surfaces which can be embedded in $L(2\alpha, \beta)$, *is* 3, *if and only if* $\alpha=4\mu_1\mu_2-1$ *and* $\beta=4\mu_1\mu_2-2\mu_1-1$.

3. System of curves on F_3. From now on we restrict ourselves to the case that $\lambda=3$. Let a^*, b^*, c^* and e_{c^*} be oriented simple closed curves and an arc on F_3, as shown in Fig. 3.1. Then we have $\{e_1a^*e_1^{-1}\}=z_1z_2$, $\{e_2b^*e_2^{-1}\}=z_2z_3$ and $\{e_{c^*}c^*e_{c^*}^{-1}\}=z_1z_2z_3$ in $\pi_1(F_3, p)$, where $\{c\}$ denotes the element of $\pi_1(F_3, p)$ represented by a p-based loop c. Note that $N(a^*\cup b^*)$ is an orientable surface of genus 1 and $F-\mathring{N}(a^*\cup b^*)$ is a Möbius band having c^* as a center-line.

For every essential simple closed curve c on F_3, there exists a homeomorphism ρ from F_3 onto itself which takes c onto either c^*, c_1, a^*, $\partial N(c^*)$ or $\partial N(c_1)$. We say that c is of type I, II, III, IV or V, according as $\rho(c)$ coincides with c^*, c_1, a^*, $\partial N(c^*)$ or $\partial N(c_1)$.

Since an autohomeomorphism of $N(a^*\cup b^*)$ can be extended to F_3, there exists a homomorphism from the homeotopy group $\mathcal{H}(N(a^*\cup b^*))$ of $N(a^*\cup b^*)$ into $\mathcal{H}(F_3)$. According to [2], the homomorphism is an isomorphism. More presisely,

Proposition 3.1. *Let* $GL(2, Z)$ *be the group of all invertible matices over* Z. *Then* $GL(2, Z)$ *is isomorphic to* $\mathcal{H}(F_3)$ *by an isomorphism which maps each matrix* $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ *to an isotopy class of an autohomeomorphism* ρ *such that* $\rho a^*\sim\alpha_{11}a^*+\alpha_{12}b^*$ *and* $\rho b^*\sim\alpha_{21}a^*+\alpha_{22}b^*$ *on* F_3.

It follows from the above proposition that every simple closed curve of type *I* on F_3 is ambient isotopic to c^* or $-c^*$ on F_3.

By a_1^*, b_1^*, a_2^* and b_2^*, we denote simple closed curves on T_2 such that $\pi^{-1}(a^*)=a_1^*\cup b_2^*$, $\pi^{-1}(b^*)=b_1^*\cup a_2^*$, $\pi^\#(\{\tilde{e}_1a_1^*\tilde{e}_1^{-1}\})=z_1z_2$ and $\pi^\#(\{\tilde{e}_2b_1^*\tilde{e}_2^{-1}\})=z_2z_3$ [Fig. 3.1].

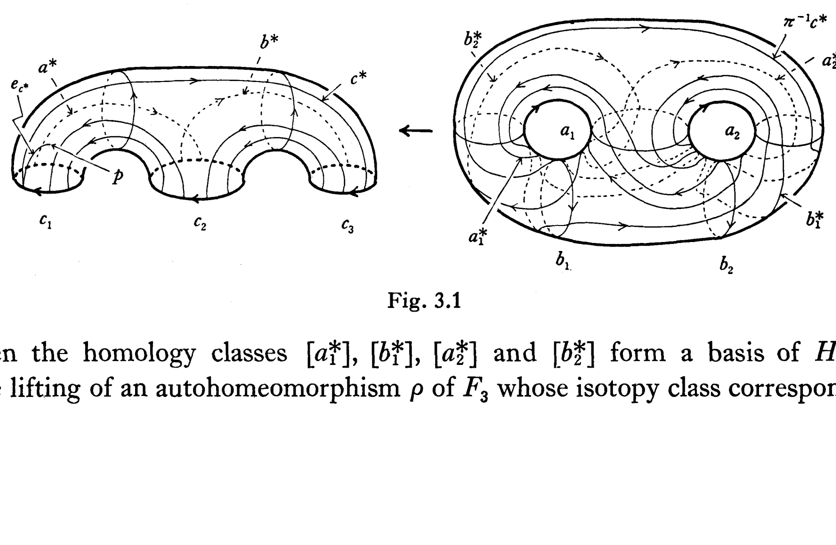

Fig. 3.1

Then the homology classes $[a_1^*]$, $[b_1^*]$, $[a_2^*]$ and $[b_2^*]$ form a basis of $H_1(T_2)$. The lifting of an autohomeomorphism ρ of F_3 whose isotopy class corresponds to

$\hat{a}_{\lambda-1}$ with coefficients $-I_1(2\alpha, \beta), \cdots, -I_{\lambda-1}(2\alpha, \beta)$. Then we can show that each a'_μ, $1 \leq \mu \leq \lambda-1$, bounds a disk in the result Q_2 of a Dehn surgery on L_2.

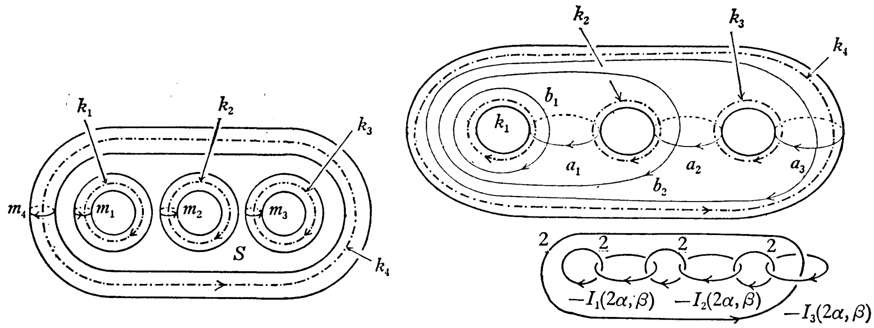

Fig. 2.3

Since $Q_2 - M(F_\lambda)$ is a solid torus of genus $\lambda-1$, Q_2 is homeomorphic to the union of $M(F_\lambda)$ and $V_{\lambda-1}$ such that $\partial D_\mu = a'_\mu$, $1 \leq \mu \leq \lambda-1$. From the definition of the sequence $\{I_\mu(2\alpha, \beta)\}$ and Lemma 2.2, it follows that Q_2 is homeomorphic to $L(2\alpha, \beta)$. The proof is completed.

Corollary 2.3. *If $\lambda=3$, there exists a homeomorphism ψ from $L(2\alpha, \beta)$ onto a Seifert fiber space such that each ψc_1, ψc_2 and ψc_3 is a fiber.*

Proof. Let L, L' and L'' be links with coefficients in S^3, as shown in Fig. 2.4.

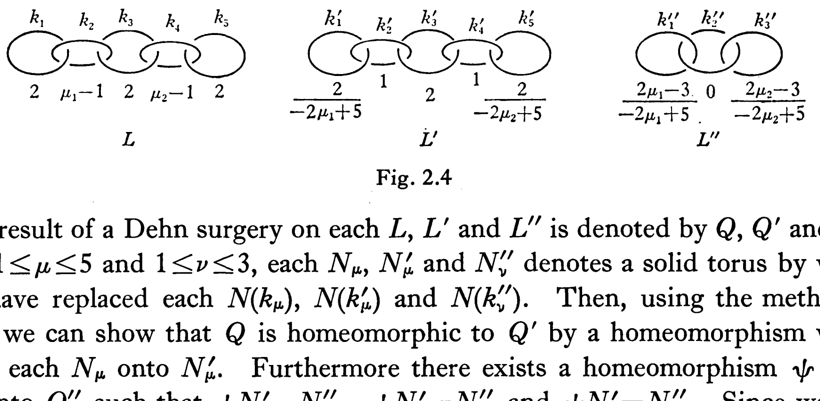

Fig. 2.4

The result of a Dehn surgery on each L, L' and L'' is denoted by Q, Q' and Q''. For $1 \leq \mu \leq 5$ and $1 \leq \nu \leq 3$, each N_μ, N'_μ and N''_ν denotes a solid torus by which we have replaced each $N(k_\mu)$, $N(k'_\mu)$ and $N(k''_\nu)$. Then, using the method in [15], we can show that Q is homeomorphic to Q' by a homeomorphism which takes each N_μ onto N'_μ. Furthermore there exists a homeomorphism ψ from Q' onto Q'' such that $\psi N'_1 = N''_1$. $\psi N'_3 = N''_2$ and $\psi N'_5 = N''_3$. Since we may consider Q'' as a Seifert fiber space having a core of each N''_μ as a fiber, the proof is completed.

Let $\mu_1 = I_1(2\alpha, \beta)+1$ and $\mu_2 = I_2(2\alpha, \beta)+1$. Then, since $N(2\alpha, \beta)$ is the minimum number of genus of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$, by [3], we have

Proposition 2.4. *The minimum number λ of the genus of non-orientable*

Theorem 2.1.[†] *Let $\lambda=N(2\alpha, \beta)$ and let $a'_1, \cdots, a'_{\lambda-1}$ be mutually disjoint simple closed curves on $T_{\lambda-1}$ with the following properties:*

1) $a'_\mu \cap (\bigcup_{\nu=1}^{\lambda-1} b_\nu \cup d_\nu) = a'_\mu \cap b_\mu = a'_\mu \cap d_\mu$.

2) *If* $x'_\mu = d_\mu a'_\mu d_\mu^{-1}$, *then* $x'_\mu = x_\mu y_\mu^{-l_\mu(2\alpha,\beta)}$.

Let $V_{\lambda-1}$ be a solid torus of genus $\lambda-1$ with meridian disks $D_1, \cdots, D_{\lambda-1}$. Then the union of $M(F_\lambda)$ and $V_{\lambda-1}$ such that $M(F_\lambda) \cap V_{\lambda-1}=T_{\lambda-1}=\partial V_{\lambda-1}$ and $\partial D_\mu = a'_\mu$, $1\leq\mu\leq\lambda-1$, is homeomorphic to $L(2\alpha, \beta)$.

Before we state the proof, we summarize notations about a surgery on links in the 3-sphere S^3[15]. A *link L with surgery coefficients* is a finite, disjoint collection of oriented simple closed curves $k_1, \cdots, k_\nu$ in S^3 with ratio γ_μ/δ_μ associated with each component k_μ. Let l_μ and m_μ be a longitude and a meridian of $N(k_\mu)$; that is, $l_\mu \sim k_\mu$ in $N(k_\mu)$, $l_\mu \sim 0$ in $S^3-\mathring{N}(k_\mu)$ and the linking number of m_μ with k_μ is 1. Let Q be the 3-manifold obtained by replacing each $N(k_\mu)$ by a solid torus N_μ with a meridian m'_μ, so that $m'_\mu \sim \gamma_\mu m_\mu + \delta_\mu l_\mu$ on $\partial N(k_\mu)$. Then we call Q the *result of a Dehn surgery on L.*

The following lemma is proved in [6].

Lemma 2.2. *Let $\gamma_1, \cdots, \gamma_\nu$ be integers and let L_0 be a link with surgery coefficients as shown in Fig. 2.2. Then the result of a Dehn surgery on L_0 is homeomorphic to $L(\gamma, \delta)$, where*

$$\frac{\gamma}{\delta} = \gamma_\nu - \cfrac{1}{\gamma_{\nu-1} - \cfrac{1}{\ddots \, \gamma_2 - \cfrac{1}{\gamma_1}}}\,.$$

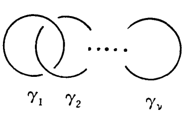

Fig. 2.2

Proof of Theorem 2.1. Let L_1 be a trivial link with the components $k_1, \cdots, k_\lambda$ such that the coefficient associated with each k_μ is 2. Then, if we perform a Dehn surgery on L_1, a longitude l_μ of each $N(k_\mu)$ bounds a Möbius band M_μ in a solid torus N_μ by which we have replaced $N(k_\mu)$. In $S^3-\mathring{N}(k_1\cup\cdots\cup k_\lambda)$, there exists a λ-punctured sphere S such that $\partial S=l_1\cup\cdots\cup l_\lambda$.

By Q_1 we denote the result of a Dehn surgery on L_1. Assume that $M(F_\lambda)$ is embedded in Q_1 so that $F_\lambda=S\cup M_1\cup\cdots\cup M_\lambda$, $M(F_\lambda)=N(S)\cup N_1\cup\cdots\cup N_\lambda$, c_μ is a centerline of M_μ and $2c_\mu \sim l_\mu$ in N_μ, $1\leq\mu\leq\lambda$. Then $V=Q_1-\mathring{M}(F_\lambda)$ is a solid torus of genus $\lambda-1$.

For $1\leq\mu\leq\lambda-1$, we take oriented simple closed curves $\hat{a}_1, \cdots, \hat{a}_{\lambda-1}$ in $\mathring{V}$ which is parallel to a_μ. Let L_2 be a link obtained from L_2 by adding $\hat{a}_1, \cdots,$

† J.S. Birman and J.H, Rubinstein have obtained independently the essentially same result as Theorem 2.1, using a different method.

itself given by $(\exp i\Theta, t) \to (\exp i(\Theta + \pi(t+1)), t)$ induces a homeomorphism τ_μ of $N(\tilde{c}_\mu)$, fixed on its boundary. Then $\tau_1 \cup \cdots \cup \tau_\lambda$ can be extended to a homeomorphism τ of $T_{\lambda-1}$ so that $\tau \mid T_{\lambda-1} - \mathring{N}(\tilde{c} \cup \cdots \cup \tilde{c}_\lambda)$ is the identity. We choose orientations so that $\tau(a_\mu) \sim a_\mu + b_\mu$ on $T_{\lambda-1}$. Clearly $\tau \cdot \iota$ is an orientation reversing, fixed point free involution on $T_{\lambda-1}$. If we denote the orbit space and the projection of $\tau \cdot \iota$ by F_λ and π, respectively, then $\pi: T_{\lambda-1} \to F_\lambda$ is an orientable double cover of a non-orientable surface of genus λ. Let $p = \pi \tilde{p}$ and $c_\mu = \pi \tilde{c}_\mu$. We take oriented arcs $e_1, \cdots, e_\lambda$ from p to a point in c_μ on F_λ, as in Fig. 2.1.

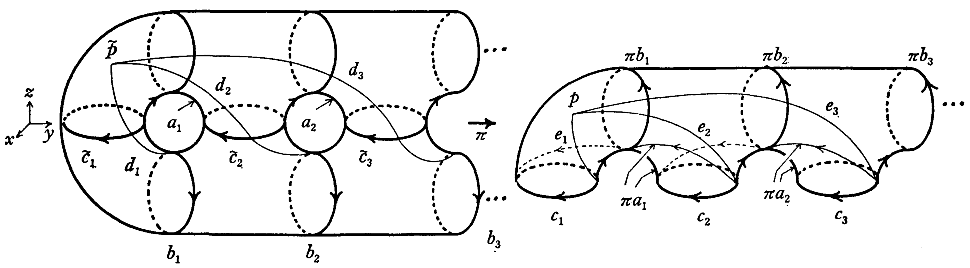

Fig. 2.1

Let z_μ, $\mu = 1, \cdots, \lambda$, be the element of $\pi_1(F_\lambda, p)$ represented by $e_\mu c_\mu e_\mu^{-1}$. By x_μ and y_μ, $\mu = 1, \cdots, \lambda - 1$, we denote the element of $\pi_1(T_{\lambda-1}, \tilde{p})$ represented by $d_\mu a_\mu d_\mu^{-1}$ and $d_\mu b_\mu d_\mu^{-1}$, respectively. Then we can show that $\pi^\sharp(x_\mu) = z_{\mu+1} z_\mu^{-1}$ and $\pi^\sharp(y_\mu) = z_\mu z_1^2 \cdots z_{\mu-1}^2 z_\mu$.

Let $(2\alpha, \beta)$ be a pair of relatively prime integers such that $\alpha\beta$ is positive and $|\beta| < 2|\alpha|$. For each pair $(2\alpha, \beta)$, we define the function $N(2\alpha, \beta)$ recursively by

$N(2, 1) = N(-2, -1) = 1$ and $N(2\alpha, \beta) = N(2\alpha', \beta') + 1$, where $\alpha' = \alpha - \beta$, $\beta' \equiv \beta \pmod{2|\alpha'|}$, $\alpha'\beta'$ is positive and $|\beta'| < 2|\alpha'|$.

By [3], we can show that $N(2\alpha, \beta)$ is the minimum number of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$. Furthermore we will define the sequence $\{I_\mu(2\alpha, \beta),\ 1 \le \mu \le N(2\alpha, \beta) - 1\}$ of integers. Since $N(2, 1) = N(-2, -1) = 1$, $\{I_\mu(2, 1)\}$ and $\{I_\mu(-2, -1)\}$ are defined to be $\emptyset$. Assume that we have defined the sequence $\{I_\mu(2\alpha', \beta')\}$. We define $\{I_\mu(2\alpha, \beta)\}$ as follows:

$$I_\mu(2\alpha, \beta) = \begin{cases} I_\mu(2\alpha', \beta') & \text{if } 1 \le \mu \le N(2\alpha, \beta) - 2\,, \\ I & \text{if } \mu = N(2\alpha, \beta) - 1\,, \end{cases}$$

where I denotes the integer such that $\beta = \beta' + 2\alpha' I$.

Note that, if we make use of the fact that $|\beta'| < 2|\alpha'|$, it follows that $I_\mu(2\alpha, \beta) \neq -1$ for each μ.

A surface F properly embedded in a 3-manifold Q is said to be *compressible* in Q, if

1) there exists a disk D such that $D\cap F=\partial D$ and ∂D is essential on F, or

2) there exists a 3-ball E in Q such that $\partial E=F$.

We say that F is incompressible in Q, if F is not compressible.

Let V and V' be a solid torus of genus 1. Let m and m' be a meridian of V and V'. Then a lens space $L(\alpha, \beta)$ of type (α, β) is the 3-manifold obtained by gluing V' and V via a homeomorphism ψ from $\partial V'$ onto ∂V such that $\psi m'\sim\alpha l+\beta m$ on ∂V.

We call the connected sum of λ-copies of a projective plane a non-orientable surface of genus λ.

R. Myers [Notices, vol. 25, 1978, A-607] and B.D. Evans [Notices, vol. 26, 1979, A-308] announced that they classifyed the fixed point free involutions on Seifert fiber spaces which have finite fundamental group. The author wish to thank the refree for bringing this to his attention.

The author would like to express his gratitude to Prof. J.S. Birman for helpful suggestions, and to Prof. F. Hosokawa and Prof. S. Suzuki for valuable discussions during the revision.

2. One-sided Heegaard splitting of $L(2\alpha, \beta)$. Let $(2\alpha, \beta)$ be a pair of integers such that $\alpha\beta$ is positive and $|\beta|<2|\alpha|$. According to [3], each $L(2\alpha, \beta)$ contains a non-orientable surface. Let λ be the minimum number of genus of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$. By F_λ we denote a non-orientable surface of genus λ embedded in $L(2\alpha, \beta)$. If $\lambda>2$ and F_λ is compressible, there exists a non-orientable surface of genus smaller than λ. If $\lambda=2$, F_λ is incompressible by [1], [12] and [7]. Hence F_λ is incompressible in $L(2\alpha, \beta)$. It follows from [4] that $L(2\alpha, \beta)$-$\mathring{N}(F_\lambda)$ is homeomorphic to a solid torus of genus $\lambda-1$. Thus we can construct $L(2\alpha, \beta)$ by gluing a regular neighbourhood $N(F_\lambda)$ of F_λ and a solid torus $V_{\lambda-1}$ of genus $\lambda-1$.

Let π: $T_{\lambda-1}\rightarrow F_\lambda$ be an orientable double covering of F_λ. We will consider $N(F_\lambda)$ as the mapping cylinder of π. For a subcomplex X of F_λ, we denote the mapping cylinder of $\pi|\pi^{-1}X$ by $M(X)$.

First we will give a description of F_λ, $T_{\lambda-1}$ and π. Let $T_{\lambda-1}$ be a closed orientable surface of genus $\lambda-1$ represented in R^3 in such a way that it is invariant under the reflection about the xy plane as illustrated in Fig. 2.1. By $\tilde{p}$, $a_1,\cdots,a_{\lambda-1},b_1,\cdots,b_{\lambda-1},\tilde{c}_1,\cdots,\tilde{c}_\lambda$ and $d_1,\cdots,d_{\lambda-1}$, we denote a base point, oriented simple closed curves and arcs, as in Fig. 2.1.

We define a homeomorphism ι: $T_{\lambda-1}\rightarrow T_{\lambda-1}$ by $\iota(x, y, z)=(x, y, -z)$. Suppose that each $N(\tilde{c}_\mu)$ is of the form $S^1\times[-1, 1]$ such that $\iota(x, t)=(x, -t)$, where $x\in S^1$ and $t\in[-1, 1]$. The homeomorphism of $S^1\times[-1, 1]$ onto

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ON ONE-SIDED HEEGAARD SPLITTINGS AND INVOLUTIONS ON A CLASS OF LENS SPACES

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1. Introduction. Let F be a closed non-orientable surface in the 3-manifold M such that the exterior of a regular neighbourhood of F is homeomorphic to a solid torus. Then the pair (M,F) is called a one-sided Heegaard splitting of M [13]. This technique is useful for studying 3-manifolds which are not sufficiently large, for example [1], [7], [12], [13] and [14]. In this paper, we will give the minimum one-sided Heegaard splitting of lens spaces [Theorem 2.1].

An *involution* φ *on a space* X is a homeomorphism from X onto itself such that φ^2 is the identity on X. Two involutions φ and φ' are said to be *equivalent* to each other, if there exists an autohomeomorphism ψ of X such that $\varphi=\psi\varphi'\psi^{-1}$. By [9], [10], [11] and [12], we can classify the fixed point free involutions on lens spaces $L(1, 0)$, $L(2, 1)$ and $L(4\alpha, 2\alpha-1)$ up to the equivalence. As an application of Theorem 2.1, we consider the fixed point free involutions on a certain family of lens spaces and will obtain

Theorem 5.1. *Let* μ_1 *and* μ_2 *be integers such that* $\mu_1\mu_2\neq 0$ *and* $\mu_1\mu_2\neq -2$. *Then the orbit space of a fixed point free involution on* $L(8\mu_1\mu_2-2, 4\mu_1\mu_2-2\mu_1-1)$ *is homeomorphic to a Seifert fiber space.*

In §2, we will give the minimum one-sided Heegaard splitting of $L(2\alpha, \beta)$. Using the lemmas proved in §3, we will find and invariant subspace under an involution on $L(8\mu_1\mu_2-2, 4\mu_1\mu_2-2\mu_1-1)$ [Lemma 4.1]. Finally the proof of Theorem 5.1 will be completed in §5.

Throughout this paper we work in the piecewise linear category. For a subcomplex X of a complex Y, the regular neighbourhood of X in Y will denoted by $N(X)$. The boundary, the interior and the closure of a manifold Q will be denoted by ∂Q, $\mathring{Q}$ and $\bar{Q}$, respectively.

Two submanifolds X and Y of Q are said to be *parallel,* if there exists an embedding ψ: $X\times I\to Q$ such that $\psi(X\times \{0\})=X$ and $\psi^{-}(\partial(X\times I)-X\times \{0\})=Y$, where I denotes the unit interval [1, 0].

$$\begin{pmatrix}[a_1^\#]\\ [b_1^\#]\\ [a_2^\#]\\ [b_2^\#]\end{pmatrix}=\begin{pmatrix}-1 & -\mu_1 & 0 & 1\\ 0 & 0 & -1 & -\mu_2\\ 0 & -1 & 1 & \mu_2\\ 1 & \mu_1 & 0 & 0\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix},$$

Using the above equation, we can compute the matrix associated with $\tilde{\rho}^\#$ with respect to $\{[a_1'], [b_1], [a_2'], [b_2]\}$.

4. Invariant subspace. The purpose of this section is to prove

Lemma 4.1. *Every involution of $L(2\alpha, \beta)$ is equivalent to φ which has one of the following properties*:

(1) *$\varphi F_3 \cap F_3$ consists of three curves of type* II.

(2) *$\varphi F_3 \cap F_3$ consists of a curve of type* I.

Assertion A. *Let F be an incompressible surface in $L(2\alpha, \beta)$ such that $F \cap F_3$ consists of simple closed curves. Then each component of $F \cap V_2$ is orientable.*

Proof. Suppose that $F \cap V_2$ is non-orientable. Let $\tilde{L}(2\alpha, \beta)$ denote the orientable double covering of $L(2\alpha, \beta)$. Then $\tilde{L}(2\alpha, \beta)$ can be considered as the union of two copies of V_2 and the double covering of $M(F_3)$. Hence the lifting $\tilde{F}_3$ of F_3 is orientable, but the lifting $\tilde{F}$ of F is non-orientable. Since F is isotptic to F_3 in $L(2\alpha, \beta)$ by [13], $\tilde{F}$ is isotopic to $\tilde{F}_3$ in $\tilde{L}(2\alpha, \beta)$. This contradicts the fact that $\tilde{F}_3$ is orientable.

Let φ_0 be an involution of $L(2\alpha, \beta)$. Then, by [10], we may suppose that $\varphi_0 F_3$ is transverse with respect to F_3, i.e., $M(c) \subset \varphi_0 F_3$ for each curve c in $\varphi_0 F_3 \cap F_3$. It follows from [12] that φ_0 is equivalent to φ_1 such that $\varphi_1 F_3 \cap F_3$ consists of essential simple closed curves on $\varphi_1 F_3$ and F_3.

Using Assertion A, we can divide our consideration into the following three cases:

Case 1: $\varphi_1 F_3 \cap F_3$ contains three curves of type II on $\varphi_1 F_3$.

Case 2: $\varphi_1 F_3 \cap F_3$ contains a curve of type I on $\varphi_1 F_3$.

Case 3: $\varphi_1 F_3 \cap F_3$ contains precisely one curve of type II on $\varphi_1 F_3$.

In the rest of this section we will give the proof of Lemma 4.1 for each case.

Case 1. In this case each curve of $\varphi_1 F_3 \cap F_3$ is of either type II or type V. Suppose that $\varphi_1 F_3 \cap F_3$ contains a curve of type V on $\varphi_1 F_3$. Let c be a simple closed curve of type V on $\varphi_1 F_3$ which bounds a Möbius band B on $\varphi_1 F_3$ such that $B \cap F_3$ consists of c and a centerline c' of B. Then c' is of type II on $\varphi_1 F_3$. On F_3, c is two-sided, so c is of type V. Hence c also bounds a Möbius band B' on F_3.

We now show that B' contains c'. Since c' and c are of type II and V on F_3, respectively, there exists an autohomeomorphism ρ of F_3 such that $\rho c' = c_1$

$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ induces an automorphism $\tilde{\rho}^{\#}$ such that

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1^*]\\ \tilde{\rho}^{\#}[b_1^*]\\ \tilde{\rho}^{\#}[a_2^*]\\ \tilde{\rho}^{\#}[b_2^*]\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0 & 0\\ \alpha_{21} & \alpha_{22} & 0 & 0\\ 0 & 0 & \alpha_{22} & \alpha_{21}\\ 0 & 0 & \alpha_{12} & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1^*]\\ [b_1^*]\\ [a_2^*]\\ [b_2^*]\end{pmatrix}$$

or

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1^*]\\ \tilde{\rho}^{\#}[b_1^*]\\ \tilde{\rho}^{\#}[a_2^*]\\ \tilde{\rho}^{\#}[b_2^*]\end{pmatrix}=\begin{pmatrix}0 & 0 & \alpha_{12} & \alpha_{11}\\ 0 & 0 & \alpha_{22} & \alpha_{21}\\ \alpha_{21} & \alpha_{22} & 0 & 0\\ \alpha_{11} & \alpha_{12} & 0 & 0\end{pmatrix}\begin{pmatrix}[a_1^*]\\ [b_1^*]\\ [a_2^*]\\ [b_2^*]\end{pmatrix} \text{ in } H_1(T_2).$$

We have an another basis $\{[a_1'], [b_1], [a_2'], [b_2]\}$ of $H_1(T_2)$ defined in §2. In this paper it is convenient to use the basis $\{[a_1'], [b_1], [a_2'], [b_2]\}$. We now find the matrix associated with $\tilde{\rho}^{\#}$ with respect to $\{[a_1'], [b_1], [a_2'], [b_2]\}$.

Lemma 3.2. *Let ρ be a homeomorphism from F_3 onto itself whose isotopy clas corresponds to* $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$. *Then*

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1']\\ \tilde{\rho}^{\#}[b_1]\\ \tilde{\rho}^{\#}[a_2']\\ \tilde{\rho}^{\#}[b_2]\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \mu_1\alpha_{11}-\alpha_{12}-\mu_1\alpha_{22} & \alpha_{12} & \mu_2\alpha_{12}+\mu_1\alpha_{21}\\ 0 & \alpha_{22} & 0 & -\alpha_{21}\\ \alpha_{21} & \mu_2\alpha_{12}+\mu_1\alpha_{21} & \alpha_{22} & -\mu_2\alpha_{11}-\alpha_{21}+\mu_2\alpha_{22}\\ 0 & -\alpha_{12} & 0 & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix}$$

or

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1']\\ \tilde{\rho}^{\#}[b_1]\\ \tilde{\rho}^{\#}[a_2']\\ \tilde{\rho}^{\#}[b_2]\end{pmatrix}=\begin{pmatrix}-\alpha_{11} & -\mu_1\alpha_{11}-\mu_1\alpha_{22} & -\alpha_{12} & \alpha_{11}-\mu_2\alpha_{12}+\mu_1\alpha_{21}\\ 0 & \alpha_{22} & 0 & -\alpha_{21}\\ -\alpha_{21} & \mu_2\alpha_{12}-\mu_1\alpha_{21}+\alpha_{22} & -\alpha_{22} & -\mu_2\alpha_{11}-\mu_2\alpha_{22}\\ 0 & -\alpha_{12} & 0 & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix},$$

where $\mu_1=I_1(2\alpha, \beta)+1$ *and* $\mu_2=I_2(2\alpha, \beta)+1$.

Proof. First we will find the matrix associated with the change of bases. Since $\pi^{\#}(x_1)=z_2z_1^{-1}$, $\pi^{\#}(y_1)=z_1^2$, $\pi^{\#}(x_2)=z_3z_2^{-1}$ and $\pi^{\#}(y_2)=z_2z_1^2z_2$, we can show that $z_1z_2=\pi^{\#}(y_1^{-1}x_1^{-1}y_2)$ and $z_2z_3=\pi^{\#}(x_2^{-1}y_2^{-1}x_1y_1x_1^{-1}y_1^{-1})$. Hence we have $a_1^*\sim a_1-b_1+b_2$ and $b_1^*\sim -a_2-b_2$. The covering transformation of π takes a_1^* and b_1^* onto b_2^* and a_2^*, respectively. Thus, by using the fact that $z_1^{-1}z_1z_2z_1=\pi^{\#}(x_1y_1)$ and $z_1^{-1}z_2z_3z_1=\pi^{\#}(y_1^{-2}x_1^{-1}y_2x_2x_1y_1)$, we obtain $a_2^*\sim -b_1+a_2+b_2$ and $b_2^*\sim a_1+b_1$. Since $a_1'\sim a_1-(\mu_1-1)b_1$ and $a_2'\sim a_2-(\mu_2-1)b_2$,

$$j_*\colon [S(E), S(E')]_G \to [S(E), V_m^\Lambda(E'\oplus\Lambda^{m-1})]_G$$

between G-homotopy sets. We are also interested in this transformation j_*.

In the non-equivariant case we already know some facts about j_*. Clearly $S^{dn-1}=S(\Lambda^n)$ where $d=1$ if $\Lambda=\boldsymbol{R}$, $d=2$ if $\Lambda=\boldsymbol{C}$, and $d=4$ if $\Lambda=\boldsymbol{Q}$. The map

$$j\colon S^{dn-1} = S(\Lambda^n) \to V_m^\Lambda(\Lambda^n\oplus\Lambda^{m-1}) = V_m^\Lambda(\Lambda^{m+n-1})$$

defined above induces a group homomorphism

$$j_*\colon \pi_i(S^{dn-1}) \to \pi_i(V_m^\Lambda(\Lambda^{m+n-1}))$$

between the i-th homotopy groups for an integer $i\geq 0$. We collect known results about the homomorphism j_* in the following:

Proposition 1 (See for example [2; Chapter 7]). (a) *j_* is an isomorphism in each case of the followings*:

(i) $m=1$,
(ii) $0\leq i\leq dn-2$,
(iii) $\Lambda=R$, *and* $i=n-1$ *is even*,
(iv) $\Lambda=\boldsymbol{C}$ *or* $\boldsymbol{Q}$, *and* $i=dn-1$.

Therefore

$$\pi_i(V_m^\Lambda(\Lambda^{m+n-1})) = \begin{cases} 0 & \textit{csae (ii)} \\ \boldsymbol{Z} & \textit{case (iii) or (iv)}. \end{cases}$$

(b) *If* $\Lambda=\boldsymbol{R}$, $m\geq 2$, *and* $i=n-1$ *is odd, then* j_* *is an epimorphism and*

$$\pi_i(V_m^\Lambda(\Lambda^{m+n-1})) = \boldsymbol{Z}/2\boldsymbol{Z}\,.$$

To state our result in the equivariant case, let us define some notations. For any closed subgroup H of G, $N(H)$ denotes the normalizer of H in G, and (H) denotes the conjugacy class of H in G. Let X be a G-space. For any $x\in X$, G_x denotes the isotropy subgroup at x. The conjugacy class of an isotropy subgroup is called an orbit type. We put

$$X^H = \{x\in X\,|\,H\subset G_x\}\,,$$
$$X_H = \{x\in X\,|\,H = G_x\}\,,\qquad \text{and}$$
$$X_{(H)} = \{x\in X\,|\,(H) = (G_x)\}\,.$$

For a representation E of G, $\mathfrak{M}(E)$ denotes the set of orbit types appearing on $S(E)$. Choose a representative of each element of $\mathfrak{M}(E)$, and denote by $\mathfrak{M}_r(E)$ the set of those representatives. For any $H\in\mathfrak{M}_r(E)$ there is a transformation

$$r_H\colon [S(E), S(E')]_G \to [S(E^H), S(E'^H)]$$

restricting to the fixed point set by H, where [,] denotes the non-equivariant

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ON THE EQUIVARIANT HOMOTOPY OF STIEFEL MANIFOLDS

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1. Introduction and results

Throughout this paper G denotes a compact Lie group, and Λ denotes one of the real numbers $\boldsymbol{R}$, the complex numbers $\boldsymbol{C}$ and the quaternions $\boldsymbol{Q}$. Let E be a representation of G over Λ. All representations considered in this paper are orthogonal if $\Lambda=\boldsymbol{R}$, unitary if $\Lambda=\boldsymbol{C}$, and symplectic if $\Lambda=\boldsymbol{Q}$. For a positive integer $m\leq\dim_\Lambda E$, the *Stiefel manifold* $V_m^\Lambda(E)$ consists of all orthonormal m-frames in E, i.e.,

$$V_m^\Lambda(E)=\{(v_1,\cdots,v_m)\,|\,v_i\in E,\ \|v_i\|=1 \quad \text{for} \quad i=1,\cdots,m, \\ \text{and} \quad v_i\perp v_j \quad \text{if} \quad i\neq j\}\,.$$

If $m=1$, then $V_1^\Lambda(E)$ is the unit sphere $S(E)$ in E. For any $g\in G$ and any orthonormal m-frame $(v_1,\cdots,v_m)$ in E, $(gv_1,\cdots,gv_m)$ is also an orthonormal m-frame in E. This induces a smooth G-action on $V_m^\Lambda(E)$.

Let E' be another representation of G over Λ. We are interested in the set of G-homotopy classes of G-maps from $S(E)$ to $V_m^\Lambda(E')$, $[S(E), V_m^\Lambda(E')]_G$. If $m=1$, this set is the set of G-homotopy classes of G-maps from sphere to sphere, $[S(E), S(E')]_G$, which was studied in Hauschild [1], Rubinsztein [3] and others. (I am grateful to the referee who informed me that there was a gap in the proof of Rubinsztein's main theorem [3; Theorem 7.2]. This information leads to an improvement of the presentation of this paper.)

For any positive integer n, let

$$\Lambda^n=\Lambda\oplus\cdots\oplus\Lambda \qquad (n \text{ summands})$$

be a representation with trivial G-action and with the standard inner product. We define a map

$$j\colon S(E')\to V_m^\Lambda(E'\oplus\Lambda^{m-1})$$

by $j(v)=(v,e_1,\cdots,e_{m-1})$ for $v\in S(E')$ and the canonical orthonormal $(m-1)$-frame $(e_1,\cdots,e_{m-1})$ in Λ^{m-1}. Then j is a G-embedding, and induces a transformation

curve k_1 on φF_3, which is ambient isotopic to c_3 in $L(2\alpha, \beta)$. Each $k_1 \cap c^*$ and $k_1 \cap g^*$ consists of a point and $k_1 \cap f^* = \emptyset$. Thus, since c^* is φ-invariant, $\varphi f^* \sim \pm b^*$ and $\varphi g^* \sim \mp a^*$ in F_3, we can show that $k_2 = \varphi k_1$ is homotopic to c_1 in F_3. Therefore a link $k_1 \cup k_2$ is ambient isotopic to $c_1 \cup c_3$ in $L(2\alpha, \beta)$, and we have proved Theorem 5.1.

References

[1] K. Asano: *Homeomorphisms of prism manifolds,* Yokohama Math. J. **26** (1978), 19–25.

[2] J.S. Birman and D.R.J. Chillingworth: *On the homeotopy group of a non-orientable surface,* Proc. Cambridge Philos. Soc. **71** (1972), 437–448.

[3] G.E. Bredon and J.W. Woods: *Non-orientable surfaces in orientable* 3-*manifolds,* Invent. Math. **7** (1969), 83–110.

[4] J. Hempel: *One-sided incompressible surfaces in* 3-*manifolds,* Leuture Notes in Math. 438, Springer-Verlag, 1974, 251–258.

[5] C. McA. Gordon and W. Heil: *Cyclic normal subgroups of fundamental groups of* 3-*manifolds,* Topology **14** (1975), 305–309.

[6] F. Hirzebruch, W.P. Neumann, and S.S. Koh: Differential manifolds and quadratic forms, M. Dekker, 1971.

[7] P.K. Kim: *Some* 3-*manifolds which admit Klein bottles,* Trans. Amer. Math. Soc. **244** (1978), 299–312.

[8] S. Kinoshita: *On Fox property of a surface in a* 3-*manifold,* Duke Math. J. **33** (1966), 791–794.

[9] G. Livesay: *Fixed point free involutions on the* 3-*sphere,* Ann. of Math. **72** (1960), 603–611.

[10] P.K.M. Rice: *Free actions of* Z_4 *on* S^3, Duke Math. J. **36** (1969), 749–751.

[11] G.X. Ritter: *Free* Z_8 *actions on* S^3, Trans. Amer. Math. Soc. **181** (1973), 195–212.

[12] J.H. Rubinstein: *On* 3-*manifolds that have finite fundamental group and contain Klein bottles,* Trans. Amer. Math. Soc. **251** (1979), 129–137.

[13] ———: *One-sided Heegaard splittings of* 3-*manifolds,* Pacific J. Math. **76** (1979), 165–175.

[14] ———: *Free actions of some finite groups on* S^3, Math. Ann. **240** (1979), 165–175.

[15] D. Rolfsen: Knots and links, Publish or Perish Inc., 1976.

[16] H. Schubert: *Konten und Vollringe,* Acta Math. **90** (1953), 131–286.

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Let f^* and g^* denote simple closed curves on G which is parallel to f and g, respectively. Since each f^* and g^* is of type III on $G\subset\varphi F_3$, there exists a matrix $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\in GL(2, Z)$ such that

$$(1)\qquad \varphi a^*\sim\alpha_{11}f^*+\alpha_{12}g^* \text{ and } \varphi b^*\sim\alpha_{21}f^*+\alpha_{22}g^* \text{ in } \varphi F_3 .$$

The union $\varphi F_3\cup F_3$ separates $L(2\alpha, \beta)$ into solid tori U_1' and U_2' such that $U_\mu'\cap V_2=U_\mu$, $\mu=1, 2$. Let $\hat{b}_1$ and $\hat{b}_2$ be simple closed curves obtained by pushing b_1 int U_2 and b_2 into U_1, respectively. Since G is ambient isotopic to G_3, $\{[f^*], [\hat{b}_2]\}$ is a basis of $H_1(U_1')$ and $\{[g^*], [\hat{b}_1]\}$ is a basis of $H_1(U_2')$. It can be shown that

$$(2)\qquad f^*\sim\mu_1\hat{b}_1 \text{ in } U_2' \text{ and } g^*\sim\mu_2\hat{b}_2 \text{ in } U_1' .$$

By the argument in the proof of Lemma 3.2, we have

$$Sc(a_1^*, b_1) = -1,\ Sc(a_1^*, a_2') = 1,\ Sc(b_1^*, b_1) = 0,\ Sc(b_1^*, a_2') = \mu_2 ,$$
$$Sc(a_2^*, a_1') = -1,\ Sc(a_2^*, b_2) = 1,\ Sc(b_2^*, a_1') = \mu_1 \text{ and } Sc(b_2^*, b_2) = 0 .$$

From this and the fact that $\pi^{-1}a^*=a_1^*\cup b_1^*$ and $\pi^{-1}b^*=b_1^*\cup a_2^*$, it follows that

$$(3)\qquad a^*\sim -f^*+\hat{b}_2 \text{ and } b^*\sim -\mu_2\hat{b}_2 \text{ in } U_1' ,$$
$$(4)\qquad a^*\sim g^*-\hat{b}_1 \text{ and } b^*\sim -\mu_1\hat{b}_1 \text{ in } U_2' .$$

In U_1', $b^*+b^*\sim 0$ and $\mu_2(a^*+f^*)+b^*\sim 0$, by (2) and (3). Using (1), (3), (4) and the fact that $\varphi U_1'=U_2'$, we can show that

$$(5)\qquad \begin{aligned}\varphi b^*+\varphi g^* &\sim\alpha_{21}f^*+\alpha_{22}g^*-\varepsilon\alpha_{21}a^*+\varepsilon\alpha_{11}b^*\\ &\sim(\alpha_{22}+\varepsilon\alpha_{11})g^*+(\mu_1\alpha_{21}-\varepsilon\alpha_{11}-\varepsilon\mu_1\alpha_{21})b^*\sim 0\end{aligned}$$

and

$$(6)\qquad \begin{aligned}&\mu_2(\varphi a^*+\varphi f^*)+\varphi b^*\\ &\sim\mu_2(\alpha_{11}f^*+\alpha_{12}g^*+\varepsilon\alpha_{22}a^*-\varepsilon\alpha_{12}b^*)+\alpha_{21}f^*+\alpha_{22}g^*\\ &\sim(\varepsilon\mu_2\alpha_{12}+\varepsilon\mu_1\mu_2\alpha_{22}+\mu_1\mu_2\alpha_{21}+\mu_1\alpha_{21})b^*+(-\varepsilon\mu_2\alpha_{12}+\mu_2\alpha_{12}+\alpha_{22})g^*\\ &\sim 0, \text{ in } U_2' ,\end{aligned}$$

where $\varepsilon=\alpha_{11}\alpha_{23}-\alpha_{12}\alpha_{21}$.

It is not difficult to show that (5) and (6) does not hold at the same time except for the case that $\varepsilon=1$, $\alpha_{11}=\alpha_{22}=0$, $\alpha_{12}\alpha_{21}=-1$ and $\mu_1=\mu_2$. Thus we obtain $\varphi f^*\sim\pm b^*$ and $\varphi g^*\sim\mp a^*$ on F_3.

Let $k=G_2'\cap c_3$ [Fig. 5.2]. Then g intersects k in a single point and $k\cap f=\emptyset$. Let k^* be an arc on G which is parallel to k. Joining the end points of k^* by a vertical line in $\varphi F_3\cap M(F_3)$, we obtain a one-sided simple closed can show that φ maps each fiber onto a fiber. Therefore the orbit space of φ is also a Seifert fiber space.

Case 2. By Corollary 2.3, we may consider $L(2\alpha, \beta)-\mathring{N}(c_1 \cup c_3)$ as a Seifert fiber space. Hence, in order to complete the proof, it suffices to show that there exists a φ-invariant link $k_1 \cup k_2$ in $L(2\alpha, \beta)$ which is ambient isotopic to $c_1 \cup c_3$.

Let $G=V_2 \cap \varphi F_3$. Then we have $\partial G=\pi^{-1}c^*$. Cutting T_2 along ∂G, we obtain G_1 and G_2 where G_μ contains a_μ^* and b_μ^*, $\mu=1, 2$. Furthermore G separates V_2 into U_1 and U_2 such that $\partial U_\mu=G_\mu \cup G$, $\mu=1, 2$. Let c be a simple closed curve on T_2 as shown in Fig. 5.2. Then c bound a disk D in V_2. Since $\pi^{-1}c^* \cap \partial D$ consists of four points, we can deform D so that it intersects G in two arcs. Each $D \cap G_\mu$ separates G_μ into a disk G'_μ and an annulus G''_μ. The union $G'_1 \cup D \cup G'_2$ is an orientable surface of genus 1. See Fig. 5.2. Let G_3 be a surface obtained by pushing $G'_1 \cup D \cup G'_2$ into V_2 so that $G_3 \cap T_2=\partial G_3$.

First we will show that G is ambient isotopic to G_3 in V_2. Let V' and V'' be solid tori in V_2 such that $\partial V'=G'_1 \cup D \cup G''_2$ and $\partial V''=G''_1 \cup D \cup G'_2$. Suppose that $U_1 \cap D$ consists of two disks Δ_1 and Δ_2. Then $G'=G \cap V'$ is a disk and $G''=G \cap V''$ is an annulus. Obviously, G' is parallel to G'_1. According to [16], G'' is parrallel to $G''_1 \cup \Delta_1 \cup \Delta_2$ or $G'_2, \cup \Delta_0$, where $\Delta_0={}^-((D-(\Delta_1 \cup \Delta_2))$. If G'' is parallel to $G''_1 \cup \Delta_1 \cup \Delta_2$, G is parallel to G_1. If G' is parallel to $G'_2 \cup \Delta_0$, G is ambient isotopic to G_3. Similarly, we can show that G is parallel to G_2 or is ambient isotopic to G_3, for the case that $U_2 \cup D$ is disconnected.

Suppose that G is parallel to G_1. Then G is parallel to G_2. Since each $\partial D_\nu \cap \partial G$, $\nu=1, 2$, consists of $|\mu_\nu|$ points, G_1 is parrallel to G_2, if and only if $|\mu_1\mu_2|=1$. Thus G is ambient isotopic to G_3.

We take oriented simple closed curves f and g on $G'_1 \cup D \cup G'_2$ so that each f and g is a centerline of an annulus $G'_\mu \cup D$, $\mu=1, 2$, and $f \cap g$ is a single point, as shown in Fig. 5.2.

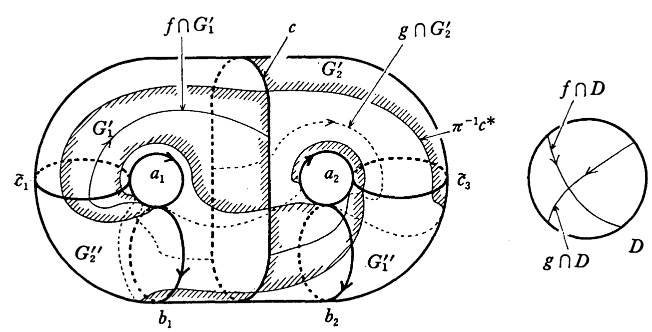

Fig. 5.2

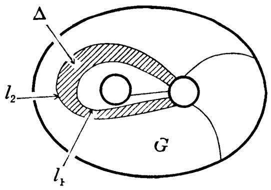

Fig. 5.1

For $\mu=1, 2$, l_μ separates D'_μ into two disks $\nabla_{\mu 1}$ and $\nabla_{\mu 2}$. Among $\nabla_{1\mu}\cup\Delta\cup\nabla_{2\nu}$, $\mu, \nu=1, 2$, there exists at least one disk $D_1'^*$ such that $\partial D_1'^*$ is not homologous to zero and ∂D_2 in T_2. Deforming $D_1'^*$ slightly, we obtain a system $\{D_1'^{**}, D_2'\}$ of V_2 such that $G\cap(D_1'^{**}\cup D_2')$ has fewer components than $G\cap(D_1'\cup D_2')$. Hence we can construct a system $\{D_1'', D_2''\}$ of meridian disks of V_2 such that at least one innermost curve b of $G\cap(D_1''\cup D_2'')$ in $D_1''\cup D_2''$ connects two points in distinct components of ∂G.

Assume that $b\subset D_1''$. Let Δ_1 be a disks in D_1'' such that $\Delta_1\cap G=b$ and $c=\partial\Delta_1-\mathring{b}$ is contained in $\partial D_1''$. Furthermore we assume that $c\subset G_1$. Cutting G and G_1 along b and c, we obtain annuli A and A'. It can be shown easily that A is incompressible. Applying Lemma 5.2 to A, we can show that the union of A, A' and two copies Δ_1' and Δ_1'' of Δ_1 bounds a solid torus U of genus 1. Thus $G\cup G_1$ bounds a solid torus U' of genus 2.

Let Δ_2 be a meridian disk of U such that $\partial\Delta_2\subset A\cup A'$. Then Δ_1 and Δ_2 form a system of meridian disks of U'. Note that each $\Delta_1\cap G$ and $\Delta_1\cap G_1$ is a single arc connecting distinct components k_1 and k_2 of $\partial G=\partial G_1$.

Suppose that G is not parallel to G_1. Then $A\cup\Delta_1'\cup\Delta_1''$ is not parallel to A'. Thus, by Lemma 5.2, we have $|Sc(k_3, \partial\Delta_2)|>1$, where $k_3=\partial G-(k_1\cup k_2)$ and $Sc(k_3, \partial\Delta_2)$ denotes the intersection number of k_3 with $\partial\Delta_2$ in $G\cup G_1$. Since k_1 and k_3 generate $H_1(G_1)$, the homomorphism Ψ from $H_1(G_1)$ into $H_1(U')$ induced by the inclusion is not onto.

From Lemma 3.1 and the fact that $\tilde{c}_1\sim b_1$ and $\tilde{c}_3\sim -b_2$, it follows that there exists a matrix $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ in $GL(2, Z)$ such that $k_1\sim\alpha_{22}b_1-\alpha_{21}b_2$ and $k_2\sim\alpha_{12}b_1-\alpha_{11}b_2$ on T_2. Since $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=\pm 1$, the inclusion from G_1 into V_2 induces an isomorphism from $H_1(G)$ onto $H_1(V_2)$. This contradicts the fact that Ψ is not onto. Thus G is parallel to G_1.

Since $\varphi M(F_3)\cup M(F_3)$ is φ-invariant, φ takes $U''={}^{-}(U'-\varphi M(F_3))$ onto U'' or ${}^{-}(V_2-(\varphi M(F_3)\cup U''))$. Using the fact that a solid torus of genus 2 does not admit a free involution, we have $\varphi U''={}^{-}(V_2-(\varphi M(F_3)\cup U''))$. Then G is parallel to G_2.

From this, it follows that $L(2\alpha, \beta)-\mathring{N}(\varphi F_3\cap F_3)$ is homeomorphic to the product of a 2-punctured disk and a circle. Hence we may consider $L(2\alpha, \beta)$ as a Seifert fiber space having each curve of $\varphi F_3\cap F_3$ as a fiber. By [5], we

If we assume that $l_1 \sim \varepsilon l_2$ in V_2, for $\varepsilon=1$ or -1, one of the following systems of equations holds.

$$\begin{cases}\alpha_{22}-\varepsilon\mu_1\alpha_{21}=0\,,\\ -\alpha_{21}-\varepsilon(\alpha_{21}-\mu_2\alpha_{22})=0\,.\end{cases}\qquad \begin{cases}\alpha_{22}+\varepsilon(\mu_1\alpha_{21}-\alpha_{22})=0\,,\\ -\alpha_{21}+\varepsilon\mu_2\alpha_{22}=0\,.\end{cases}$$

Each of the above systems does not hold except for $\varepsilon=1$ and $\mu_1\mu_2=-2$. Hence the proof is completed.

By making use of Assertion C and the same method as in the proof for Case 1, we can show that φ_1 is equivalent to φ_2 such that $\varphi_2F_3\cap F_3$ does not contain a curve of type V. Let B be a Möbius band in φ_2F_3 such that B intersects F_3 in ∂B and a centerline of B. Then ∂B is of type III on F_3. If $\mu_1\mu_2\neq-2$, this contradicts Assertion C.

5. Orbit space. In this section we will complete the proof of the following main theorem.

Theorem 5.1. *Let μ_1 and μ_2 be integers such that $\mu_1\mu_2\neq 0$ and $\mu_1\mu_2\neq-2$. Then the orbit space of a fixed point free involution on $L(8\mu_1\mu_2-2,\ 4\mu_1\mu_2-2\mu_1-1)$ is homeomorphic to a Seifert fiber space.*

By Lemma 4.1, we can divide the proof into the following two cases:

Case 1: $\varphi F_3\cap F_3$ consists of three curves of type II on φF_3 and F_3.
Case 2: $\varphi F_3\cap F_3=c^*$.

Case 1. To prove Theorem 5.1 for Case 1, we need the following lemma which can be shown easily.

Lemma 5.2. *Let A be an annulus properly embedded in V_2 such that A is incompressible and ∂A bounds an annulus A' on T_2. Then $A\cup A'$ bounds a solid torus U in V_2. Furthermore A is parallel to A', if and only if the inclusion from A' into U induces an isomorphism from $H_1(A')$ onto $H_1(U)$.*

Let G_1 and G_2 be 2-punctured disks obtained by cutting T_2 along $\varphi F_3\cap T_2$. First we will show that $G=\varphi F_3\cap V_2$ is parallel to G_1 and G_2. Each G, G_1 and G_2 is incompressible in V_2. Hence we can deform D_1 and D_2 so that $G\cap(D_1\cup D_2)$ consists of arcs, where D_1 and D_2 are meridian disks of V_2, as in §2. Since V_2 is irreducible, we can construct a system $\{D_1', D_2'\}$ of meridian disks of V_2 such that each curve $G\cap(D_1'\cup D_2')$ is not parallel to ∂G in G.

From the fact that G is incompressible, it follows that $G\cap D_\mu'\neq\emptyset$, for $\mu=1, 2$. Suppose that each of the innermost curves of $G\cap(D_1'\cup D_2')$ on $D_1'\cup D_2'$ connects two points in the same component of ∂G. Then there exists a disk Δ on G such that each $l_1=\Delta\cap D_1'$ and $l_2=\Delta\cap D_2'$ is an arc in $\partial\Delta$ and $\partial\Delta-{}^\circ(\Delta\cup(D_1'\cup D_2'))\subset\partial G$.