

LEFT-INVARIANT LORENTZ METRICS ON LIE GROUPS^{*})

KATSUMI NOMIZU

(Received October 7, 1977)

With J. Milnor [2] we consider a special class \mathfrak{S} of solvable Lie groups. A non-commutative Lie group G belongs to \mathfrak{S} if its Lie algebra \mathfrak{g} has the property that $[x, y]$ is a linear combination of x and y for any elements x and y in \mathfrak{g} . It is shown that \mathfrak{g} has this property if and only if there exist a commutative ideal \mathfrak{u} of codimension 1 and an element $b \in \mathfrak{u}$ such that $[b, x] = x$ for every $x \in \mathfrak{u}$.

Milnor has shown that if $G \in \mathfrak{S}$, then every left-invariant (positive-definite) Riemannian metric on G has negative constant sectional curvature. The simplest example is given by

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}; a > 0, a, b \in \mathbb{R} \right\}.$$

On the other hand, Wolf [3, p. 58] showed that this group G admits a left-invariant Lorentz metric which is flat (that is, with zero sectional curvature).

Our first and main objective in this paper is to prove the following theorem.

Theorem 1. *If a Lie group G belongs to the class \mathfrak{S} , then*

(1) *every left-invariant Lorentz metric (of signature $(-, +, \dots, +)$) has constant sectional curvature;*

(2) *given any arbitrary constant k , $k > 0$, $k = 0$, or $k < 0$, one can find a left-invariant Lorentz metric on G with k as constant sectional curvature.*

Unlike the Riemannian case, the existence of a flat left-invariant Lorentz metric seems to be a more frequent phenomenon. Our second objective is to prove

Theorem 2. *Each of the following 3-dimensional Lie groups admits a flat left-invariant Lorentz metric:*

(1) $E(2)$: *group of rigid motions of Euclidean 2-space;*

(2) $E(1, 1)$: *group of rigid motions of Minkowski 2-space;*

^{*}) This work was supported in part by an NSF grant, (MCS 76-06324 A01).

(3) *The Heisenberg group consisting of all real matrices of the form*

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that Milnor has shown that the group (1) admits a flat left-invariant Riemannian metric but not the groups (2) and (3); see Corollaries 4.6, 4.7 and 4.8 in [2].

1. Proof of Theorem 1

We take a commutative ideal \mathfrak{u} of codimension 1 and an element $b \notin \mathfrak{u}$ such that $[b, x] = x$ for every $x \in \mathfrak{u}$. Let \langle, \rangle be the Lorentz inner product in \mathfrak{g} coming from a given left-invariant Lorentz metric on G .

Case I. \mathfrak{u} is nondegenerate (that is, the restriction of the inner product to \mathfrak{u} is nondegenerate).

We may choose b' such that $\langle b', \mathfrak{u} \rangle = 0$ and $\mathfrak{g} = \{b'\} + \mathfrak{u}$ (direct sum). Then we can write $b' = \alpha b + x_0$ with some $\alpha \neq 0$ and $x_0 \in \mathfrak{u}$. For every $x' \in \mathfrak{u}$, we have

$$[b', x] = \alpha[b, x] + [x_0, x] = \alpha x.$$

We may now take b'/α and rename it b . Then $\langle b, \mathfrak{u} \rangle = 0$ and $[b, x] = x$ for every $x \in \mathfrak{u}$. We now consider two subcases: Ia (b is time-like) and Ib (b is space-like).

Subcase Ia. Let $\langle b, b \rangle = -\lambda^2$ ($\lambda > 0$). We use the formula

$$(1) \quad 2\langle \nabla_x y, z \rangle = \langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle$$

to determine the covariant derivative $\nabla_x y$ of the left-invariant Lorentz metric (formula (5.3) in [2]). Easy computation shows

$$\nabla_b b = 0, \quad \nabla_b x = 0 \quad \text{for } x \in \mathfrak{u}$$

so that

$$\nabla_x b = \nabla_b x + [x, b] = -x.$$

For $x, y \in \mathfrak{u}$ we have

$$2\langle \nabla_x y, b \rangle = \langle [x, y], b \rangle - \langle [y, b], x \rangle + \langle [b, x], y \rangle = 2\langle x, y \rangle$$

and

$$2\langle \nabla_x y, z \rangle = 0 \quad \text{for } z \in \mathfrak{u}.$$

Hence

$$\nabla_x y = -\langle x, y \rangle b / \lambda^2,$$

since $\langle b, b \rangle = -\lambda^2$.

For $x, y, z \in \mathfrak{u}$, we obtain for the curvature tensor R

$$\begin{aligned} R(x, y)z &= \nabla_x \nabla_y z - \nabla_y \nabla_x z \\ &= \nabla_x (-\langle y, z \rangle b / \lambda^2) - \nabla_y (-\langle x, z \rangle b / \lambda^2) \\ &= \langle y, z \rangle x / \lambda^2 - \langle x, z \rangle y / \lambda^2, \\ R(x, y)b &= \nabla_x \nabla_y b - \nabla_y \nabla_x b \\ &= \nabla_x (-y) - \nabla_y (-x) = [x, y] = 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} R(x, b)b &= -x \\ R(x, b)y &= -\langle x, y \rangle b / \lambda^2. \end{aligned}$$

Thus

$$(2) \quad \begin{aligned} R(x, y) &= \frac{1}{\lambda^2} x \wedge y \\ R(x, b) &= \frac{1}{\lambda^2} x \wedge b, \end{aligned}$$

where $u \wedge v$ denotes the endomorphism

$$(u \wedge v)w = \langle v, w \rangle u - \langle u, w \rangle v.$$

The equations (2) imply that our metric has constant sectional curvature $1/\lambda^2$.

Subcase Ib. Let $\langle b, b \rangle = \lambda^2$ ($\lambda > 0$). Since our inner product is Lorentzian, we have a time-like unit vector, say, c in \mathfrak{u} : $\langle c, c \rangle = -1$. We can write $\mathfrak{u} = \{c\} + \mathfrak{u}_1$, where \mathfrak{u}_1 is the orthogonal complement of $\{c\}$. We write y, z, u, \dots for elements in \mathfrak{u}_1 in the following computation.

We have, again by means of (1),

$$\nabla_b b = \nabla_b c = \nabla_b y = 0, \quad \text{where } y \in \mathfrak{u}_1.$$

Thus

$$\nabla_c b = -c, \quad \nabla_c b = -y, \quad \text{where } y \in \mathfrak{u}_1.$$

From

$$2\langle \nabla_c c, b \rangle = -\langle [c, b], c \rangle + \langle [b, c], c \rangle = -2$$

and

$$2\langle \nabla_c c, c \rangle = 2\langle \nabla_c c, y \rangle = 0,$$

we obtain

$$\nabla_c c = -b/\lambda^2.$$

we have easily

$$\nabla_c y = 0 \quad \text{and} \quad \nabla_y c = 0, \quad \text{where } y \in \mathfrak{u}_1.$$

For $y, z, u \in \mathfrak{u}_1$ we get

$$2\langle \nabla_y z, c \rangle = 2\langle \nabla_y z, u \rangle = 0$$

and

$$2\langle \nabla_y z, b \rangle = -\langle [z, b], y \rangle + \langle b, y \rangle, z \rangle = 2\langle y, z \rangle$$

and hence

$$\nabla_y z = \langle y, z \rangle b / \lambda^2.$$

We may now compute

$$\begin{aligned} R(b, c)c &= b/\lambda^2, & R(b, c)b &= c, & R(b, c)y &= 0 \\ R(b, y)b &= y, & R(b, y)c &= 0, & R(b, y)z &= -\langle (y, z) \rangle b / \lambda^2 \\ R(c, y)b &= 0, & R(c, y)c &= -y/\lambda^2, & R(c, y)z &= -\langle y, z \rangle c / \lambda^2 \\ R(y, z)b &= 0, & R(y, z)c &= 0, & R(y, z)u &= -\langle z, u \rangle y / \lambda^2 + \langle y, u \rangle z / \lambda^2. \end{aligned}$$

We have thus

$$(3) \quad R(u, v) = \frac{-1}{\lambda^2} u \wedge v \quad \text{for all } u, v \in \mathfrak{g},$$

which means that our metric has constant sectional curvature $-1/\lambda^2$.

Case II it is degenerate (that is, restriction of the inner product to \mathfrak{u} is degenerate).

According to [1], Theorem 1.1, \mathfrak{u} contains a light-like vector c and an $(n-2)$ -dimensional subspace \mathfrak{u}_1 on which the inner product is positive-definite such that $\mathfrak{u} = \{c\} + \mathfrak{u}_1$ (direct sum) and $\langle c, \mathfrak{u}_1 \rangle = 0$. In the orthogonal complement \mathfrak{u}_1^\perp of \mathfrak{u}_1 in \mathfrak{g} we can find a vector b' such that

$$\langle b', b' \rangle = 0 \quad \text{and} \quad \langle b', c \rangle = -1.$$

We can write $b' = \alpha b + x_0$ for some $\alpha \neq 0$ and $x_0 \in \mathfrak{u}$. Then $[b', x] = \alpha [b, x] + [x_0, x] = \alpha x$ for every $x \in \mathfrak{u}$. Now if we denote b'/α and αc by b and c , then our new b and c satisfy the following conditions:

$$\begin{aligned} \mathfrak{g} &= \{b\} + \{c\} + \mathfrak{u}_1, & \mathfrak{u} &= \{c\} + \mathfrak{u}_1 \text{ (direct sums);} \\ \langle b, b \rangle &= 0, & \langle b, c \rangle &= -1, & \langle c, c \rangle &= 0, & \langle b, \mathfrak{u}_1 \rangle &= \langle c, \mathfrak{u}_1 \rangle = 0; \\ [b, c] &= c, & [b, y] &= y & \text{for } y \in \mathfrak{u}_1. \end{aligned}$$

We find

$$\begin{aligned} \nabla_b b &= -b, & \nabla_b c &= c, & \nabla_b y &= 0 & \text{for } y \in \mathfrak{u}_1 \\ \nabla_c b &= 0, & \nabla_c c &= 0, & \nabla_c y &= 0 & \text{for } y \in \mathfrak{u}_1 \\ \nabla_y b &= -y, & \nabla_y c &= 0, & \nabla_y z &= -\langle y, z \rangle c & \text{for } y, z \in \mathfrak{u}_1. \end{aligned}$$

From these we obtain

$$R(b, c) = R(b, y) = R(c, y) = R(y, z) = 0 \quad \text{for } y, z \in \mathfrak{u}_1,$$

that is, our metric is flat.

We have thus concluded the proof of the first part of Theorem 1. The second part does not require much extra work as we see in the following.

We start with $\mathfrak{g} = \{b\} + \mathfrak{u}$, where \mathfrak{u} is a commutative ideal of codimension 1 and $[b, x] = x$ for every $x \in \mathfrak{u}$.

If $k > 0$, then take $\lambda > 0$ such that $k = 1/\lambda^2$. Take any positive-definite inner product in \mathfrak{u} and extend it to a Lorentz inner product in \mathfrak{g} by

$$\langle b, \mathfrak{u} \rangle = 0 \quad \text{and} \quad \langle b, b \rangle = -1/\lambda^2.$$

The computation in Case Ia shows that the resulting left-invariant Lorentz metric on G has constant sectional curvature $k = 1/\lambda^2$.

If $k < 0$, then take $\lambda > 0$ such that $k = -1/\lambda^2$. Take any Lorentz inner product in \mathfrak{u} and extend it to \mathfrak{g} by

$$\langle b, \mathfrak{u} \rangle = 0 \quad \text{and} \quad \langle b, b \rangle = -1/\lambda^2.$$

The computation in Case Ib shows that the resulting Lorentz metric on G has constant sectional curvature $k = -1/\lambda^2$.

Finally, suppose $k = 0$. Take an element $c \neq 0$ in \mathfrak{u} and an $(n-2)$ -dimensional subspace \mathfrak{u}_1 of \mathfrak{u} . We extend any positive-definite inner product in \mathfrak{u}_1 to a Lorentz metric in \mathfrak{g} by

$$\langle b, b \rangle = \langle c, c \rangle = \langle b, \mathfrak{u}_1 \rangle = \langle c, \mathfrak{u}_1 \rangle = 0 \quad \text{and} \quad \langle b, c \rangle = -1.$$

The computation in Case II shows that we get a flat Lorentz metric on G .

2. Proof of Theorem 2

For each group, we describe its Lie algebra and show how to define a Lorentz inner product which will give rise to a flat left-invariant Lorentz metric on the group.

E(2): This consists of all matrices of the form

$$\begin{bmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Its Lie algebra has a basis consisting of

$$x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for which

$$[x, y] = 0, \quad [z, x] = y, \quad [y, z] = x.$$

We define a Lorentz inner product by

$$\langle z, z \rangle = -1, \quad \langle x, x \rangle = \langle y, y \rangle = 1, \quad \langle z, x \rangle = \langle z, y \rangle = \langle x, y \rangle = 0.$$

Computation shows

$$\nabla_x x = \nabla_x y = \nabla_y x = \nabla_x z = \nabla_y y = \nabla_y z = \nabla_z z = 0$$

and

$$\nabla_z x = y, \quad \nabla_z y = -x.$$

It follows that

$$R(x, y) = R(y, z) = R(z, x) = 0,$$

that is, the metric is flat.

E(1, 1): This is the group of all matrices of the form

$$\begin{bmatrix} \cosh t & \sinh t & a \\ \sinh t & \cosh t & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Its Lie algebra has a basis consisting of

$$x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for which

$$[x, y] = 0, \quad [z, x] = y, \quad [z, y] = x.$$

We define a Lorentz inner product by

$$\langle x, x \rangle = \langle z, z \rangle = 1, \quad \langle y, y \rangle = -1, \quad \langle z, x \rangle = \langle z, y \rangle = \langle x, y \rangle = 0.$$

Computation shows

$$\nabla_x = 0, \quad \nabla_y = 0, \quad \nabla_z x = y, \quad \nabla_z y = x, \quad \nabla_z z = 0.$$

It follows that

$$R(x, y) = R(y, z) = R(z, x) = 0.$$

The Heisenberg group: Its Lie algebra has a basis consisting of

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for which

$$[x, y] = z, \quad [z, x] = [z, y] = 0.$$

We consider the Lorentz inner product given by

$$\begin{aligned} \langle z, z \rangle &= \langle x, x \rangle = 0, & \langle z, x \rangle &= -1 \\ \langle x, y \rangle &= \langle z, y \rangle = 0, & \langle y, y \rangle &= 1. \end{aligned}$$

Computation shows that

$$\begin{aligned} \nabla_x x &= y, & \nabla_x y &= z, & \nabla_x z &= 0 \\ \nabla_y x &= \nabla_y y = \nabla_y z = \nabla_z x = \nabla_z y = \nabla_z z = 0, \end{aligned}$$

and consequently,

$$R(x, y) = R(y, z) = R(z, x) = 0.$$

REMARK 1. The group $SO(3)$ (or $SU(2)$) does not admit a left-invariant flat Lorentz metric. Suppose it does. Then $x \rightarrow \nabla_x$ is a homomorphism of the Lie algebra $\mathfrak{o}(3)$ into the Lie algebra $\mathfrak{o}(1, 2)$ of all skew-symmetric endomorphisms of the 3-dimensional flat Lorentz space. The kernel has to be (0) because $\mathfrak{o}(3)$ is simple. This means that there is an isomorphism of $\mathfrak{o}(3)$ onto $\mathfrak{o}(1, 2)$, a contradiction.

REMARK 2. The group $\mathfrak{o}(1, 2)$ (or $SL(2, R)$) admits a left-invariant Lorentz metric with constant sectional curvature -1 . For the Lie algebra $\mathfrak{sl}(2, R)$ of $SL(2, R)$, we define

$$\langle x, y \rangle = \frac{1}{2} \text{trace}(xy).$$

The matrices

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are orthogonal and have length square $-1, 1, 1$, respectively. Thus we have a Lorentz inner product. One can compute to see that the resulting left-invariant Lorentz metric has constant sectional curvature -1 . Another geometric way is the following. We consider the vector space R_2^4 with indefinite inner product

$$\langle x, y \rangle = -x_1 y_1 - x_2 y_2 + x_3 y_3 + x_4 y_4.$$

It is well known that the hypersurface $H_1^3 = \{x \in R_2^4; \langle x, x \rangle = -1\}$ has constant sectional curvature -1 with respect to the induced Lorentz metric (so-called anti De Sitter space). Now the mapping

$$(x_1, x_2, x_3, x_4) \in R_2^4 \rightarrow \begin{bmatrix} x_1 + x_3 & x_4 - x_2 \\ x_2 + x_4 & x_1 - x_3 \end{bmatrix} \in \mathfrak{gl}(2, R)$$

gives a one-to-one correspondence between H_1^3 and $SL(2, R)$, and the Lorentz metric on H_1^3 corresponds to the left-invariant Lorentz metric (also right-invariant) on $SL(2, R)$ which we defined earlier. Our metric is essentially the same as the Killing-Cartan form.

BROWN UNIVERSITY

References

- [1] L. Graves: *Codimension one isometric immersions between Lorentz spaces*, Thesis, Brown University, 1977.
- [2] J. Milnor: *Curvatures of left invariant metrics on Lie groups*, *Advances in Math.* **21** (1976), 293–329.
- [3] J.A. Wolf: *Isotropic manifolds of indefinite metrics*, *Comment. Math. Helv.* **39** (1964), 21–64.

Added in Proof. Concerning Remark 1, we learned that Professor Y. Matsushima had the following result (unpublished): A semi-simple Lie group does not admit a left-invariant torsion-free flat linear connection.