (i.e., matrix* $(K^{(n)}_{p,q})$ *is invertible) and* $|det\ K^{-1}| \equiv [|det\ K^{(n)}_{p,q}|^{-1}] \in {}^*R$.

(iv) (*integration by parts*).

For $G(\phi) \in C^1(E_h)$, $F(\phi) \in C^1_0(E_h)$, *we have*

(3.7) $$\int G(\phi)\frac{\delta F(\phi)}{\delta\phi(t)}\delta\phi = -\int F(\phi)\frac{\delta G(\phi)}{\delta\phi(t)}\delta\phi .$$

(v) (*the functional Fourier transform*). *For* $F(\phi) \in \mathcal{S}'(E_h)$

(3.8) $$\mathcal{F}\left(\frac{\delta F(\phi)}{\delta\phi(t)}\right) = -i\lambda(t)\mathcal{F}(F(\phi)) .$$

Theorem 2 (properties of the functional derivative).

(i) *If* $G(\phi)$, $F(\phi)$ *are differentiable* $\mathcal{F}_0$ *a.s., then*

(3.9) $$\frac{\delta}{\delta\phi(t)}\{G(\phi)F(\phi)\} = F(\phi)\frac{\delta G(\phi)}{\delta\phi(t)} + G(\phi)\frac{\delta F(\phi)}{\delta\phi(t)} .$$

(ii) (*chain rule*). *Let F be differentiable* $\mathcal{F}_0$ *a.s. If u is of class* $C^1(E_h \to E_h)$, *then the following formula holds*:

(3.10) $$\frac{\delta F(u(\phi))}{\delta\phi} = \int \frac{\delta u(\phi)(x)}{\delta\phi}\frac{\delta F(u)}{\delta u}(x)dx .$$

(iii) *If $f(x)$ is a standard function* $f \in C^1(R)$ *and* $G(\phi) \in C^1(E_h)$, *then*

(3.11) $$\frac{\delta f(G(\phi))}{\delta\phi(t)} = f'(x)|_{x=G(\phi)}\frac{\delta G(\phi)}{\delta\phi(t)} .$$

By definition, we can easily evaluate the functional derivatives:

(3.12) $$\frac{\delta\phi(t)}{\delta\phi(s)} = \delta(t, s) ,$$

(3.13) $$\frac{\delta}{\delta\phi(t)}\iint\phi(t)K(t, t')\phi(t')dtdt' = 2\int\phi(t')K(t, t')dt' ,$$

whenever $K(t, t') = K(t', t)$.

We will show now that the evaluation of the integral can be done in some special cases.

EXAMPLE 1.

$$\int e^{-1/2\int\phi^2(t)dt}\delta\phi = [\int_{R^{n+1}} e^{-1/2(\xi,\xi)}d\xi] = [(2\pi)^{n+1/2}]\,. \tag{3.14}$$

Symbolically, the expression (3.14) can be written in the form

$$\int e^{-1/2\int\phi^2(t)dt}\delta\phi = \Pi(2\pi)^{1/2}\,.$$

EXAMPLE 2. We examine the integral $\int \exp(-\iint \phi(t)A(t, s)\phi(s)dsdt)\delta\phi$, where $A(t, s)=[\sum_{p,q\leqq n}a^{(n)}_{p,q}h_p(t_n)h_p(s_n)]$ and $(a^{(n)}_{j,k})$ is positive definite. By making use of the well known formula

$$\int\cdots\int \exp(-\sum_{j,k}^{n} a^{(n)}_{jk}x_jx_k)dx_1\cdots dx_n = \pi^{n+1/2}[\det(a^{(n)}_{jk})]^{-1/2}\,,$$

we easily obtain the result

$$\begin{aligned}\int \exp(-\iint\phi(t)A(t, s)\phi(s)dsdt)\delta\phi &= [\,|\det(a^{(n)}_{jk})|^{-1/2}\pi^{n+1/2}]\\ &= |\det A|^{-1/2}\Pi(\pi^{1/2})\,.\end{aligned}$$

In the case that $A(\xi, \eta)$ is a nonstandard version of the Green's function of Klein-Gordon equation, it is necessary to evaluate the integral of the functional $\exp(\pm\frac{i}{2}\iint\phi(\xi_1)A(\xi_1, \xi_2)\phi(\xi_2)d\xi_1d\xi_2)$ $(i=\sqrt{-1})$. The calculation is slightly more complicated. By definition the integral becomes an ordinary Fresnel integral. If $(a^{(n)}_{jk})$ is a positive definite Hermitian matrix and $\det(a^{(n)}_{jk})\neq 0$, then we have

$$\begin{aligned}&\int \exp(\frac{i}{2}\iint\phi(\xi_1)A(\xi_1, \xi_2)\phi(\xi_2)d\xi_1d\xi_2)\delta\phi\cdot\\ &\int \exp(-\frac{i}{2}\iint\phi(\xi_1)A(\xi_1, \xi_2)\phi(\xi_2)d\xi_1d\xi_2)\delta\phi\\ &= [\,|\det(a^{(n)}_{jk})|^{-1}(2\pi)^{n+1}] = |\det A|^{-1}\Pi(2\pi)\,.\end{aligned} \tag{3.15}$$

We shall use this formula later.

4. A justification of calculations

In the section we will give mathematical foundations of the calculations which have been done by Edwards and Peierls. We simplyfy the equations of Schwinger as same as they have done to linear one by assuming that $\Delta(x, x'; \phi)$ is independent of the external ϕ, and hence that (1.2) reduces to

$$(-\frac{\partial^2}{\partial x^2}+k^2)\Delta(x, x') = \delta(x-x') .$$

Then we need only consider the equation (1.1). First of all, we will give a rigorous definition of the linearized Schwinger's equation as follows:

DEFINITILN 6. For neutral pseudoscalar mesons, *the linearized Schwinger's equation* is

$$(4.1) \quad \langle[-i\gamma_\mu\frac{\partial}{\partial x_{\mu,n}}+m-g\gamma_5\phi_n(x_n)+ig\gamma_5\int d\xi_n\Delta_n(x_n, \xi_n)\sum_{|p|\leq n}h_p(\xi_n)\frac{\partial}{\partial a_p^{(n)}}] \times G_n(x_n, x_n'; a^{(n)}; \Delta_n), \psi(x_n, x_n')\rangle = \langle\delta(x_n-x_n'), \psi(x_n, x_n')\rangle\mathscr{F}_0 \text{ a.s.}$$

for all $\psi\in\mathcal{S}(R^4_{x_n}\times R^4_{x_n'})$, where $\Delta(x, \xi)$ is nonstandard version of the Green's function, $G(x, x'; \phi; \Delta)$ is of class $C^1(E_h)$ and absolutely integrable $\mathscr{F}_0$ a.s., $G_n(x_n, x_x'; \phi_n; \Delta_n)$ is a standard tempered distribution for fixed ϕ_n, Δ_n, $\phi(\xi)=[\sum_{|p|\leq n}a_p^{(n)}h_p(\xi_n)]$, $\delta(x_n-x_n')$ is the standard Dirac function and $p=(p_1, p_2, p_3, p_4)$, $|p|=\max(p_1, p_2, p_3, p_4)$.

In the frame of our theory (4.1) can be expressed in the following form.

$$(4.2) \quad [-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\phi(x)+ig\gamma_5\int d\xi\Delta(x, \xi)\frac{\delta}{\delta\phi(\xi)}]G(x, x'; \phi) = \delta(x-x') .$$

Secondly, if we put

$$(4.3) \quad G(x, x'; \phi) = G_1(x, x'; \phi)R_1^{-1}(\phi) ,$$

where $R_1^{-1}(\phi)=\exp(-\frac{i}{2}\iint\phi(\xi)\Delta^{-1}(\xi, \xi')\phi(\xi)d\xi\, d\xi')$, then from (4.2) we get

$$\{-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m+ig\gamma_5\int d\xi\Delta(x, \xi)\frac{\delta}{\delta\phi(\xi)}\}G_1(x, x'; \phi) = \delta(x-x')R_1 ,$$

here we have made use of (2.3), (3.9) and (3.11). If we put

$$G_2(x, x'; \lambda) = \mathscr{F}(G_1(x, x'; \phi)) = \int G_1(x, x'; \phi)\exp\{i\int\phi(\xi)\lambda(\xi)d\xi\}\delta\phi ,$$

then from (3.8) we get

$$(4.4) \quad \{-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\int d\xi\Delta(x, \xi)\lambda(\xi)\}G_2(x, x'; \lambda) = \delta(x-x')R_2(\lambda) ,$$

where $R_2(\lambda)=\int\exp\{\frac{i}{2}\iint\phi(\xi)\Delta^{-1}(\xi, \xi')\phi(\xi')d\xi\, d\xi'\}\cdot\exp\{i\int\phi(\xi)\lambda(\xi)d\xi\}\delta\phi$. To evaluate R_2, we write

$$\phi(\xi) = \phi'(\xi)-\int\Delta(\xi, \xi')\lambda(\xi')d\xi' .$$

Then from (2.3) and (3.5) we have

$$(4.5)\qquad R_2(\lambda) = \exp\{-\frac{i}{2}\int\lambda(\xi)\Delta(\xi, \xi')\lambda(\xi')d\xi\, d\xi'\}$$

$$\times \int\exp\{\frac{i}{2}\int\phi'(\eta)\Delta^{-1}(\eta, \eta')\phi'(\eta')d\eta\, d\eta'\}\delta\phi'$$

$$\equiv \exp\{-\frac{i}{2}\int\lambda(\xi)\Delta(\xi, \xi')\lambda(\xi')d\xi\, d\xi'\}C\,.$$

Returning to G_2,

$$(4.6)\qquad [-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\int d\xi\Delta(x, \xi)\lambda(\xi)]G_2(x, x'; \lambda)$$

$$= \delta(x-x')C\exp\{-\frac{i}{2}\int\Omega(\xi)\Delta(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\,.$$

The Δ on the left can be absorbed by introducing

$$\Omega(\xi) = \int\Delta(\xi, \xi')\lambda(\xi')d\xi'\,,$$

giving

$$(4.7)\qquad [-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\Omega(x)]G_3(x, x'; \Omega)$$

$$= \delta(x-x')C\exp\{-\frac{i}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\,,$$

where $G_3(x, x'; \Omega)=G_2(x, x'; \lambda)$.
It is also useful to introduce G_4 by

$$(4.8)\quad G_3(x, x'; \Omega) = CG_4(x, x'; \Omega)\exp\{-\frac{i}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\,.$$

Combining (4.8) with (4.7), we get

$$(4.9)\qquad [-i\gamma_\mu\frac{\partial}{\partial x_\mu}+m-g\gamma_5\Omega(x)]G_4(x, x'; \Omega) = \delta(x-x')\,.$$

Thus, now returning to equation (4.1), from (3.6) and (4.8) we see that

$$G(x, x'; \phi) = R_1^{-1}(\phi)\Pi(2\pi)^{-1}\int G_2(x, x'; \lambda)\exp\{-i\int\lambda(\xi)\phi(\xi)d\xi\}\delta\lambda$$

$$= R_1^{-1}(\phi)\Pi(2\pi)^{-1}\int G_3(x, x'; \Omega)\exp\{-i\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\phi(\xi')d\xi\, d\xi'\}\frac{\partial(\lambda)}{\partial(\Omega)}\delta\Omega$$

$$= CR_1^{-1}(\phi)\Pi(2\pi)^{-1}|\det\Delta^{-1}|\int G_4(x, x'; \Omega)\exp\{-i\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\phi(\xi')d\xi\, d\xi'\}$$

$$\times\exp\{-\frac{i}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\delta\Omega\,.$$

Here we will treat the function S' (S' is the S'_F of Dyson),

$$S'(x, x') \equiv G(x, x'; 0)$$

Since $R_1^{-1}(0)=1$,

$$S'(x, x') = C\,|\det\Delta^{-1}|\,\Pi(2\pi)^{-1}\int G_4(x, x'; \Omega) \times \exp\{-\frac{i}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\delta\Omega\,. \tag{4.10}$$

Hence the function $S'(x, x')$ can be obtained from G_4 by integrating over all Ω with the weight factor. In case we have an explicit solution G_4 to (4.9), the Green's function $S'(x, x')$ is expressed in closed form. Our formalism can be extended without difficulty to more complicated Green's functions, corresponding to many-meson many-nucleon systems.

To illustrate our method we shall consider the case of the scalar mesons when nucleon recoil is neglected. In this case (4.2) and (4.9) become

$$\{i\frac{\partial}{\partial t}+m-g\phi(t)+ig\int d\xi\Delta(t, \xi)\frac{\delta}{\delta\phi(\xi)}\}G(t, t'; \phi) = \delta(t-t') \tag{4.11}$$

and

$$(i\frac{\partial}{\partial t}+m-g\Omega(t))G_4(t, t'; \Omega) = \delta(t-t'), \text{ respectively.} \tag{4.12}$$

The latter immediately gives

$$G_4(t, t'; \Omega) = S(t-t')\exp\{-gi\int_{t'}^{t}\Omega(s)ds\}\,, \tag{4.13}$$

where S is defined by

$$(i\frac{\partial}{\partial t}+m)S(t-t') = \delta(t-t')\,.$$

S' is now given by

$$S'(t, t') = S(t-t')C\,|\det\Delta^{-1}|\,\Pi(2\pi)^{-1}\int\delta\Omega\exp\{-ig\int_{t'}^{t}\Omega(\xi)d\xi -\frac{i}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'\}\,. \tag{4.14}$$

To evaluate the functional integral, put

$$\Omega(\xi) = \Omega'(\xi)-g\int_{t'}^{t}\Delta(\xi, \eta)d\eta\,,$$

giving

$$g\int_{t'}^{t}\Omega(\xi)d\xi+\frac{1}{2}\int\Omega(\xi)\Delta^{-1}(\xi, \xi')\Omega(\xi')d\xi\, d\xi'$$

$$= \frac{1}{2}\int \Omega'(\xi)\Delta^{-1}(\xi, \xi')\Omega'(\xi')d\xi d\xi' - \frac{1}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta(\xi, \xi') .$$

Now applying (3.15) we have

$$(4.15) \quad S'(t, t') = S(t-t')\exp\{\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta(\xi, \xi')\}$$

$$\times C\Pi(2\pi)^{-1}|\det\Delta^{-1}| \int \exp\{-\frac{i}{2}\int \Omega'(\xi)\Delta^{-1}(\xi, \xi')\Omega'(\xi')d\xi d\xi'\}\delta\Omega'$$

$$= \Pi(2\pi)^{-1}|\det\Delta^{-1}|\,|\det\Delta^{-1}|^{-1}\Pi(2\pi)S(t-t')\exp\{\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta(\xi, \xi')\}$$

$$= S(t-t')\exp\{\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta(\xi, \xi')\} ,$$

that is,

$$\langle S'(t_n, t_n'), \psi(t_n, t_n')\rangle$$
$$= \langle S(t_n - t_n')\exp\{\frac{i}{2} g^2 \int_{t_n'}^{t_n} d\xi_n \int_{t_n'}^{t_n} d\xi_n' \Delta_n(\xi_n, \xi_n'), \psi(t_n, t_n')\rangle$$

$\mathcal{F}_0$ a.s. for all $\psi \in \mathcal{S}(R_{t_n} \times R_{t_n'})$.
If we put $S'(t-t') = S'(t,t')|_R$, then

$$(4.16) \quad \langle S'(t-t'), \psi(t, t')\rangle = \langle S(t-t')\exp(\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta_n(\xi, \xi')), \psi(t, t')\rangle$$

$\mathcal{F}_0$ a.s. for all $\psi \in \mathcal{S}(R_t \times R_{t'})$.
To evaluate $\langle S'(t-t'), \psi(t, t')\rangle$ further, we insert the one-dimensional Green's function Δ

$$(\frac{d^2}{dt^2} + k^2)\Delta(t-t') = \delta(t-t'), \quad \Delta(t-t') = Re(\frac{1}{ik} e^{ik|t-t'|}) .$$

When $\Delta(t-t') = Re(\frac{1}{ik} e^{ik|t-t'|})$, $\exp(\int \frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta_n(\xi, \xi'))$ is bounded $\mathcal{F}_0$ a.s. Thus, by taking standard parts of both side of the equation (4.16) we have

$$(4.17) \quad \lim_{\mathcal{F}_0} \langle S'(t-t'), \psi(t, t')\rangle$$
$$= {}^\circ[\langle S(t-t')\exp\{\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \Delta_n(\xi, \xi')\}, \psi(t, t')\rangle] .$$

In this simple case $\lim_{\mathcal{F}_0} \langle S'(t-t'), \psi(t, t')\rangle$ becomes a distribution.

Theorem 3. *We now have the equality*

$$(4.18) \quad {}^\circ S'(t-t') = S(t-t')\exp\{\frac{i}{2} g^2 \int_{t'}^{t} d\xi \int_{t'}^{t} d\xi' \, Re(\frac{e^{ik|\xi-\xi'|}}{ik})\}$$

in the tempered distributions space $\mathcal{S}'$.

Before proving this, we need a lemma for the Cesaro sum:

$$C_n^{(1)}=\frac{1}{(n+1)^2}\sum_{\max(p,q)\leqq n} S_{p,q}^{(0)}$$

where $S_{p,q}^{(0)}$=the partial sum of Hermite expansion, $\sum_{i=0}^{p}\sum_{j=0}^{q} f_{ij}h_i(x)h_j(y)$.

Lemma. *If $f(x, y)$ is continuous in R^2 and there exist constants γ, K such that $|f(x,y)|\leqq K(1+|x|^{\gamma})(1+|y|^{\gamma})$, $2>\gamma\geqq 0$ for all $(x,y)\in R^2$, then the $(C, 1)$ sum of Hermite series of $f(x, y)$ converges uniformly to $f(x, y)$ on any compact sets in R^2.*

Outline of Proof.

The formal expansion of a function in a Hermite series is

$$f(x,y)\sim\sum_{p,q} h_p(x)h_q(y)\iint f(u,v)h_p(u)h_q(v)dudv\,.$$

If we put $f(x,y)\equiv f_0(x,y)e^{(-x^2/2)-(y^2/2)}$, then

$$f_0(x,y)\sim\sum_{p,q}\frac{H_p(x)H_q(y)}{2^p p!\sqrt{\pi}\,2^q q!\sqrt{\pi}}\iint e^{-u^2-v^2}f_0(u,v)H_p(u)H_p(v)dudv\,. \tag{4.19}$$

The series (4.19) is called of type H, according to Charlier. For the $(C, 1)$ sum of Hermite series of type H we have

$$f_n^{(1)}(x,y)=\iint e^{-u^2-v^2}f_0(u,v)S_n^{(1)}(x,u)S_n^{(1)}(y,v)dudv \tag{4.20}$$

where

$$S_n^{(1)}(x,u)=\sum_{p=0}^{n}\frac{n-(p-1)}{n+1}\frac{H_p(x)H_p(u)}{2^p p!\sqrt{\pi}}$$

By the using the estimations in the paper of Kogbetliantz [8] for the Hermite series in one variable, we have the following formulas for asymptotic approximation of the $S_n^{(1)}(x, u)$.

$$|S_n^{(1)}(x,u)|=O\left(\frac{e^{(x2+u2)/2}}{|u-x|^2\sqrt{n}}\right)\qquad\text{for}\quad |u|\leqq O(\sqrt{n}),\quad |x|\leqq b\leqq O(\sqrt{n}) \tag{4.21}$$

$$|S_n^{(1)}(x,u)|=O\left(\frac{e^{u^2/2}}{|u|^3}\right)\qquad\text{for}\quad 0<a\leqq|u|,\quad |x|\leqq b\leqq\sqrt{n}-a\,. \tag{4.22}$$

We begin by proving the pointwise convergence of the $(C, 1)$ sum of the Hermite series of type H. In particular if $f_0(x, y)$ is continuous, then series (4.20) converges uniformly to $f_0(x, y)$ on any compact sets.

It is easy to see that the series is $(C, 1)$ summable for the Green's function $\Delta(x-y)=Re\left(\frac{e^{ik|x-y|}}{ik}\right)$ and is uniformly convergent on any compact sets. Therefore the Theorem is proved.

REMARK 3. In (3.15) the assumption $\det(\Delta_{jk}^{(n)})\neq 0$ is not essential for this problem. When $\det(\Delta_{jk}^{(n)})=0$, we can choose $\varepsilon_n>0$ such that $(\Delta_{jk}^{(n)}(\varepsilon_n))\equiv(\Delta_{jk}^{(n)})+\varepsilon_n I$, $\det(\Delta_{jk}^{(n)})\neq 0$ and $(\Delta_{jk}^{(n)}(\varepsilon_n))\rightarrow(\Delta_{jk}^{(n)})$ as $\varepsilon_n\rightarrow 0$. We apply (3.15) to $\Delta_{jk}^{(n)}(\varepsilon_n)$ instead of $\Delta_{jk}^{(n)}$ and take standard parts. This gives the same result, (4.17).

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