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EQUIVARIANT IMMERSIONS AND EMBEDDINGS OF SMOOTH G-MANIFOLDS

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Introduction

Let f: $P\rightarrow Q$ be a smooth G-map between smooth G-manifolds P, Q where G is a compact Lie group. The aim of this paper is to give sufficient conditions for f to be equivariantly homotopic to an orthogonally isovariant map, an isovariant immersion, and an equivariant embedding. We work in the smooth category. In the PL category with G finite there is a result of Illman [3]. In the smooth category, when Q is a euclidean representation space, there are also results of Wasserman [6], Marchow and Pulikowski [4]. As Illman pointed out in his paper the existence of equivariant PL and equivariant smooth embeddings are questions of completely different natures. In the PL category, conditions for the dimensions of fixed point sets of each subgroup of G give a sufficient condition for the existence of equivariant embeddings. In the smooth category, however, those are not sufficient. Conditions also for the normal representations around fixed point sets are needed.

Let G_x denote the isotropy subgroup of G at $x\in P$. A G-map $f: P\rightarrow Q$ is called *isovariant* if $G_x=G_{f(x)}$ for every $x\in P$. Let $\mathcal{J}(P)$ denote the set of isotropy types (i.e., conjugacy classes of isotropy subgroups) on P. For any $(H)\in\mathcal{J}(P)$ define

$$P^{(H)}=\{x\in P\,|\,H \text{ is conjugate to a subgroup of } G_x\}\ ,$$
$$P_{(H)}=\{x\in P\,|\,H \text{ is conjugate to } G_x\}\ .$$

$\nu(P_{(H)})$ denotes the normal bundle of $P_{(H)}$ in P. The differential df of a smooth isovariant map f: $P\rightarrow Q$ induces a bundle homomorphism

$$\tilde{d}f\colon \nu(P_{(H)})\rightarrow\nu(Q_{(H)})\,.$$

If, for every $(H)\in\mathcal{J}(P)$, $\tilde{d}f$ is a bundle monomorphism (i.e., a monomorphism on each fibre), f is called *orthogonally isovariant.*

Our main results are Theorem 7.2, Corollary 7.3 and Corollary 7.4 stated in section 7. Theorem 7.2 shows that, under conditions for the dimensions of fixed point sets and also for the normal representations around them, a smooth

G-map $f: P \to Q$ may be equivariantly homotoped to an orthogonally isovariant map. Corollary 7.3 (resp., Corollary 7.4) shows that, under extra conditions for the dimensions of fixed point sets, f may be equivariantly homotoped to an isovariant immersion (resp., an equivariant embedding).

Theorem 7.2 will be proved as follows. Let

$$\mathcal{I}(P) = \{(H_1), (H_2), \cdots, (H_a)\}$$

be ordered in such a way that if H_i is conjugate to a subgroup of H_j then $j \leq i$. Assume f is already orthogonally isovariant on a neighborhood of $A = \partial P \cup P^{(H_1)} \cup \cdots \cup P^{(H_{i-1})}$. First, using the condition (7.2.1), we show in section 4 that f may be equivariantly homotoped relative to a neighborhood of A to a smooth G-map $g: P \to Q$ which is isovariant on $P^{(H_i)}$. Next we want to homotope g to a map which is orthogonally isovariant on a neighborhood of $P^{(H_i)}$. In section 5 we will introduce obstructions to do so. We see that these obstructions vanish under the condition (7.2.2). Thus g may be equivariantly homotoped relative to a neighborhood of A to a map which is orthogonally isovariant on a neighborhood of $P^{(H_i)}$. Thus f is equivariantly homotopic to a map which is orthogonally isovariant on a neighborhood of $\partial P \cup P^{(H_1)} \cup \cdots \cup P^{(H_i)}$. Continuing as above, f may be equivariantly homotoped to an orthogonally isovariant map, and Theorem 7.2 may be proved. Corollary 7.3 and Corollary 7.4 may also be proved by homotoping an orthogonally isovariant map inductively as above.

1. Preliminaries

G always denotes a compact Lie group. Let P be a smooth G-manifold. The conjugacy class (G_x) of the isotropy subgroup at $x \in P$ is called an *isotropy type* on P. All smooth G-manifolds considered in this paper have finite isotropy types. If P is compact then P has finite isotropy types. Denote by $\mathcal{I}(P)$ the set of all isotropy types on P. For (H), $(K) \in \mathcal{I}(P)$ define $(H) \leq (K)$ if and only if H is conjugate to a subgroup of K. Then $\mathcal{I}(P)$ becomes an ordered set with the order $\leq$. For $(H) \in \mathcal{I}(P)$ define

$$\begin{aligned} P^H &= \{x \in P \mid H \subset G_x\}, \\ P_H &= \{x \in P \mid H = G_x\}, \\ P^{(H)} &= \{x \in P \mid (H) \leq (G_x)\}, \\ P_{(H)} &= \{x \in P \mid (H) = (G_x)\}. \end{aligned}$$

Then

$$\begin{aligned} P^{(H)} &= GP^H = \{gx \mid g \in G,\ x \in P^H\}, \\ P_{(H)} &= GP_H \approx G \times_{N(H)} P_H, \end{aligned}$$

where $N(H)$ is the normalizer of H in G. P^H, P_H, and $P_{(H)}$ are $N(H)$-, $N(H)$-, and G-invariant smooth submanifolds of P, respectively. If (H) is maximum in $\mathcal{J}(P)$ then $P^{(H)}$ is also a G-invariant smooth submanifold of P.

Let π: $P^H \to P^H/N(H)$ be the canonical projection. The inverse image of a connected component of $P^H/N(H)$ by π is called an *$N(H)$-component* of P^H. Let $\{P^H_\alpha \mid \alpha \in C_{N(H)}(P^H)\}$ be the family of all $N(H)$-components of P^H. Let Q be another smooth G-manifold, and f: $P \to Q$ a G-map. For $\alpha \in C_{N(H)}(P^H)$, $Q^H_{f(\alpha)}$ denotes the $N(H)$-component of Q^H containing $f(P^H_\alpha)$. If G-maps f, g: $P \to Q$ are G-homotopic (denote by $f \simeq_G g$), then $Q^H_{f(\alpha)} = Q^H_{g(\alpha)}$. If $H \subset K$ then $Q^K \subset Q^H$, and define

$$(Q^H_{f(\alpha)})^K = \{x \in Q^H_{f(\alpha)} \mid K \subset G_x\} = Q^H_{f(\alpha)} \cap Q^K .$$

Then, since $(Q^H_{f(\alpha)})^K$ is a sum of connected components of Q^K, it is a smooth submanifold of Q.

The dimensions of fixed point sets P^H, $(Q^H_{f(\alpha)})^K$ and so on are not homogeneous, that is, the dimensions may vary with connected components. If a manifold M has not a homogeneous dimension, we denote by dim M the maximum of dimensions of M. In this connection note that $N(H)$-components P^H_α, $Q^H_{f(\alpha)}$ have homogeneous dimensions.

A G-map f: $P \to Q$ is called *isovariant* if $G_x = G_{f(x)}$ for every $x \in P$. Equivariant embeddings are isovariant. However, equivariant immersions are in general not isovariant.

Smooth G-manifolds P, Q considered in this paper have or have not boundary. If P or Q has boundary, immersions and embeddings f: $P \to Q$ satisfy $\partial P = f^{-1}(\partial Q)$. Immersions and embeddings in this paper are smooth ones, that is, we work in the smooth category.

2. Simple case

In this section we consider the case in which P is of one isotropy type, and prove the following three results.

Theorem 2.1. *Let P be a compact smooth G-manifold with only one isotropy type (H), and Q a smooth G-manifold with $\mathcal{J}(Q)$ finite and containing (H). Let f: $P \to Q$ be a smooth G-map isovariant on a neighborhood of ∂P in P. For any $\alpha \in C_{N(H)}(P^H)$ suppose that*

$$\dim P^H_\alpha < \dim Q^H_{f(\alpha)} - \dim (Q^H_{f(\alpha)})^K + \dim (N(H) \cap N(K))/H$$

for any $(K) \in \mathcal{J}(Q)$ with $H \subsetneqq K$ and with $(Q^H_{f(\alpha)})^K \neq \phi$. Then there is a smooth G-map g: $P \to Q$ such that

(1) *g is isovariant,*

(2) *$g = f$ on a neighborhood of ∂P in P,*

(3) $g \simeq_G f$ *rel the neighborhood of* ∂P *in* P.

Corollary 2.2. *In Theorem* 2.1, *suppose furthermore that* f *is an isovariant immersion on a neighborhood of* ∂P *in* P, *and suppose that*

$$2 \dim P^H_\alpha \leq \dim Q^H_{f(\alpha)} + \dim N(H)/H$$

for any $\alpha \in C_{N(H)}(P^H)$. *Then* g *may be chosen so as to be an isovariant immersion.*

Corollary 2.3. *In Theorem* 2.1, *suppose furthermore that* f *is an embedding on a neighborhood of* ∂P *in* P, *and suppose that*

$$2 \dim P^H_\alpha < \dim Q^H_{f(\alpha)} + \dim N(H)/H$$

for any $\alpha \in C_{N(H)}(P^H)$. *Then* g *may be chosen so as to be an embedding.*

Proof of Theorem 2.1. We proceed by induction on the number

$$n(Q) = \#\{(K) \in \mathcal{J}(Q) \mid H \subsetneqq K\} .$$

If $n(Q)=0$, since

$$f(P) = f(P_{(H)}) \subset Q^{(H)} = Q_{(H)} ,$$

then f is isovariant in itself.

Suppose that $n(Q) \geq 1$ and that the theorem is valid for Q_1 with $n(Q_1) < n(Q)$. Let (K) be an isotropy type maximal in $\mathcal{J}(Q)$ with $H \subsetneqq K$. Consider smooth $N(H)$-maps

$$f^H_\alpha = f \mid P^H_\alpha \colon P^H_\alpha \to Q^H_{f(\alpha)} , \qquad \text{and}$$
$$1 \times f^H_\alpha \colon P^H_\alpha \to P^H_\alpha \times Q^H_{f(\alpha)}, \quad x \mapsto (x, f^H_\alpha(x)) .$$

Passing $1 \times f^H_\alpha$ to orbit spaces, we obtain a smooth map

$$1 \times_{N(H)} f^H_\alpha \colon P^H_\alpha / N(H) \to P^H_\alpha \times_{N(H)} Q^H_{f(\alpha)} .$$

Since (K) is maximal in $\mathcal{J}(Q)$, $N(H)\,(Q^H_{f(\alpha)})^K$ is an $N(H)$-invariant smooth submanifold of $Q^H_{f(\alpha)}$, and we see

$$N(H)\,(Q^H_{f(\alpha)})^K \approx N(H) \times_{N(H) \cap N(K)} (Q^H_{f(\alpha)})^K .$$

Thus

$$\dim N(H)\,(Q^H_{f(\alpha)})^K = \dim (Q^H_{f(\alpha)})^K + \dim N(H)/N(H) \cap N(K) .$$

$P^H_\alpha \times_{N(H)} N(H)(Q^H_{f(\alpha)})^K$ is a smooth submanifold of $P^H_\alpha \times_{N(H)} Q^H_{f(\alpha)}$. If f is isovariant on a neighborhood A of ∂P in P, then

$$1 \times_{N(H)} f^H_\alpha((A \cap P^H_\alpha)/N(H)) \cap P^H_\alpha \times_{N(H)} N(H)\,(Q^H_{f(\alpha)})^K = \phi .$$

From the dimension condition of the theorem we see

$$\dim P_\alpha^H/N(H) < \dim P_\alpha^H \times_{N(H)} Q_{f(\alpha)}^H - \dim P_\alpha^H \times_{N(H)} N(H)\,(Q_{f(\alpha)}^H)^K\,.$$

Thus we obtain a smooth G-map

$$g_1\colon P_\alpha^H/N(H) \to P_\alpha^H \times_{N(H)} Q_{f(\alpha)}^H$$

such that

(1) $\operatorname{Im} g_1 \cap P_\alpha^H \times_{N(H)} N(H)(Q_{f(\alpha)}^H)^K = \phi$,

(2) $g_1 = f$ on $(B \cap P_\alpha^H)/N(H)$, where B $(\subset A)$ is a neighborhood of ∂P in P,

(3) $g_1 \simeq 1 \times_{N(H)} f_\alpha^H$ rel $(B \cap P_\alpha^H)/N(H)$.

The canonical projections

$$P_\alpha^H \to P_\alpha^H/N(H), \qquad \text{and}$$
$$P_\alpha^H \times Q_{f(\alpha)}^H \to P_\alpha^H \times_{N(H)} Q_{f(\alpha)}^H$$

are smooth $N(H)$-fibre bundles. Thus, by the equivariant covering homotopy property (a relative version of Bierstone [1]), g_1 lifts to a smooth $N(H)$-map

$$P_\alpha^H \to P_\alpha^H \times Q_{f(\alpha)}^H\,.$$

This map is followed by the projection

$$P_\alpha^H \times Q_{f(\alpha)}^H \to Q_{f(\alpha)}^H\,,$$

and induces a smooth $N(H)$-map

$$g_2\colon P_\alpha^H \to Q_{f(\alpha)}^H$$

such that

(1) $\operatorname{Im} g_2 \cap N(H)(Q_{f(\alpha)}^H)^K = \phi$,

(2) $g_2 = f_\alpha^H$ on $C \cap P_\alpha^H$, where C $(\subset B)$ is a G-invariant neighborhood of ∂P in P,

(3) $g_2 \simeq_{N(H)} f_\alpha^H$ rel $C \cap P_\alpha^H$.

Performing the same as above over all $N(H)$-components of P^H, we obtain a smooth $N(H)$-map $P^H \to Q^H$. Since $P \approx G \times_{N(H)} P^H$, this map extends equivariantly to a smooth G-map

$$g_3\colon P \to Q\,,$$

which satisfies

(1) $\operatorname{Im} g_3 \cap Q^{(K)} = \phi$,

(2) $g_3 = f$ on C,

(3) $g_3 \simeq_G f$ rel C.

Let $Q_1 = Q - Q^{(K)}$. Then Q_1 is a smooth G-manifold with $n(Q_1) = n(Q)$

-1. g_3 is isovariant on a neighborhood of ∂P in P, and $\operatorname{Im} g_3 \subset Q_1$. We may see that

$$(Q_1)^H_{g_3(\alpha)} \subset Q^H_{f(\alpha)}, \qquad \text{and}$$
$$\dim (Q_1)^H_{g_3(\alpha)} = \dim Q^H_{f(\alpha)},$$

also see that

$$((Q_1)^H_{g_3(\alpha)})^{K'} \subset (Q^H_{f(\alpha)})^{K'}, \qquad \text{and}$$
$$\dim ((Q_1)^H_{g_3(\alpha)})^{K'} \leq \dim (Q^H_{f(\alpha)})^{K'}.$$

Thus g_3: $P \to Q_1$ satisfies the hypothesis of the theorem. So, by the hypothesis of induction, we obtain a smooth G-map

$$g_4: P \to Q_1 \subset Q$$

such that

(1) g_4 is isovariant,

(2) $g_4 = g_3 = f$ on a neighborhood of ∂P in P,

(3) $g_4 \simeq_G g_3 \simeq_G f$ rel a neighborhood of ∂P in P.

Thus g_4 satisfies the conclusions of the theorem. Q.E.D.

Proof of Corollary 2.2. We may assume f: $P \to Q$ is isovariant. Then, by restricting f a smooth $N(H)$-map

$$f^H = f \mid P^H: P^H = P_H \to Q_H$$

is induced. Passing this map to orbit spaces, we obtain a smooth map

$$f^H/N(H): P^H/N(H) \to Q^H/N(H).$$

By the hypothesis of the corollary this is an immersion on a neighborhood of $\partial P^H/N(H)$ in $P^H/N(H)$. For any $\alpha \in C_{N(H)}(P^H)$ let

$$Q_H^{f(\alpha)} = Q_H \cap Q^H_{f(\alpha)},$$

then

$$\dim Q_H^{f(\alpha)} = \dim Q^H_{f(\alpha)}.$$

Thus, from the hypothesis it follows

$$2 \dim P^H_\alpha/N(H) \leq \dim Q_H^{f(\alpha)}/N(H).$$

So $f^H/N(H)$ may be homotoped to a smooth map

$$g_1: P^H/N(H) \to Q_H/N(H)$$

such that

(1) g_1 is an immersion,

(2) $g_1 = f^H/N(H)$ on a neighborhood of $\partial P^H/N(H)$ in $P^H/N(H)$,

(3) $g_1 \simeq f^H/N(H)$ rel the neighborhood of $\partial P^H/N(H)$ in $P^H/N(H)$.

By the equivariant covering homotopy property (a relative version of Bierstone [1]) g_1 lifts to a smooth $N(H)$-map $P^H \to Q_H$. Since $P \approx G \times_{N(H)} P^H$, this map extends equivariantly to a smooth G-map

$$g_2 \colon P \to Q$$

such that

(1) g_2 is an isovariant immersion,

(2) $g_2 = f$ on a neighborhood of ∂P in P,

(3) $g_2 \simeq_G f$ rel the neighborhood of ∂P in P.

Thus the corollary is proved. Q.E.D.

Proof of Corollary 2.3. This is essentially the same as for Corollary 2.2. Q.E.D.

3. Technical lemmas

Let us number all the isotropy types on a compact smooth G-manifold P,

$$\mathcal{J}(P) = \{(H_1), (H_2), \cdots, (H_a)\} ,$$

in such a way that $(H_i) \leq (H_j)$ implies $j \leq i$. In the following we fix once and for all such a numbering of $\mathcal{J}(P)$. If $(H) = (H_i) \in \mathcal{J}(P)$, then define

$$\begin{aligned} {}^{(H)}P &= P^{(H_1)} \cup P^{(H_2)} \cup \cdots \cup P^{(H_{i-1})} \\ &= P_{(H_1)} \cup P_{(H_2)} \cup \cdots \cup P_{(H_{i-1})} . \end{aligned}$$

Lemma 3.1. *Let (H) be an isotropy type on a compact smooth G-manifold P, and X a G-invariant neighborhood of $\partial P \cup {}^{(H)}P$ in P. Then we obtain G-invariant compact smooth submanifolds M, $\bar{P}$ of P such that*

(1) $P = M \cup \bar{P}$, $\partial M = \partial P \cup \partial \bar{P}$, $\partial \bar{P} = M \cap \bar{P}$,

(2) $\dim P = \dim M = \dim \bar{P}$,

(3) *M is contained in* $\operatorname{Int} X$ *and is a neighborhood of $\partial P \cup {}^{(H)}P$ in P.*

Proof. Let $(H) = (H_i)$. Then, for any integer j with $1 \leq j \leq i$, consider the following assertion A(j).

A(j). *There are G-invariant compact smooth submanifolds M_j, $\bar{P}_j$ of P such that*

(1) $P = M_j \cup \bar{P}_j$, $\partial M_j = \partial P \cup \partial \bar{P}_j$, $\partial \bar{P}_j = M_j \cap \bar{P}_j$,

(2) $\dim P = \dim M_j = \dim \bar{P}_j$,

(3) *M_j is containd in* $\operatorname{Int} X$ *and is a neighborhood of $\partial P \cup {}^{(H_j)}P$ in P.*

A(i) is equivalent to the lemma. We prove all A(j) by induction.

There is a G-invariant collar $\partial P\times[0, 1]$ of ∂P in P which is contained in $\mathrm{Int}\, X$. Then

$$M_1=\partial P\times[0, 1]\,,\qquad \text{and}$$
$$\bar{P}_1=P-\partial P\times[0, 1)$$

satisfy the conclusions of A(1).

Assume A(j) is valid. $\bar{P}_j^{(H_j)}$ is a G-invariant smooth submanifold of $\bar{P}_j$, since $\bar{P}_j$ has no isotropy type larger than (H_j). There is a G-invariant closed tubular neighborhood T of $\bar{P}_j^{(H_j)}$ in $\bar{P}_j$ which is contained in $\mathrm{Int}\, X$. Let T^o be the open tubular neighborhood with the same radius as T, and let

$$L=\bar{P}_j-T^o\,.$$

Then L is a G-invariant submanifold (with corner) of $\bar{P}_j$. Take a G-invariant collar $\partial L\times[0, 1]$ of ∂L in L which is contained in $\mathrm{Int}\, X$. Then

$$M_{j+1}=M_j\cup T\cup\partial L\times[0, 1]\,,\qquad \text{and}$$
$$\bar{P}_{j+1}=\bar{P}_j-(T\cup\partial L\times[0, 1))$$

satisfy the conclusions of A($j+1$), that is, A($j+1$) is valid.

Thus the lemma is proved. Q.E.D.

Lemma 3.2. *Let P, Q be smooth G-manifolds, and P compact. Let M be a G-invariant compact smooth submanifold of P with $\partial M\subset\partial P$, and let A be either a G-invariant open neighborhood of M in P, or M itself. Let $f\colon P\to Q$ and $g\colon A\to Q$ be smooth G-maps such that*

(1) *$g=f$ on $A\cap B$, where B is a G-invariant neighborhood of ∂P in P,*

(2) *$g|M\simeq_G f|M$ rel $M\cap B$.*

Then we obtain a smooth G-map $h\colon P\to Q$ such that

(1) *$h=f$ on a neighborhood of ∂P in P,*

(2) *if A is a neighborhood of M then $h=g$ on a neighborhood of M in P, and if $A=M$ then $h=g$ on M,*

(3) *$h\simeq_G f$ rel the neighborhood of ∂P in P, moreover, if $f=g$ on M then $h\simeq_G f$ rel the neighborhood and rel M.*

Proof. We prove the case in which A is a neighborhood of M. (The proof for the case $A=M$ is essentially the same (but easier).)

There is a G-invariant collar $\partial P\times[0, 2]$ of ∂P in P which restricts to a collar $\partial M\times[0, 2]$ of ∂M in M and which is contained in $\mathrm{Int}\, B$. Let

$$P'=P-\partial P\times[0, 2)\,,\qquad \text{and}$$
$$M'=M\cap P'=M-\partial M\times[0, 2)\,.$$

There is a G-invariant closed tubular neighborhood $T_{4\delta}(M)$, with radius $4\delta>0$, of M in P which restricts to a tubular neighborhood $T_{4\delta}(M')$ of M' in P' and which is contained in A. By the hypothesis of the lemma there is a G-homotopy

$$H: M\times[0, 1] \to Q$$

such that

(1) $H_0=f\,|\,M$, $H_1=g\,|\,M$,

(2) H is constant for $t\in[0, 1]$ on $M\cap B$.

Define a G-map

$$h_1: P' \to Q$$

as follows. For any $x\in T_{4\delta}(M')$ denote by $||x||$ the length of x as a normal vector. If $0\leq||x||\leq\delta$, define

$$h_1(x) = g(x)\,,$$

if $\delta\leq||x||\leq 2\delta$, define

$$h_1(x) = g((\frac{2\delta}{||x||}-1)x)\,,$$

if $2\delta\leq||x||\leq 3\delta$, define

$$h_1(x) = H(\pi(x),\, 3-\frac{||x||}{\delta})\,,$$

where π: $T_{4\delta}(M')\to M'$ is the canonical projection, if $3\delta\leq||x||\leq 4\delta$, define

$$h_1(x) = f((4-\frac{12\delta}{||x||})x)\,,$$

and if $x\in P'-T^{\circ}_{4\delta}(M')$, define

$$h_1(x) = f(x)\,,$$

where $T^{\circ}_{4\delta}(M')$ is the open tubular neighborhood with radius 4δ. Then h_1: $P'\to Q$ is well-defined. Consider the subspace $\partial P\times[1, 2]$ of $\partial P\times[0, 2]$. By using the fact that H is a constant homotopy of $f=g$ on $\partial M\times\{2\}=(\partial P\times\{2\})\cap M$, we obtain a G-map

$$h_2: \partial P\times[1, 2] \to Q$$

such that

(1) $h_2=f$ on $\partial P\times\{1\}$,

(2) $h_2=h_1$ on $\partial P\times\{2\}$,

(3) $h_2=f=g$ on $T_\delta(M)\cap\partial P\times[1, 2]$.

Define a G-map

$$h_3\colon P \to Q$$

as $h_3=f$ on $\partial P\times[0,\ 1]$, $h_3=h_2$ on $\partial P\times[1,\ 2]$, and $h_3=h_1$ on P'. Then h_3 is G-homotopic rel $\partial P\times[0,\ 1]$ to f, and is smooth on $\partial P\times[0,\ 1)\cup T^{\circ}_{\delta}(M)$. The desired smooth G-map $h\colon P\to Q$ is obtained by equivariant smoothing of h_3 (cf. Wasserman [6; Corollary 1.12]). Q.E.D.

4. Extension of isovariancy (1)

In this section we prove

Theorem 4.1. *Let P, Q be smooth G-manifolds with $(H)\in\mathcal{I}(P)\subset\mathcal{I}(Q)$, P compact, and $\mathcal{I}(Q)$ finite. Let $f\colon P\to Q$ be a smooth G-map isovariant on a neighborhood of $\partial P\cup{}^{(H)}P$ in P. For any $\alpha\in C_{N(H)}(P^H)$ suppose that*

$$\dim P^H_\alpha<\dim Q^H_{f(\alpha)}-\dim (Q^H_{f(\alpha)})^K+\dim (N(H)\cap N(K))/H$$

for any $(K)\in\mathcal{I}(Q)$ with $H\subsetneqq K$ and with $(Q^H_{f(\alpha)})^K\neq\phi$. Then the isovariancy of f extends over $P^{(H)}$, that is, there is a smooth G-map $g\colon P\to Q$ such that

(1) *g is isovariant on a neighborhood of $\partial P\cup{}^{(H)}P$ and on $P^{(H)}$,*
(2) *$g=f$ on a neighborhood of $\partial P\cup{}^{(H)}P$,*
(3) *$g\simeq_G f$ rel the neighborhood of $\partial P\cup{}^{(H)}P$.*

Proof. Let X be a G-invariant neighborhood of $\partial P\cup{}^{(H)}P$ on which f is isovariant. Let M ($\subset\operatorname{Int}X$) and $\bar{P}$ be G-invariant compact smooth submanifolds of P obtained from Lemma 3.1. The smooth G-map $f|\bar{P}\colon \bar{P}\to Q$ is isovariant on $X\cap\bar{P}$ a neighborhood of $\partial\bar{P}$ in $\bar{P}$. If $P^{(H)}\subset X$ then there remains nothing to be proved, and if not then (H) is maximal in $\mathcal{I}(\bar{P})$, and $\bar{P}^{(H)}$ is a compact smooth G-manifold with the only one isotropy type (H). We may apply Theorem 2.1 to a smooth G-map $f|\bar{P}^{(H)}\colon \bar{P}^{(H)}\to Q$, and obtain a smooth G-map

$$g_1\colon \bar{P}^{(H)}\to Q$$

such that

(1) g_1 is isovariant,
(2) $g_1=f$ on a neighborhood of $\partial\bar{P}^{(H)}$ in $\bar{P}^{(H)}$,
(3) $g\simeq_G f|\bar{P}^{(H)}$ rel the neighborhood.

By Lemma 3.2, $f|\bar{P}^{(H)}$ and g_1 give a smooth G-map

$$g_2\colon \bar{P}\to Q$$

such that

(1) $g_2=f$ on a neighborhood of $\partial\bar{P}$ in $\bar{P}$,
(2) $g_2=g_1$ on $\bar{P}^{(H)}$, and hence g_2 is isovariant on $\bar{P}^{(H)}$,

(3) $g_2 \simeq_G f \mid \bar{P}^{(H)}$ rel the neighborhood of $\partial\bar{P}$ in $\bar{P}$.
Define

$$g: P \to Q$$

by $g=f$ on $P-\mathrm{Int}\,\bar{P}$, and $g=g_2$ on $\bar{P}$. Then g is the desired smooth G-map.
Q.E.D.

5. Orthogonal isovariance and obstructions

Let $f: P \to Q$ be a smooth G-map isovariant on a G-invariant open subset X of P. Let $\tau(P)$, $\tau(Q)$ be the tangent bundles of P, Q, respectively, and df: $\tau(P) \to \tau(Q)$ the differential of f. Let

$$d_x f = df \mid \tau_x(P): \tau_x(P) \to \tau_{f(x)}(Q),$$

where $\tau_x(P)$ and $\tau_{f(x)}(Q)$ are the tangent spaces at $x \in P$ and $f(x) \in Q$, respectively. For $x \in X$ let $K=G_x=G_{f(x)}$. $\tau_x(P_{(K)})$ and $\tau_{f(x)}(Q_{(K)})$ are subvector spaces of $\tau_x(P)$ and $\tau_{f(x)}(Q)$, respectively, and we see

$$d_x f(\tau_x(P_{(K)})) \subset \tau_{f(x)}(Q_{(K)}).$$

So $d_x f$ induces a homomorphism of quotient vector spaces,

$$\tilde{d}_x f: \tau_x(P)/\tau_x(P_{(K)}) \to \tau_{f(x)}(Q)/\tau_{f(x)}(Q_{(K)}).$$

If $\tilde{d}_x f$ is injective for any $x \in X$, f is called *orthogonally isovariant* on X.

Now suppose that P is compact, $(H) \in \mathcal{I}(\mathrm{P}) \subset \mathcal{I}(\mathrm{Q})$, and that $f: P \to Q$ is isovariant on $P^{(H)}$ and orthogonally isovariant on a G-invariant open neighborhood X of $\partial P \cup {}^{(H)}P$ in P. In this section we introduce obstructions for extending the orthogonal isovariancy of f over a neighborhood of $P^{(H)}$.

By Lemma 3.1 we obtain G-invariant compact smooth sbumanifolds M, $\bar{P}$ of P such that

(1) $P=M \cup \bar{P}$, $\partial M=\partial P \cup \partial\bar{P}$, $\partial\bar{P}=M \cap \bar{P}$,

(2) $\dim P=\dim M=\dim \bar{P}$,

(3) M is a neighborhood of $\partial P \cup {}^{(H)}P$, and $M \subset X$.

Since $\bar{P}$ has no isotropy type larger than (H),

$$\bar{P}^{(H)} = \bar{P}_{(H)}.$$

Let

$$\nu(\bar{P}_{(H)}) = (\tau(\bar{P}) \mid \bar{P}_{(H)})/\tau(\bar{P}_{(H)}), \qquad \text{and}$$
$$\nu(Q_{(H)}) = (\tau(Q) \mid Q_{(H)})/\tau(Q_{(H)})$$

be the normal bundles of $\bar{P}_{(H)}$ in $\bar{P}$, and of $Q_{(H)}$ in Q, respectively. Since f is isovariant on $P^{(H)}$, then

$$df(\tau(\bar{P}_{(H)}))\subset\tau(Q_{(H)})\,.$$

So df induces a smooth G-vector bundle homomorphism

$$\tilde{d}f\colon \nu(\bar{P}_{(H)}) \to \nu(Q_{(H)})\,.$$

Then $\tilde{d}f$ is a bundle monomorphism over some G-invariant open neighborhood $Y\,(\subset X)$ of $\partial\bar{P}_{(H)}$ in $\bar{P}_{(H)}$, since f is orthogonally isovariant on a neighborhood of $\partial\bar{P}$ in $\bar{P}$. Let

$$f_{(H)}=f\,|\,\bar{P}_{(H)}\colon \bar{P}_{(H)} \to Q_{(H)}\,,$$

and consider the induced smooth G-vector bundle

$$f^*_{(H)}\nu(Q_{(H)}) \to \bar{P}_{(H)}\,.$$

If there exist H-monomorphisms from fibres of $\nu(\bar{P}_{(H)})\,|\,\bar{P}_H$ to fibres of $f^*_{(H)}\nu(Q_{(H)})\,|\,\bar{P}_{(H)}$ over the same points of $\bar{P}_H$, then denote the set of all those H-monomorphisms by

$$\mathrm{Mon}^H(\nu(\bar{P}_{(H)})\,|\,\bar{P}_H, f^*_{(H)}\nu(Q_{(H)}\,|\,\bar{P}_H)\,.$$

By the standard manner we may give this a smooth $N(H)$-fibre bundle structure over $\bar{P}_H$. Passing this bundle to orbit spaces, we obtain a smooth fibre bundle

$$B_H = \mathrm{Mon}^H(\nu(\bar{P}_{(H)})\,|\,\bar{P}_H, f^*_{(H)}\nu(Q_{(H)})\,|\,\bar{P}_H)/N(H) \to \bar{P}_H/N(H)\,.$$

For a nonnegative integer n, denote by $B_H(\pi_n)$ the associated bundle to B_H by the n-th homotopy groups of fibres. (See Steenrod [5; 30.2] for its definition.) Since $\tilde{d}f$ is a bundle monomorphism over Y, $\tilde{d}f$ induces a smooth cross section

$$s_f\colon Y_H/N(H) \to B_H\,|\,(Y_H/N(H))$$

of B_H on $Y_H/N(H)$. There is an obstruction theory for extending s_f over $\bar{P}_H/N(H)$ (see [5]). The obstructions are denoted by

$$\mathfrak{o}^n_H(f)\in H^n(\bar{P}_H/N(H),\, \partial\bar{P}_H/N(H);\, B_H(\pi_{n-1}))$$
$$(n = 1, 2, \cdots, \dim \bar{P}_H/N(H))$$

Note that $\mathfrak{o}^n_H(f)$ is defined if and only if the following (1), (2) and (3) are satisfied:

(1) $f\colon P\to Q$ is isovariant on $P^{(H)}$ and orthogonally isovariant on a G-invariant open neighborhood of $\partial P\cup {}^{(H)}P$ in P,

(2) B_H is defined, that is, there exist H-monomorphisms from fibres of $\nu(\bar{P}_{(H)})$ to fibres of $f^*_{(H)}\nu(Q_{(H)})$ over the same points of $\bar{P}_H$,

(3) for any i with $i<n$, $\mathfrak{o}^i_H(f)$ is defined and vanishes.

Lemma 5.1. *The vanishing of $o_H^n(f)$ for any n with $1 \leq n \leq \dim \bar{P}_H/N(H)$ is a necessary and sufficient condition for the existence of a smooth G-vector bundle monomorphism*

$$\varphi\colon \nu(\bar{P}_{(H)}) \to \nu(Q_{(H)})$$

such that

(1) *φ covers $f_{(H)}$,*

(2) *$\varphi = \tilde{d}f$ over a neighborhood of $\partial \bar{P}_{(H)}$ in $\bar{P}_{(H)}$.*

Proof. If such φ exists, we easily see that all $o_H^n(f)$ vanish. Conversely, if all $o_H^n(f)$ vanish, then there is a cross section

$$s\colon \bar{P}_H/N(H) \to B_H$$

which coincides with s_f on a neighborhood of $\partial \bar{P}_H/N(H)$ in $\bar{P}_H/N(H)$. By the differentiable approximation theorem [5; 6.7] we may assume s is smooth. s induces a smooth $N(H)$-cross section

$$\tilde{s}\colon \bar{P}_H \to \mathrm{Mon}^H(\nu(\bar{P}_{(H)})\,|\,\bar{P}_H, f_{(H)}^*\nu(Q_{(H)})\,|\,\bar{P}_H)\,,$$

and $\tilde{s}$ also induces a smooth $N(H)$-vector bundle monomorphism

$$\rho\colon \nu(\bar{P}_{(H)})\,|\,\bar{P}_H \to \nu(Q_{(H)})$$

such that

(1) ρ covers $f_{(H)}\,|\,\bar{P}_H$,

(2) $\rho = \tilde{d}f$ on a neighborhood of $\partial \bar{P}_H$ in $\bar{P}_H$.

Since

$$\nu(\bar{P}_{(H)}) \cong G \times_{N(H)} (\nu(\bar{P}_{(H)})\,|\,\bar{P}_H)\,,$$

then ρ extends equivariantly to the desired smooth G-vector bundle monomorphism

$$\varphi\colon \nu(\bar{P}_{(H)}) \to \nu(Q_{(H)})\,. \qquad \text{Q.E.D.}$$

Let $\{V_j \,|\, j \in J(H)\}$ be a complete set of nontrivial, nonisomorphic, irreducible, real representations of H. For any $j \in J(H)$ let

$$\boldsymbol{F}_j = \mathrm{Hom}^H(V_j, V_j)\,,$$

which is the reals $\boldsymbol{R}$, the complexes $\boldsymbol{C}$, or the quaternions $\boldsymbol{Q}$. For nonnegative integers $m \leq n$ denote by $V(m, n; \boldsymbol{F}_j)$ the Stiefel manifold of m-frames (not necessarily orthonormal) in the n-dimensional vector space $n\boldsymbol{F}_j$ over $\boldsymbol{F}_j$. Note that $V(m, n; \boldsymbol{F}_j)$ is $(d_j(n-m+1)-2)$-connected, where $d_j = \dim_{\boldsymbol{R}} \boldsymbol{F}_j$.

The fibre $\nu_x(P_{(H)})$ of $\nu(P_{(H)})$ over $x \in P_H$ is a representation of H, and splits into

$$\nu_x(P_{(H)}) \cong \bigoplus_{j \in J(H)} m_{x,j} V_j ,$$

where $m_{x,j}$ are nonnegative integers. Similarly the fibre $\nu_y(Q_{(H)})$ of $\nu(Q_{(H)})$ over $y \in Q_H$ splits into

$$\nu_y(Q_{(H)}) \cong \bigoplus_{j \in J(H)} n_{y,j} V_j ,$$

where $n_{y,j}$ are nonnegative integers. Note

$$\nu(\bar{P}_{(H)}) \cong \nu(P_{(H)}) \,|\, \bar{P}_{(H)} .$$

The fibre of B_H over $[x] \in \bar{P}_H / N(H)$,

$$\mathrm{Mon}^H(\nu_x(\bar{P}_{(H)}),\ \nu_{f(x)}(Q_{(H)})) ,$$

may be identified with

$$\prod_{j \in J(H)} V(m_{x,j},\ n_{f(x),j};\ \boldsymbol{F}_j) .$$

For $x \in P_H$ let

$$\lambda^H(f, x) = \min \{ d_j(n_{f(x),j} - m_{x,j} + 1) \mid j \in J(H),\ m_{x,j} \neq 0 \} + \dim N(H)/H .$$

Lemma 5.2. *For any $x \in P_H \cap P^H_\alpha$, $\lambda^H(f, x)$ takes the same value.*

According to the lemma, we define $\lambda^H_\alpha(f)$ to be the same value of $\lambda^H(f, x)$ for any $x \in P_H \cap P^H_\alpha$.

Proof. (i) Suppose that x and y belong to the same connected component of $P_H \cap P^H_\alpha$. Then $f(x)$ and $f(y)$ belong to the same connected component of $Q_H \cap Q^H_{f(\alpha)}$, since f is isovariant on P_H. Thus we see that $\nu_x(P_{(H)})$ and $\nu_{f(x)}(Q_{(H)})$, respectively, are isomorphic to $\nu_y(P_{(H)})$ and $\nu_{f(y)}(Q_{(H)})$ as representations of H, This implies $\lambda^H(f, x) = \lambda^H(f, y)$.

(ii) Suppose that x and y belong to distinct connected components of $P_H \cap P^H_\alpha$. $P_H \cap P^H_\alpha$ is the space of points whose isotropy subgroups are the principal isotropy subgroup H on the $N(H)$-manifold P^H_α. Since $P^H_\alpha / N(H)$ is connected, then $P_H \cap P^H_\alpha / N(H)$ is so (see Bredon [2; Theorem 3.1, p 179]). Thus it is sufficient that we only consider the case where $y = ax$ for some $a \in N(H)$. In this case, for any $h \in H$ and any $v \in \nu_x(P_{(H)})$, we see $ahv = \psi(h)av$ in $\nu_{ax}(P_{(H)})$, where ψ is the automorphism of H defined by $\psi(h) = aha^{-1}$. We also see the similar for $\nu_{f(x)}(Q_{(H)})$ and $\nu_{af(x)}(Q_{(H)}) = \nu_{f(ax)}(Q_{(H)})$. These imply

$$\{ n_{f(x),j} - m_{x,j} \mid j \in J(H),\ m_{x,j} \neq 0 \} = \{ n_{f(ax),j} - m_{ax,j} \mid j \in J(H),\ m_{ax,j} \neq 0 \} .$$

Thus $\lambda^H(f, x) = \lambda^H(f, ax)$ follows. Q.E.D.

Proposition 5.3. *Suppose* $\dim P^H_\alpha < \lambda^H_\alpha(f)$ *for any* $\alpha \in C_{N(H)}(P^H)$. *Then the bundle B_H is defined, and*

$$H^n(\bar{P}_H/N(H), \partial\bar{P}_H/N(H); B_H(\pi_{n-1})) = 0$$

for any $n \geq 1$. *Hence all the obstructions* $o_H^n(f)$ *vanish.*

Proof. Since $N(H)/H$ acts freely on $P_H \cap P_\alpha^H$,

$$\dim N(H)/H \leq \dim P_H \cap P_\alpha^H = \dim P_\alpha^H .$$

So the hypothesis of the proposition implies $m_{x,j} \leq n_{f(x),j}$ for any $x \in P_H \cap P_\alpha^H$ and any $j \in J(H)$. Thus B_H may be defined. From

$$\dim P_\alpha^H = \dim \bar{P}_H \cap P_\alpha^H$$

we see

$$\dim \bar{P}_H/N(H) = \max \{\dim P_\alpha^H \mid \alpha \in C_{N(H)}(P^H)\} - \dim N(H)/H .$$

This and the hypothesis imply

$$\pi_{n-1}(\prod_{j \in J(H)} V(m_{x,j}, n_{f(x),j}; \boldsymbol{F}_j)) = 0$$

for any n with $1 \leq n \leq \dim \bar{P}_H/N(H)$. Thus the cohomology groups in question vanish. Q.E.D.

6. Extension of isovariancy (2)

In this section we prove

Theorem 6.1. *Let P, Q be smooth G-manifolds with $(H) \in \mathcal{J}(P) \subset \mathcal{J}(Q)$, and P compact. Let $f: P \to Q$ be a smooth G-map which is isovariant on $P^{(H)}$ and is orthogonally isovariant on a G-invariant open neighborhood of $\partial P \cap {}^{(H)}P$. Then the vanishing of $o_H^n(f)$ for any n with $1 \leq n \leq \dim P^H - \dim N(H)/H$ is a necessary and sufficient condition for the existence of a smooth G-map $g: P \to Q$ such that*

(1) *g is orthogonally isovariant on a neighborhood of $\partial P \cup {}^{(H)}P \cup P^{(H)}$ in P,*
(2) *$g = f$ on $P^{(H)}$ and on a neighborhood A of $\partial P \cup {}^{(H)}P$ in P,*
(3) *$g \simeq_G f$ rel $P^{(H)} \cup A$.*

Proof. Suppose that such a smooth G-map $g: P \to Q$ exists. Then the differential of g induces a smooth G-vector bundle monomorphism

$$\tilde{d}g: \nu(P_{(H)}) \to \nu(Q_{(H)}) ,$$

and this gives φ such as in Lemma 5.1. Thus every $o_H^n(f)$ vanishes.

Let X be a G-invariant open neighborhood of $\partial P \cap {}^{(H)}P$ in P on which f is orthogonally isovariant. Let M $(\subset X)$ and $\bar{P}$ be G-invariant compact smooth submanifolds obtained from Lemma 3.1. Suppose, conversely, that

$$o_H^n(f) \in H^n(\bar{P}_H/N(H), \partial\bar{P}_H/N(H); B_H(\pi_{n-1}))$$

vanishes for any n with

$$1 \leq n \leq \dim \bar{P}_H/N(H) = \dim P^H - \dim N(H)/H .$$

Then, from Lemma 5.1 we obtain a smooth G-vector bundle monomorphism

$$\varphi\colon \nu(\bar{P}_{(H)}) \to \nu(Q_{(H)})$$

such that

(1) φ covers $f_{(H)} = f \mid \bar{P}_{(H)}\colon \bar{P}_{(H)} \to Q_{(H)}$,

(2) $\varphi = \tilde{d}f$ over a neighborhood of $\partial \bar{P}_{(H)}$ in $\bar{P}_{(H)}$.

Making use of φ and exponential maps, for a G-invariant tubular neighborhood T of $\bar{P}_{(H)}$ in $\bar{P}$, we obtain a smooth G-map

$$g_1\colon T \to Q$$

such that

(1) g_1 is orthogonally isovariant,

(2) $g_1 = f$ on $\bar{P}_{(H)}$ and on $T \cap B$, where B is a G-invariant neighborhood of $\partial \bar{P}$ in $\bar{P}$.

By Lemma 3.2, f and g_1 give a smooth G-map

$$g_2\colon \bar{P} \to Q$$

such that

(1) $g_2 = f$ on $\bar{P}_{(H)}$ and on a neighborhood of $\partial \bar{P}$ in $\bar{P}$,

(2) $g_2 = g_1$ on a neighborhood of $\bar{P}_{(H)}$ in $\bar{P}$,

(3) $g_2 \simeq_G f \mid \bar{P}$ rel $\bar{P}_{(H)}$ and rel the neighborhood of $\partial \bar{P}$ in $\bar{P}$.

Define

$$g\colon P \to Q$$

as $g = f$ on M, and $g = g_2$ on $\bar{P}$. Then g is the desired smooth G-map. Q.E.D.

7. Main results

Let $f\colon P \to Q$ be a smooth G-map. Denote by $\mathcal{E}_f^H$ the set of smooth G-maps $g\colon P \to Q$ which are G-homotopic to f and isovariant on P_H. For $g \in \mathcal{E}_f^H$ and $x \in P_H$, we obtain the splittings into irreducible representations of H,

$$\nu_x(P_{(H)}) \cong \bigoplus_{j \in J(H)} m_{x,j} V_j ,$$

$$\nu_{g(x)}(Q_{(H)}) \cong \bigoplus_{j \in J(H)} n_{g(x),j} V_j ,$$

and define

$$\lambda^H(f)(g, x) = \min \{ d_j (n_{g(x),j} - m_{x,j} + 1) \mid j \in J(H),\ m_{x,j} \neq 0 \} + \dim N(H)/H .$$

Lemma 7.1. *For any $g \in \mathcal{E}_f^H$ and any $x \in P_H \cap P_\alpha^H$, $\lambda^H(f)(g, x)$ takes the* *same value.*

According to the lemma, for a smooth G-map $f: P \to Q$ we define $\lambda_\alpha^H(f)$ to be the same value of $\lambda^H(f)(g, x)$ for $g \in \mathcal{E}_f^H$ and $x \in P_H \cap P_\alpha^H$. Note that $\lambda_\alpha^H(f)$ may be defined when $\mathcal{E}_f^H$ is not empty.

Proof. In Lemma 5.2 it is already shown that if g is fixed then $\lambda^H(f)(g, x)$ takes the same value for any $x \in P_H \cap P_\alpha^H$. Thus it only remains to show that $\lambda^H(f)(g, x) = \lambda^H(f)(k, x)$ for $g, k \in \mathcal{E}_f^H$. Since g and k are G-homotopic, $g(x)$ and $k(x)$ belong to the same connected component of Q^H. Thus $\nu_{g(x)}(Q^H)$ and $\nu_{k(x)}(Q^H)$ are isomorphic as representations of H. We see

$$\nu_{g(x)}(Q^H) \cong \nu_{g(x)}(Q_{(H)}) \oplus V, \quad \text{and}$$
$$\nu_{k(x)}(Q^H) \cong \nu_{k(x)}(Q_{(H)}) \oplus V$$

for some representation V of H. These imply $\nu_{g(x)}(Q_{(H)}) \cong \nu_{k(x)}(Q_{(H)})$. Hence $\lambda^H(f)(g, x) = \lambda^H(f)(k, x)$. Q.E.D.

Now we may state the main results of this paper.

Theorem 7.2. *Let P, Q be smooth G-manifolds with $\mathcal{J}(P) \subset \mathcal{J}(Q)$, P compact, and $\mathcal{J}(Q)$ finite. Let $f: P \to Q$ be a smooth G-map which is orthogonally isovariant on a neighborhood of ∂P. For any $(H) \in \mathcal{J}(P)$ and any $\alpha \in C_{N(H)}(P^H)$, suppose that*

(7.2.1) $\dim P_\alpha^H < \dim Q_{f(\alpha)}^H - \dim (Q_{f(\alpha)}^H)^K + \dim (N(H) \cap N(K))/H$ *for any* $(K) \in \mathcal{J}(Q)$ *with* $H \subsetneq K$ *and with* $(Q_{f(\alpha)}^H)^K \neq \phi$,

(7.2.2) $\dim P_\alpha^H < \lambda_\alpha^H(f)$.

Then there exists a smooth G-map $g: P \to Q$ such that

(1) *g is orthogonally isovariant,*

(2) *$g = f$ on a neighborhood of ∂P in P,*

(3) *$g \simeq_G f$ rel the neighborhood of ∂P in P.*

Corollary 7.3. *In Theorem 7.2, suppose furthermore that f is an isovariant immersion on a neighborhood of ∂P, and suppose that*

$$2 \dim P_\alpha^H \leq \dim Q_{f(\alpha)}^H + \dim N(H)/H$$

for any $(H) \in \mathcal{J}(P)$ and any $\alpha \in C_{N(H)}(P^H)$. Then g may be chosen so as to be an isovariant immersion.

Corollary 7.4. *In Theorem 7.2, suppose furthermore that f is an embedding on a neighborhood of ∂P, and suppose that*

$$2 \dim P_\alpha^H < \dim Q_{f(\alpha)}^H + \dim N(H)/H$$

for any $(H) \in \mathcal{J}(P)$ and any $\alpha \in C_{N(H)}(P^H)$. Then g may be chosen so as to be *an embedding.*

Proof of Theorem 7.2. By repeated applications of Theorem 4.1 and Theorem 6.1, we may inductively construct the desired smooth G-map $g: P \to Q$. Q.E.D.

Next we prove Corollary 7.4. Proof of Corollary 7.3 is the same (or easier) as for Corollary 7.4.

Proof of Corollary 7.4. We may assume $f: P \to Q$ is orthogonally isovariant. Let

$$\mathcal{I}(P) = \{(H_1), (H_2), \cdots, (H_a)\}$$

be numbered in such a way that $(H_i) \leq (H_j)$ implies $j \leq i$. For any i with $1 \leq i \leq a+1$, consider the following assertion:

$A(i)$. *There exists a G-invariant open neighborhood U_i of $\partial P \cup {}^{(H_i)}P$ in P, and exists a smooth G-map g_i: $P \to Q$ such that*

(1) *g_i is orthogonally isovariant on P, and is an embedding on U_i,*

(2) *$g_i = f$ on a neighborhood of ∂P in P,*

(3) *$g_i \simeq_G f$ rel the neighborhood of ∂P in P.*

(Here, if $i = a+1$, consider $P = P^{(H_1)} \cup \cdots \cup P^{(H_a)}$ as ${}^{(H_i)}P$.)

Since U_{a+1} must be P, $A(a+1)$ implies the corollary. We prove all $A(i)$ by induction. Letting $g_1 = f$, we see $A(1)$ is valid.

We suppose $A(i)$ is valid, and want to prove $A(i+1)$. As in the proof of Lemma 3.1, we obtain a G-invariant smooth submanifold M of Q such that

(1) $\dim M = \dim Q$,

(2) M is a closed neighborhood of $\partial Q \cup Q^{(H_1)} \cup \cdots \cup Q^{(H_{i-1})}$,

(3) $g_i^{-1}(M) \subset U_i$, (This is possible from the fact that g_i is isovariant and that P is compact.)

(4) $g_i^{-1}(M)$ is a G-invariant compact smooth submanifold of P. (This is also possible by making $\operatorname{Im} g_i$ and ∂M intersect transversally.)

Note that $g_i^{-1}(M)$ is a neighborhood of $\partial P \cup {}^{(H_i)}P$. Let

$$L_1 = P - \operatorname{Int} g_i^{-1}(M), \quad \text{and}$$
$$L_2 = Q - \operatorname{Int} M,$$

then both L_1 and L_2 are smooth G-manifolds, and satisfy $g_i(L_1) \subset L_2$ and $\partial L_1 = g_i^{-1}(\partial L_2)$. The smooth G-map $g_i | L_1$: $L_1 \to L_2$ is isovariant on L_1, and is an embedding on a neighborhood of ∂L_1 in L_1. If $P^{(H_i)} \subset U_i$, then U_i and g_i insure $A(i+1)$. And if not, then (H_i) is maximal in $\mathcal{I}(L_1)$, and $L_1^{(H_i)}$ is a G-invariant compact smooth submanifold with the only one isotropy type (H_i). We may apply Corollary 2.3 to

$$g_i \mid L_1^{(H_i)}\colon L_1^{(H_i)} \to L_2 .$$

So we obtain a smooth G-map

$$h_1\colon L_1^{(H_i)} \to L_2$$

such that

(1) h_1 is an embedding,
(2) $h_1 = g_i$ on a neighborhood of $\partial L_1^{(H_i)}$ in $L_1^{(H_i)}$,
(3) $h_1 \simeq_G g_i \mid L_1^{(H_i)}$ rel the neighborhood.

We apply Lemma 3.2 to $g_i \mid L_1$ and h_1, and obtain a smooth G-map

$$h_2\colon L_1 \to L_2$$

such that

(1) $h_2 = g_i \mid L_1$ on a neighborhood of ∂L_1 in L_1,
(2) $h_2 = h_1$ on $L_1^{(H_i)}$,
(3) $h_2 \simeq_G g_i \mid L_1$ rel the neighborhood of ∂L_1 in L_1.

Define

$$h_3\colon P \to Q$$

as $h_3 = g_i$ on $g_i^{-1}(M)$, and $h_3 = h_2$ on L_1. Then h_3 is a smooth G-map, and is an embedding on $P^{(H_i)}$ and on a neighborhood of $\partial P \cup {}^{(H_i)}P$. In virtue of Proposition 5.3 and the hypothesis of Theorem 7.2, we see that the obstructions $\sigma^n_{H_i}(h_3)$ vanish. So, from Theorem 6.1, we obtain a smooth G-map

$$h_4\colon P \to Q$$

such that

(1) h_4 is orthogonally isovariant on a neighborhood A of $\partial P \cap {}^{(H_{i+1})}P$,
(2) $h_4 = h_3$ on $P^{(H_i)}$ and on a neighborhood B of $\partial P \cup {}^{(H_i)}P$,
(3) $h_4 \simeq_G h_3$ rel $P^{(H_i)} \cup B$.

Since h_4 is an embedding on $P^{(H_i)}$ and is orthogonally isovariant on a neighborhood of $P^{(H_i)}$, then h_4 is an embedding on a neighborhood of $P^{(H_i)}$. Furthermore, since $h_4 = h_3$ on B, h_4 is an embedding on a neighborhood of $\partial P \cup {}^{(H_{i+1})}P$. Let $\bar{P}$ be a G-invariant compact smooth submanifold of P obtained from Lemma 3.1 for the neighborhood A of $\partial P \cup {}^{(H_{i+1})}P$. Applying Theorem 7.2 to $h_4 \mid \bar{P}\colon \bar{P} \to Q$, we obtain a smooth G-map

$$h_5\colon \bar{P} \to Q$$

such that

(1) h_5 is orthogonally isovariant,
(2) $h_5 = h_4 \mid \bar{P}$ on a neighborhood of $\partial \bar{P}$ in $\bar{P}$,
(3) $h_5 \simeq_G h_4 \mid \bar{P}$ rel the neighborhood.

Define

$$g_{i+1}\colon P \to Q$$

as $g_{i+1}=h_4$ on $P-\operatorname{Int}\bar{P}$, and $g_{i+1}=h_5$ on $\bar{P}$. Then g_{i+1} has the desired property. Thus we see that $A(i)$ implies $A(i+1)$, and this completes the proof of Corollary 7.4. Q.E.D.

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