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SELF MINI-INJECTIVE RINGS

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Let R be a ring. We have studied rings whose projective modules have the extending property of simple modules in [3] and [5]. In this note, we shall further study those rings when R is an artinian ring and give some relations between those rings and mini-injectivity (see §1).

If R is a QF-ring [8], every projective has the extending property of direct decompositions of the socle [3]. In order to characterize artinian rings with above property, we have defined the condition (** 2) in [3]. We shall introduce new concepts: (weakly) mini-injective module and (weakly) uni-injective module. We shall show, for a left and right artinian ring R, that R is a QF-ring if and only if R is mini-injective as a both left and right R-module and if and only if R is uni-injective as a right R-module and right QF-2. When R is right artinian, we shall show that the above extending property for right R-projectives is valid if and only if R is right QF-2 and right R mini-injective.

We can consider the dual property, namely the lifting property of simple modules. However, when R is right artinian, every R-projective P has the lifting property of simple modules and further the lifting property of direct decompositions of $P/J(P)$ [5], where $J(P)$ is the Jacobson radical of P.

1 Definitions

Throughout this note, R is a ring with identity and every module M is a unitary right R-module. We shall denote the *Jacobson radical*, an *injective envelope* and the *socle* of M by $\mathrm{J}(M)$, $\mathrm{E}(M)$ and $\mathrm{S}(M)$, respectively. If for any simple (resp. uniform) submodule A of M there exists a (completely indecomposable) direct summand M_1 of M such that $\mathrm{S}(M_1)=A$ (resp. A is an essential submodule of M_1), then we say that M has the *extending property of simple modules* (resp. *uniform submodules*). Futhermore, if for any direct decomposition of $\mathrm{S}(M)$: $\mathrm{S}(M)=\sum_I \oplus A_\alpha$ (resp. any independent set of uniform submodules B_α such that $\sum_I \oplus B_\alpha$ is essential in M) there exists a direct decomposition $M=\sum_I \oplus M_\alpha$ of M such that $\mathrm{S}(M_\alpha)=A_\alpha$ (resp. B_α is an essential submodule in M_α) for all $\alpha \in I$, then we say that M has the *extending property of direct decompositions* *of* S(M) (resp. *direct sum of uniform submodules*).

In this note, we consider only artinian rings and so from now on we understand that a ring R is always right artinian. We note that most results in this note are true for left and right perfect rings. Let

$$R=\sum_{i=1}^{n}\sum_{j=1}^{p(i)}\oplus e_{ij}R$$

be the standard decomposition, namely the e_{ij} are primitive idempotents and $e_{ij}R\approx e_{i1}R$, $e_{j1}R\not\approx e_{i1}R$ if $i\neq j$. If S($e_{i1}R$) is simple for each i, then we say R is *right QF*-2 [3] and [9]. If E(R) is right R-pojective, R is called a *right QF*-3 ring [7] and [9]. Finally if $e_{i1}R$ is a serial module for each i, we call R a *right generalized uniserial ring* [8] and [5].

First we shall generalize the concept of injectivity. Let M be an R-module and I a right ideal in R. We take an R-homomorphism f of I to M. Put $M_1=\operatorname{im} f$ and consider a diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"] & R \arrow[ddl, dashed, bend left, "h"] \\ & M_1 \arrow[d, "i'"] \arrow[ur, dashed, "h'"] & \\ & M & \end{tikzcd} .$$

We shall introduce two conditions.

(I) *There exists* $h\in \operatorname{Hom}_R(R, M)$ *such that* $hi=f$.

(II) *There exists either* $h\in \operatorname{Hom}_R(R, M)$ or $h'\in \operatorname{Hom}_R(M, R)$ *such that* $hi=f$ *or* $if^{-1}=h'\mid M_1$ *provided* f *is an monomorphism.*

If M satisfies (I) (resp. (II)) for every minimal right ideal I in R and any f in $\operatorname{Hom}_R(I, M)$, we say R is *right* (resp. *weakly*) *mini-injective*. Similarly if M satisfies (I) (resp. (II)) for every uniform right ideal I in R and any f in $\operatorname{Hom}_R(I, M)$, then we say M is right (resp. *weakly*) *uni-injective*.

It is clear that every injective is uni-injective and uni-injective is mini-injective. The converse is not true in general (see Example 5 below). Every semi-simple module is weakly mini-injective, but not mini-injective. If R is a right QF-2 ring, every uni-injective is injective (see the proof of 7)$\to$1) in Theorem 13 below).

2 Mini-injective modules

We shall study some elementary properties of the mini-injective modules. From the definitions and the standard argument [1], we have

Proposition 1. *Let M be an R-module and $M=M_1\oplus M_2$. Then*

1) *M is mini-injective (resp. uni-injective) if and only if so is ecah M_i.*

2) *If M is weakly mini-injective (resp. weakly uni-injective), then so is each M_i.*

Theorem 2. *Let R be a right artinian ring and M an R-module. Then M is mini-injective (resp. uni-injective) if and only if any minimal (resp. uniform) right ideal I in e_iR and any f in* $\mathrm{Hom}_R(I,M)$, *f is extendable to an element in* $\mathrm{Hom}_R(e_iR, M)$, *where e_i runs through all primitive idempotents.*

Proof. "If" part. First we take a minimal right ideal I in $R=\sum_{i=1}^{m}\oplus e_iR$. Let f be in $\mathrm{Hom}_R(I, M)$ and $\pi_i\colon R\to e_iR$ projection. We may assume $I_i=\pi_i(I)\neq 0$ for $i\leq$ some t and $I_j=0$ for $j>t$. Since $\pi_1|I$ is an monomorphism, put $f_1=f(\pi_1|I)^{-1}$. Then there exists F_1 in $\mathrm{Hom}_R(e_1R, M)$ such that $F_1|I_1=f_1$ by the assumption. Put $F_j=0$ ($\in\mathrm{Hom}_R(e_jR, M)$) for $j\neq 1$ and $F=\sum F_i$. Let x be in I and $x=\sum_{i=1}^{t}\pi_i(x)$. Then $F(x)=\sum F_i\pi_1(x)=f(\pi_1|I)^{-1}\pi_1(x)=f(x)$. If I is uniform, $\bigcap_i \ker(\pi_i|I)=0$ implies that some $\pi_i|I$ is an monomorphism. Hence, we can use the same argument in this case, too.

3 Self mini-injective rings

Let R be a right artinian ring. We assume that every idempotent in this note is always primitive and we denote it by e. We put $R/J=\bar{R}$ and $\bar{e}$ means the residue class of e in $\bar{R}$, where $J=J(R)$.

First we shall study the extending property for R-projectives.

Theorem 3. *Let R be right artinian. Then*

1) *Every projective has the extending property of simple modules if and only if R is right* QF-2 *and R is weakly mini-injective as a right R-module* (cf. [3], Theorem 2).

2) *Every projective has the extending property of direct decompositions of the socle if and only if R is right* QF-2 *and mini-injective as a right R-module.*

Proof. 1) We assume that every projective has the extending property of simple modules. Then R is right QF-2. Let $R=\sum_{i=1}^{m}\oplus e_iR$ with e_i primitive and let $\pi_i\colon R\to e_iR$ be the projection. We take two minimal right ideals K_1 and K_2 and assume $f\colon K_1\to K_2$ is an isomorphism. We assume $K_i\subseteq\sum_{j=1}^{t_i}\oplus I_{i,j(i)}$, where $I_{ij(i)}=\pi_{j(i)}(K_i)\neq 0$. Since $I_{ij(i)}\approx K_1$ for all i, j, from [6], Corollary 8 we can find minimal one among $e_{j(i)}R$ with respect to the order $<^*$ in [6], say $e_{j(i)}R=e_1R$ and $i=1$. We consider $p_k=\pi_k f\pi_1^{-1}\colon I_{11}\to e_kR$. If $k\notin\{2(1), 2(2), \cdots, 2(t_2)\}$, $p_k=0$. Hence, since e_1R is minimal, there exists $F_k\in\mathrm{Hom}_R(e_1R, e_kR)$

such that $F_k | I_{11} = p_k$ by [6]. Corollary 8. Put $h = (\sum_{k=1}^{m} F_k)\pi_1 \in \mathrm{Hom}_R(R, R)$. Then $h | K_1 = ((\sum_k F_k)\pi_1) | K_1 = (\sum_k \pi_k f) | K_1 = f$. If the minimal one above $e_{j(i)}R$ is equal to $e_{j(2)}R$, we take f^{-1} in the above. Then we can find $h' \in \mathrm{Hom}_R(R, R)$ such that $h' | K_2 = f^{-1}$. The converse is clear from [6], Corollary 8.

2) We can similarly show it by making use of [6], Corollary 20 instead of Corollary 8.

Let $\mathrm{S}(R) = \sum_{i=1}^{k} \oplus S_i$ and the S_i simple. If $S_1 \not\approx S_j$ for any $j \neq 1$, S_1 is called *isolated*. From the similar argument to the above we have

Theorem 3′ *Let R be as above. Then R has the extending property of direct decompositions of the socle* (*resp. of simple modules*) *as a right R-module if and only if R is right* QF-2 *and* (I) (*resp.* (II)) *is satisfied for non-isolated minimal right ideals.*

For the uni-injective case, we have

Theorem 4. *Let R be right artinian. Then*

1) *Every projective has the extending property of uniform submodules if and only if R is right* QF-2 *and weakly uni-injective as a right R-module.*

2) *Every projective has the extending property of direct sums of uniform submodules if and only if R is right* QF-2 *and uni-injective as a right R-module.*

Proof. First we note that every uniform submodule in a projective module P is finitely generated. Let $P = \sum_I \oplus P_\alpha$ and $P_\alpha \approx e_{i(\alpha)}R$ and U a uniform submodule. Let $x \neq 0$ be in U. Then $x = \sum_{i=1}^{n} p_{\alpha_i}$; $p_{\alpha_i} \in P_{\alpha_i}$. Hence, $U \cap \sum_{i=1}^{n} \oplus P_{\alpha_i} \neq 0$ and so $U \cap \sum_{I-\{\alpha_i\}} \oplus P_\beta = 0$. Accordingly, U is isomorphic to a submodule of $\sum_{i=1}^{n} \oplus P_{\alpha_i}$. Furthermore, $U \approx \pi_i(U)$ for some i, where $\pi_i : P \to P_{\alpha_i}$ is the projection. Therefore, we can apply the same argument given in the proof of Theorem 3 by making use of [6], Theorems 10 and 22.

Next we shall study self (resp. weakly) mini-injective rings.

Theorem 5. *Let R be right artinian and mini-injective as a right R-module. Then*

1) *If $e_1R \not\approx e_2R$, no minimal submodule in e_1R is isomorphic to any minimal one in e_2R.*

2) $\mathrm{S}(e_1R) = e_1J^k$ *and every minimal submodule in e_1R is isomorphic to one another.*

3) $\mathrm{r}(J) \supseteq \mathrm{l}(J)$ *and* $J = \mathrm{Z}(R)$.

Where $J = \mathrm{J}(R)$, the e_i are primitive idempotents, $\mathrm{r}(J) = \{x \in R \mid Jx = 0\}$ *and*

$1(J)=\{x\in R\,|\,xJ=0\}$. $Z(R)$ *is the right singular ideal of* R.

Proof. Let $e_1R\not\approx e_2R$ and I_i a minimal right ideal in e_iR for $i=1, 2$. If $I_1\approx I_2$, there exist y in $e_2Re_1=e_2Je_1$ and z in e_1Je_2 such that $I_2=yI_1$, $I_1=zI_2$ by the assumption. Hence, $I_1=zyI_1$ and $zy\in J$, which is a contradiction. Therefore, $\{I_i\}_{i=1}^{n}$ is the representative set of minimal R-modules. Let S be a minimal right ideal in e_1R. Then S must be isomorphic to I_1 from the above. Let $e_1J^k\neq 0$ and $e_1J^{k+1}=0$. We take a minimal right ideal K in e_1J^k. Since $K\approx S$, there exists x in e_1Re_1 such that $S=xK\subseteq e_1J^k$. Hence, $S(e_1R)=e_1J^k$. We have obtained 1) and 2).

3) We take I_1 in $S(e_1R)$. Let $I_1=xR$ and $x\in e_1R$. Now $Jx\subseteq\sum_{i=1}^{m}e_iJx=\sum_{i=1}^{m}e_iJe_1x$. If $e_jR\not\approx e_1R$, $e_jJe_1xR=0$ by 1). If $e_jR\approx e_1R$, we take z in e_1Re_j which induces an isomorphism of e_jR to e_1R. Then $ze_jJe_1xR\subseteq e_1Je_1xR=0$ by 2). Hence, $e_jJe_1xR=0$. Therefore, $Jx=0$ and $1(J)=S(R_R)\subseteq r(J)$. Furthermore, $Z(R)=\{x\subseteq R\,|\,x1(J)=0\}\supseteq J$ and so $Z(J)=J$, since every ideal properly containing J contains a projective submodule.

Proposition 6. *Let R be a right artinian ring. Then R is mini-injective as a right R-module if and only if R is weakly mini-injective as a right R-module and* $1(J)\subseteq r(J)$.

Proof. "If" part. We assume $I_1\approx I_2$ for minimal right ideals I_i in e_iR. Then there exists an element x in either e_1Re_2 or e_2Re_1 which induces an isomorpism between I_1 and I_2. Hence, $x\notin J$ by the assumption. Therefore, x induces an isomorphism between e_1R and e_2R. Accordingly, R is mini-injective for $\mathrm{Hom}_R(e_iR, e_jR)=e_jRe_i$. The converse is clear from Theorem 5.

Similarly to the above

Proposition 7. *Let R be right artinian. Then R is uni-injective as a right R-module if and only if R is weakly uni-injective as a right R-module and* $1(J)\subseteq r(J)$.

Proof. Since uni-injective is mini-injective, the "Only if" part is clear from Theorem 5. Let U_i be a uniform submodule of e_iR and $f\colon U_1\to U_2$ a homomorphism. If $\ker f\neq 0$, f is extendable to an element in $\mathrm{Hom}_R(e_1R, e_2P)$ by the assumption. We assume $\ker f=0$. We know from Proposition 6 that R is mini-injective as a right R-module. Hence, $e_1R\approx e_2R$ by Theorem 5. Therefore, f and f^{-1} are extendable to elements in $\mathrm{Hom}_R(e_1R, e_2R)$ and $\mathrm{Hom}_R(e_2R, e_1R)$, respectively. Thus R is uni-injective by Theorem 2.

The author can not find an artinian ring which is self mini-injective but not self uni-injective

We consider algebras over a field.

Proposition 8. *Let K be a field and R a K-algebra with finite dimension. If R is mini-injective as a right R-module, then R is right* QF-2.

Proof. Let I_1 be a minimal right ideal in e_1R, where e_1 is primitive. We assume $I_1 \approx \overline{e_2'R}$. Since $I_1 \subseteq e_1J^{k_1}$ and $e_1J^{k_1+1}=0$, each element in $\overline{e_1Re_1}$ gives an element in $\mathrm{Hom}_R(I_1, e_1R)$ $(=\mathrm{Hom}_R(I_1, \mathrm{S}(e_1R))$ via the left multiplication and $\mathrm{Hom}_R(I_1, e_1R)=\overline{e_1Re_1}$ as a K-module by the assumption. Put $I_1=x\bar{e}_2'R$ and consider an isomorphism f of I_1 by setting $f(x)=xa$ for $a \in \overline{e_2'Re_2'}$. Then f is extendable to an element in $\mathrm{Hom}_R(e_1R, e_1R)$ by the assumption. Hence, $xa=bx$ for some b in $\overline{e_1Re_1}$. This relation gives us a K-monomorphism of $\overline{e_2'Re_2'}$ to $\overline{e_1Re_1}$. Hence, $[\overline{e_1Re_1}: K] \geqslant [\overline{e_2'Re_2'}: K]$. Repeating those arguments, we obtain a chain of primitive idempotents $e_1, e_2', \cdots, e_i', \cdots$ such that a minimal right ideal I_i in $e_i'R$ is isomorphic to $\overline{e_{i+1}'R}$ and $[\overline{e_i'Re_i'}: K] \geqslant [\overline{e_{i+1}'Re_{i+1}'}: K]$. We may assume $e_i'R \approx e_j'R$ for some $i<j$. Then $I_{i-1} \approx \overline{e_i'R} \approx \overline{e_j'R} \approx I_{j-1}$. Hence, $e_{i-1}'R \approx e_{j-1}'R$ by Theorem 5. Therefore, $e_1R \approx e_k'R$ for some k. Accordingly, $[\overline{e_1Re_1}: K]=[\overline{e_2'Re_2'}: K]=[\overline{e_k'Re_k'}: K]$. Hence, $\mathrm{Hom}_R(I_1, e_1R)=\overline{e_2'Re_2'}$. Let S be a minimal right ideal in e_1R. Then there exists b in $\overline{e_1Re_1}$ such that $bI_1=S$ by the assumption. However, since $\mathrm{Hom}_R(I_1, e_1R)=\overline{e_2'Re_2'}$ as above, there exists a in $\overline{e_2'Re_2'}$ such that $bx=xa$. Hence, $S=bI_1=bxR=xaR \subseteq I_1$. Therefore, $\mathrm{S}(e_1R)$ is simple.

REMARK. If $\mathrm{End}_R(\overline{eR})$ is given by the multiplication of the central elements in R for each idempotent e, Proposition 8 is valid for such artinian rings from the above proof.

Proposition 9. *Let R be a K-algebra as above. We assume $[\overline{eRe}: K]=[\overline{e'Re'}: K]$ for any primitive idempotents e and e'. Then every projective has the extending property of simple modules (resp. direct decompositions of the socle) if and only if R is right* QF-2 *and if $\mathrm{S}(e_1R) \approx \mathrm{S}(e_2R)$, either $e_2R\mathrm{S}(e_1R)=\mathrm{S}(e_2R)$ or $e_1R\mathrm{S}(e_2R)=\mathrm{S}(e_1R)$ (resp. $e_2R\mathrm{S}(e_1R)=\mathrm{S}(e_2R)$), where the e_i are primitive.*

Proof. "If" part. Since R is right QF-2, $e_1Je_1\mathrm{S}(e_1R)=0$. Hence, $I_1=\mathrm{S}(e_1R)$ is a left $\overline{e_1Re_1}$-module. We assume $I_1 \approx \overline{e_2R}$ and so $\mathrm{End}_R(I_1) \approx \overline{e_2Re_2}$. Since I_1 is a left $\overline{e_1Re_1}$-module, each element x in $\overline{e_1Re_1}$ induces an element in $\mathrm{End}_R(I_1)$ by the left multiplication. Now, $[\overline{e_1Re_1}: K]=[\overline{e_2Re_2}: K]$ from the assumption. Hence, we may assume $\mathrm{End}_R(I_1)=\overline{e_1Re_1}$. Let $I_3=\mathrm{S}(e_3R)$ and $I_3 \approx I_1$. If $e_3RI_1=I_3$, $yI_1=I_3$ for some $y \in e_2Re_1$. Then $g: I_1 \to I_3$ given by setting $g(x)=yx$; $x \in I_1$ is an isomorphism. Let f be any isomorphism of I_1 to I_3. Then $g^{-1}f \in \mathrm{End}_R(I_1)=\overline{e_1Re_1}$. Hence, $f(x)=yzx$ for some z in e_1Re_1. Therefore, f is extendable to an element in $\mathrm{Hom}_R(e_1R, e_3R)$. Thus, every projective has the extending property of simple modules (resp. direct decompositions of the socle) by [3], Theorem 2 (resp. [6], Corollary 20).

Since the extending property is preserved by Morita equivalence, if R/J is a simple ring, we may assume R is a local ring.

Proposition 10. *Let R be a right artinian and local ring. Then every projective has the extending property of uniform submodules if and only if R is a* QF-*ring.*

Proof. If R has the extending property, R is right QF-2. Since every projective is a direct sum of copies of R, R is a QF-ring by [6], Theorem 10.

Proposition 11. *Let R be a right artinian and local ring. We assume that every monomorphism of R/J into itself as a field is an isomorphism. Then every projective has the extending property of simple modules (and hence of direct decompositions of the socle) if and only if R is right* QF-2.

Proof. "If" part. Since R is local QF-2, $\mathrm{S}(R)=I$ is a unique minimal right ideal and a left ideal in R. Let $I=xR$. Then since $JI=0$, for any element a in $\bar{R}$, there exists b in $\bar{R}$ such that $ax=xb$. Hence, the correspondence $\sigma: a \to b$ gives us a monomorphism of $\bar{R}$ into $\bar{R}$. Therefore, σ is onto by the assumption, which means that R is right mini-injective. Accordingly, every projective has the extending property of direct decompositions of the socle by Theorem 3.

Finally we shall give an additional result to [5].

Proposition 12. *Let R be a right artinian, generalized uniserial and right* QF-3 *ring. Then every R-projective module has the extending property of simple modules.*

Proof. Let $\mathrm{S}(R)=\sum_{i=1}^{m} \oplus S_i$ and $S_i=\mathrm{S}(e_iR)$. We assume $S_1 \approx S_2 \approx \cdots \approx S_i$ and $S_j \not\approx S_1$ for $j>i$. Since R is right QF-3, $\mathrm{E}(S_1)$ is isomorphic to some e_kR. Hence, e_pR is isomorphic to some submodule of e_kR for $p \leqslant i$. Now e_kR is serial and injective by the assumption. Hence, each submodule of e_kR is a character submodule and $\mathrm{End}_R(S_k)$ is extendable to $\mathrm{End}_R(e_kR)$. Therefore, every R-projective has the extending property of simple modules by [3], Theorem 2.

4 QF-rings

We shall give some characterizations of QF-rings in terms of extending property of projectives.

Theorem 13. *Let R be left and right artinian. Then the following condi-*

tions are equivalent.

1) *R is a* QF-*ring.*

2) *Every right (and left) R-projective has the extending property of direct decompositions of the socle.*

3) *Every right R-projective has the extending property of direct decompositions of the socle and* $\mathrm{r}(J)\subseteq \mathrm{l}(J)$.

4) *Every right R-injective E has the lifting property of direct decompositions of $E/J(E)$ and R is a right* QF-2 (see [4]).

5) *R is right and left* QF-2 *and mini-injective as a right R-module.*

6) *R is mini-injective as a left and right R-module.*

7) *R is uni-injective as a right R-module and right* QF-2.

Proof. 1)→2)~7), 2)→1) and 5)→1). They are clear from Theorems 3 and 5, [2], Theorem 3, [3], Theorem 2 and [8].

3)→1). It is sufficient to show that R is left QF-2 ,since R is right QF-2 and R-mini-injective by Theorem 3. We take a unique minimal right ideal x_1R in e_1R. We may assume $x_1\in e_1Re_2'$ as the proof of Proposition 8. Since $\mathrm{r}(J)\supseteq \mathrm{l}(J)$, $Jx_1=0$. Hence, Rx_1 is semi-simple. On the other hand, since $Rx_1=Re_1x_1$, Rx_1 is a minimal left ideal in Re_2'. Let Rx_2 be another minimal one in Re_2' and $x_2\in e_3'Re_2'$. Then $\mathrm{S}(e_1R)=x_1R\approx\overline{e_2'R}\approx x_2R=\mathrm{S}(e_3'R)$ since $\mathrm{r}(J)\subseteq \mathrm{l}(J)$ by the assumption. Hence, $e_1R\approx e_3'R$ by Theorem 5. Noting that x_1R is minimal, we obtain an isomorphism $f\colon x_1R\to x_2R$ with $f(x_1)=x_2$. f is extendable to an element $y\in\mathrm{Hom}_R(e_1R, e_3'R)$ by [6], Corollary 20. Hence, $x_2=yx_1$ and so $Rx_2=Rx_1$. The above correspondence $e_1\to e_2'$ gives a permutation of the set $\{e_{i1}\}_{i}^{n=1}$ by Theorem 5. Hence, R is left QF-2.

4)→1). We know from [2], Theorem 3 that there exists the representative set $\{e_{i1}R/e_{i1}A_{ij}\}_{i=1}^{p}{}_{j=1}^{\kappa(i)}$ of indecomposable injectives. Since R is artinian, $\kappa(i)=1$ for all i by [4], Theorem 2. $e_{i1}R$ is uniform by the assumption. Hence, $\mathrm{E}(e_{i1}R)\approx e_{j1}R/e_{j1}A_{j1}$ for some j. We consider a diagram, where $e_k=e_{k1}$, $A_k=A_{k1}$ and φ is the natural epimorphism:

$$\begin{tikzcd} 0 \arrow[r] & e_iR \arrow[r] \arrow[d, "\varphi"] & \mathrm{E}(e_iR)\approx e_jR/e_jA_j \arrow[dl, dashed, "h"] \\ & e_iR/e_iA_i & \end{tikzcd}$$

Since e_iR/e_iA_i is injective, we obtain an epimorphism $h\colon e_jR/e_jA_j\to e_iR/e_iA_i$. Hence, $i=j$ and $e_iA_i=0$. Since $\kappa(i)=1$ for all $i, p=n$. Therefore, $R=\sum_{i=1}^{n}\sum_{j=1}^{\rho(i)}\oplus e_{ij}R$ is self injective as a right R-module.

6)→1). We assume that R is self mini-injective. Let xR be a minimal right ideal in e_1R, where e_1 is primitive. Then $xR=xe_2'R$ and $x\in e_1Re_2'$. Since $Jx=0$ by Theorem 5, Rx is minimal in Re_2' as above. Therefore, for any element b

in $\overline{e_1Re_1}$ there exists a in $\overline{e_2'Re_2'}$ such that $bx=xa$ as the proof of Proposition 8 for R is left mini-injective. Again using the same argument, we know $xR=S(e_1R)$. Hence, R is QF-2. Therefore, R is a QF-ring by Theorem 5 and [8].

7)$\to$1). We shall show that R is self-injective as a right R-module. We can use the standard argument [1]. Let I be a right ideal in R and $f\in \mathrm{Hom}_R(I, R)$. We can find a maximal one among the set of extensions of f by Zorn's Lemma, say $(I_0, f_0\colon I_0\to R)$. We assume $I_0\neq R$. Then there exists a primitive idempotent e such that $e\notin I_0$. Put $K=eR\cap I_0$ and $I_1=I_0+eR$. We take an extension f_1 of $f_0|K$ from the assumption. We put $g(x)=f_0(x_1)+f_1(er)$, where $x_1\in I_0$ and $r\in R$. Then $g\in \mathrm{Hom}_R(I_1, R)$, which contradicts the assumption of I_0. Hence, $I_0=R$ and R is self-injective.

Theorem 14. *Let R be a K-algebra with $[R\colon K]<\infty$. Then the following conditions are equivalent.*

1) *R is a* QF*-ring.*
2) *R is mini-injective as a right R-module and* $\mathrm{r}(J)=\mathrm{l}(J)$.
3) *R is uni-injective as a right R-module.*

Proof. It is clear from Proposition 8 and Theorem 13.

5 Examples

Let K be a field.

1. Put

$$R=\begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{pmatrix}.$$

Then $\mathrm{Hom}_R(S(e_{22}R), S(e_{33}R))$ is not extendable to $\mathrm{Hom}_R(e_{22}R, e_{33}R)$. Hence, R is right generalized uniserial, but does not have the extending property of simple modules as a right R-module (cf. Proposition 12).

2. We shall give an example, where artinian and right self mini-injective rings are not right QF-2 in general. Let x be an indeterminate and Q a field. Put $L=Q(x)$ and $K=Q(x^2)$. Then we have an isomorphism σ of L onto K and $[L\colon K]=2$. Let $R=L1\oplus Lu$ be a left vector space over L. We put $(Lu)^2=0$ and $ul=\sigma(l)u$ for $l\in L$. Then R is a ring and $[R\colon L]=2$ as a left L-module and $[R\colon L]=3$ as a right L-module. Hence, R is a left and right artinian ring. $J=Lu$ contains minimal right ideals Ku and xKu. Let I be a minimal right ideal in J. Then $I=aL$; $a=lu$ and $\mathrm{End}_R(Ku)=K$. Therefore, R is self right mini-injective (and uni-injective). We note that $\mathrm{End}_R(J)$ as a left R-module $\supsetneqq$ {the right multiplications of R} and R is left QF-2. Furthermore, R satisfies the

conditions in Theorem 5 as a left R-module. However, R is not left mini-injective (cf. Theorems 13 and 14).

In case of QF-rings, right artinian and right self-injective rings satisfy the same conditions on the left side. However, this fact is not true for self mini-injective rings from this example.

3. Let K and L be as in Example 2. Put

$$R=\begin{pmatrix} L & L \\ 0 & L \end{pmatrix}.$$

Then R is right weakly mini-injective. However R is not right QF-2 and hence not right mini-injective. $e_{22}R$ is weakly uni-injective, but not mini-injective. (cf. Proposition 8).

4. Put

$$R=\left\{\begin{pmatrix} a & b & c \\ o & d & e \\ o & o & a \end{pmatrix}\middle|\, a\sim e\in K\right\}.$$

Then R is weakly mini-injective but not weakly uni-injective for $f\colon e_{11}R\to e_{11}J^2$ is not extendable.

5. Put

$$R=\begin{pmatrix} K & uK+vK & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

and $e_{12}(uk_1+vk_2)e_{23}k_3=e_{13}(k_1k_3+k_2k_3)$ for $k_i\in K$. Then $e_{11}R$ is mini-injective. On the other hand, $e_{11}R$ contains two isomorphic uniform modules $(0, uK, K)$, $(0, vK, K)$. The above isomorphism is not extendable to an element in $\mathrm{Hom}_R(e_{11}R, e_{11}R)$. Hence $e_{11}R$ is not uni-injectcetve.

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