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ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE SCHRÖDINCER EQUATION $(-\Delta+Q(y)-k^2)V=F$

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1. Introduction

Let us consider the Schrödinger operator

$$(1.1)\qquad S=-\Delta+Q(y)\qquad (y\in\boldsymbol{R}^N)$$

in $\boldsymbol{R}^N$. The purpose of this work is to show an asymptotic formula for the solution V of the equation $(S-k^2)V=F$ under the assumption that $Q(y)$ is a long-range potential, i.e., $Q(y)=O(|y|^{-\varepsilon})$ $(\varepsilon>0)$ as $|y|\to\infty$. Here $k\in\boldsymbol{R}-\{0\}$ and $F(y)$ is a given function on $\boldsymbol{R}^N$, and the solution V satisfies the "radiation condition"

$$(1.2)\qquad \frac{\partial V}{\partial|y|}-ikV(y)\to 0\qquad (|y|\to\infty).$$

The exact definition of the radiation condition will be given below (Definition 2.1).

Our method has its origin in the works of W. Jäger ([4]~[7]). He considered the differential operator with operator-valued coefficients

$$(1.3)\qquad L=-\frac{d^2}{dr^2}+B(r)+C(r)\qquad r\in I=(0,\infty),$$

where for each $r\in I$ $B(r)$ is a non-negative definite, self-adjoint operator in a Hilbert space X and $C(r)$ is a symmetric operator in X. L acts on X-valued functions on I. In the above papers Jäger, among others, has established the limiting absorption principle for L and an asymptotic formula for the solution v of the equation $(L-k^2)v=f$, which were used to develop an eigenfunction expansion theory associated with L. These results can be applied to the Schrödinger operator as follows: Let $X=L_2(S^{N-1})$, S^{N-1} denoting the $(N-1)$-sphere, and let $L_2(I, X)$ be the Hilbert space of all X-valued functions $F(r)$ on I such that $|f(r)|_X$ is square integralbe on I, where $|\ |_X$ means the norm of X. Then the multiplication operator U of the form

$$(1.4)\qquad U: L_2(\boldsymbol{R}^N)\ni f(y)\mapsto r^{(N-1)/2}f(r\omega)\in L_2(I, X)\qquad (r=|y|,\ \omega=y/r\in S^{N-1})$$

gives a unitary equivalence between $L_2(\boldsymbol{R}^N)$ and $L_2(I, X)$. Further, we have

$$(1.5)\qquad S=U^{-1}LU$$

with

$$(1.6)\qquad B(r)=r^{-2}A=r^{-2}\Big(-\Lambda_N+\frac{(N-1)(N-3)}{4}\Big),\qquad C(r)=Q(r\omega)\times,$$

Λ_N being the Laplace-Beltrami operator on S^{N-1}. Thus all results obtained for L can be applied to S by the use of the unitary operator U. Saitō [8]~[12] have extended Jäger's results to apply to the Schrödinger operator. [10] gives an asymptotic formula for the solution v of the equation $(L-k^2)v=f$ which can be applied to the Schrödinger equation $(S-k^2)V=F$ with $Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>1/2$. On the other hand the Schrödinger operator can be treated directly by using essentially the same idea as the above works. Along this line Ikebe-Saitō [3] has shown the limiting absorption principle for S with $Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>0$, and the asymptotic behavior of the solutions of the equation $(S-k^2)V=F$ and spectral representations for S have been investigated in Ikebe [2] with $Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>1/2$.

In this work we shall restrict ourselves to the case that the potential $Q(y)$ satisfies $Q(y)=O(|y|^{-\varepsilon})$ at infinity with $0<\varepsilon\leqq 1/2$. More precisely $Q(y)$ is assumed to satisfy the following

Assumption 1.1.

(Q) $Q(y)$ can be decomposed as $Q(y)=Q_0(y)+Q_1(y)$ such that Q_0 and Q_1 are real-valued functions on $\boldsymbol{R}^N$, N being an integer with $N\geqq 2$.

(Q_0) There exist constants $C>0$ and $0<\varepsilon\leqq 1/2$ such that $Q_0\in C^m(\boldsymbol{R}^N)$ and

$$(1.7)\qquad |D^jQ_0(y)|\leqq C(1+|y|)^{-j-\varepsilon}\qquad (y\in\boldsymbol{R}^N,\ j=0, 1, 2, \cdots, m),$$

where D^j denotes an arbitrary derivative of j-th order and

$$(1.8)\qquad m=\begin{cases}[2/\varepsilon] & (\text{if } 2/\varepsilon \text{ is an integer}),\\ [2/\varepsilon]+1 & (\text{otherwise}),\end{cases}$$

$[a]$ denoting the greatest integer n such that $n\leqq a$.

$(Q)_1$ $Q_1\in C^0(\boldsymbol{R}^N)$ and

$$(1.9)\qquad |Q_1(y)|\leqq C(1+|y|)^{-2}\qquad (y\in\boldsymbol{R}^N)$$

with the same C, ε as in (Q_0).

In §2~§4 we shall consider the operator L given by (1.3), where $B(r)$ and

$C(r)$ satisfy (1.6) with $N\geqq 3$. The argument have much similarity to the one used in [10]. But we have to newly construct a function $\lambda(y, k)$ ($y\in \boldsymbol{R}^N$, $k\in \boldsymbol{R}-\{0\}$) which is introduced as a solution of the following problem: Find a function $\lambda(y, k)$ such that $(L-k^2)(e^{i\mu}x)=O(r^{-1-\varepsilon})$ at infinity for any smooth $x(\omega)\in X=L_2(S^{N-1})$, where

$$(1.10) \qquad \mu(y, k) = rk-\lambda(y, k) \qquad (r=|y|).$$

The function $\int_0^r Q_0\left(t\frac{y}{|y|}\right)dt$, which was used in Saitō [10], [11] (or Ikebe [2]), will be turn out to be the "first approximation" to $\lambda(y, k)$. The case of $N=2$ will be discussed briefly in §5.

Using the results obtained in this work we can develop an eigenfunction expansion theory for the Schrödinger operator (1.1) with $Q(y)=O(|y|^{-\varepsilon})$ with $0<\varepsilon\leqq 1/2$. We shall discuss this in [13].

2. The limiting absorption principle and the main theorem

In this and the succeeding two sections we shall assume the spatial dimension $N\geqq 3$. Then $B(r)$ defined by (1.6) is a non-negative self-adjoint operator in X for each $r\in I=(0, \infty)$. Corresponding to the decomposition $Q(y)=Q_0(y)+Q_1(y)$, we set $C(r)=C_0(r)+C_1(r)$, i.e., $C_j(r)=Q_j(r\omega)\times$ $(j=1, 2)$.

Now we shall list the notation which will be employed in the sequel without further reference. Many of these were used in [10] and [11].

$$\boldsymbol{C}^+ = \{k = k_1+ik_2\in \boldsymbol{C}/k_1\neq 0,\ k_2\geqq 0\}\ .$$

$X=L_2(S^{N-1})$. Its norm and inner product are denoted by and $|\ |_X$ and $(\ ,\ )_X$.

$L_{2,\beta}(J, X)$ ($\beta\in R$) is the Hilbert space of all X-valued functions $f(r)$ on an open interval J such that $(1+r)^\beta|f(r)|_X$ is square integrable on J. The inner product and norm are defined by

$$(f, g)_{\beta,J} = \int_J (1+r)^{2\beta}(f(r), g(r))_X\, dr$$

and

$$\|f\|_{\beta,J} = [(f, f)_{\beta,J}]^{1/2}\,,$$

respectively. When $\beta=0$ or $J=I=(0, \infty)$ the subscript 0 or I may be omitted as in $L_2(I, X)$, $\|\ \|_\beta$ etc.

$H_0^{1,B}(J, X)=UH_1(B_R)$, where $J=(0, R)$, $B_R=\{y\in \boldsymbol{R}^N/|y|\geqq R\}$, U is given by (1.4) and $H_m(\Omega)$ is the Hilbert space obtained by the completion of $C_0^\infty(\Omega)$ by the norm

$$||u||_m = \sum_{j\leqq m} [\int_\Omega |D^j u(y)|^2 dy]^{1/2}.$$

For $N \geqq 3$ $H_0^{1,B}(I, X)$ is a Hilbert space with its inner product

$$(f, g)_{B,J} = (f', g')_{0,J} + (B^{1/2}f, B^{1/2}g)_{0,J} + (f, g)_{0,J} \qquad \left(f' = \frac{\partial}{\partial r} f\right)$$

and norm $||f||_{B,J} = [(f, f)_{B,J}]^{1/2}$. When $J=I$ we shall omit the subscript I as in $||\ ||_B$ etc.

$F_\gamma(J, X)$ $(\gamma \geqq 0)$ is the set of all anti-linear continuous functionals ℓ on $H_0^{1,B}(J, X)$, i.e.,

$$\ell : H_0^{1,B}(J, X) \ni v \mapsto \langle \ell, v \rangle \in \boldsymbol{C},$$

such that

$$|||\ell|||_{\gamma,J} = \sup \{|\langle \ell, (1+r)^\gamma v\rangle| / ||v||_{B,J} = 1\}$$

is finite. $F_\gamma(J, X)$ is a Banach space with its norm $|||\ |||_{\gamma,J}$. When $J=I$ the subscript I will be omitted as in $|||\ |||_\gamma$.

$L_2(I, X)_{loc}$ $(H_0^{1,B}(I, X)_{loc})$ is the set of all X-valued functions f such that $\xi f \in L_2(I, X)$ $(H_0^{1,B}(I, X))$ for any real-valued, smooth function ξ on $\bar{I} = [0, \infty)$ with compact support in $\bar{I}$.

D denotes the domain of the Laplace-Beltrami operator Λ_N (as a self-adjoint operator in X).

$C(A, B, \cdots)$ denotes a positive constant depending only on $A, B, \cdots$. But very often symbols indicating obvious dependence will be omitted.

$C^m(\boldsymbol{R}^N)$, $C_0^\infty(\boldsymbol{R}^N)$, $H_2(\boldsymbol{R}^N)_{loc}$ etc. will be employed as usual.

Let us first show the limiting absorption principle for L which is our main tool. Throughout this paper a number δ will be fixed such that $1/2 < \delta < 1/2 + \varepsilon/4$.

Definition 2.1 (radiative function). Let $\ell \in F_0(I, X)$ and $k \in \boldsymbol{C}^+$ be given. Then an X-valued function $v(r)$ on I is called the *radiative function* for $\{L, k, \ell\}$, if the following three conditions hold:

1) $v \in H_0^{1,B}(I, X)_{loc}$.
2) $v' - ikv \in L_{2,\delta-1}(I, X)$.
3) v satisfies the equation

$$(2.1) \qquad (v, (L-k^2)\phi)_0 = \langle \ell, \phi \rangle \qquad (\phi \in UC_0^\infty(\boldsymbol{R}^N)).$$

For the proof of the limiting absorption principle it suffices to replace (Q_1) in Assumption 1.1 by

$(Q_1)'$ $Q_1 \in C^0(\boldsymbol{R}^N)$ and

$$|Q_1(y)| \leqq C(1+|y|)^{-1-\varepsilon} \qquad (y \in \boldsymbol{R}^N) \tag{2.2}$$

with C and ε as in Assumption 1.1.

Theorem 2.2 (limiting absorption principle). *Let* (Q), (Q_0) *in Assumption and* (Q_1) *be satisfied and let* $N \geqq 3$.

(i) *Let* $(k, \ell) \in \boldsymbol{C}^+ \times F_0(I, X)$ *be given. Then the radiative function for* $\{L, k, \ell\}$ *is unique.*

(ii) *For given* $(k, \ell) \in \boldsymbol{C}^+ \times F_\delta(I, X)$ *there exists a unique radiative function* $v = v(\cdot, k, \ell)$ *for* $\{L, k, \ell\}$ *which belongs to* $L_{2,-\delta}(I, X) \cap H_0^{1,B}(I, X)_{loc}$. *The mapping*

$$\boldsymbol{C}^+ \times F_\delta(I, X) \ni (k, \ell) \mapsto v(\cdot, k, \ell) \in L_{2,-\delta}(I, X) \cap H_0^{1,B}(I, X)_{loc} \tag{2.3}$$

is continuous as a mapping from $\boldsymbol{C}^+ \times F_\delta(I, X)$ *into* $L_{2,-\delta}(I, X)$ *and is also continuous as a mapping from* $\boldsymbol{C}^+ \times F_\delta(I, X)$ *into* $H_0^{1,B}(I, X)_{loc}$.

(iii) *Let K be a compact set in* $\boldsymbol{C}^+$. *Let* $v = v(\cdot, k, \ell)$ *be the radiative function for* $\{L, k, \ell\}$ *with* $k \in K$ *and* $\ell \in F_\delta(I, X)$. *Then there exists a positive constant* $C = C(K)$, *depending only on* K (*and* L), *such that*

$$||v||_{-\delta} + ||v' - ikv||_{\delta-1} + ||B^{1/2}v||_{\delta-1} \leqq C|||\ell|||_\delta \tag{2.4}$$

and

$$||v||^2_{-\delta,(r,\infty)} \leqq C^2 r^{-(2\delta-1)} |||\ell|||^2_\delta \qquad (r \geqq 1). \tag{2.5}$$

Before proving this theorem we prepare

Lemma 2.3. *Let* $v \in H_0^{1,B}(I, X)_{loc}$ *be a solution of the equation* (2.1) *with* $k \in \boldsymbol{C}$ *and* $\ell = \ell[f]$ $(f \in L_2(I, X)_{loc})$, *where we set*

$$\langle \ell[f], \phi \rangle = (f, \phi)_0 . \tag{2.6}$$

Then v satisfies following (1)$\sim$(4):

(1) $v(r)$ *is an X-valued, strongly continuously differentiable function on* $\bar{I}$ *with its derivative* $v'(r)$. *We have* $v(0) = 0$.

(2) $v'(r)$ *is an X-valued, strongly absolutely continuous function on every compact interval in I, and* $v'(r)$ *is strongly differentiable almost everywhere on I with its derivative* $v''(r) \in L_2((a, b), X)$ *for any* $0 < a < b < \infty$.

(3) $v(r) \in D$ *for almost all* $r \in I$, *and* $B(r)v \in L_2((a, b), X)$ *for any* $0 < a < b < \infty$.

(4) *We have*

$$-v''(r) + B(r)v(r) + C(r)v(r) - k^2 v(r) = f(r) \qquad (\text{a.e. } r \in I). \tag{2.7}$$

Proof. Using (1.5), we obtain from the relation (2.1) with $l=l[f]$

$$(2.8)\qquad (U^{-1}v, (S-k^2)\varphi)_{L_2}=(U^{-1}f, \varphi)_{K_2}\qquad (\varphi\in C_0^\infty(\boldsymbol{R}^N)),$$

where $(\ ,\ )_{L_2}$ means the inner product of $L_2(\boldsymbol{R}^N)$. Then, as is well-known, $\varphi_0=U^{-1}v$ belongs to $H_2(\boldsymbol{R}^N)_{loc}$ (see Ikebe-Saitō [3], p. 536). Therefore there exists a sequence $\{\varphi_n\}\subset C_0^\infty(\boldsymbol{R}^N)$ such that $\varphi_n\to\varphi_0$ in $H_2(\boldsymbol{R}^N)_{loc}$. We set $v_n=U\varphi_n$. From the relations $v_n\to v$, $v_n'\to v'$ in $L_2((0, b), X)$ and $v_n''\to v''$, $Bv_n\to Bv$ in $L_2((a, b), X)$ $(0<a<b<\infty)$ we can easily obtain (1)~(4). Q.E.D.

Proof of Theorem 2.2. $B(r)$ and $C(r)$ satisfy Assumption 1.1 of [10] except the smoothness of $C_1(r)$[1]. Hence we can proceed as in the proof of Theorem 1.3 of [10], if we use Lemma 2.3 in place of Proposition 2.4 of [10]. Q.E.D.

Now we are in a position to state the main theorem. Here we may assume with no loss of generality that

$$(2.9)\qquad Q_0(y)=0\qquad (|y|\leqq 1)$$

Theorem 2.4 (asymptotic behavior of the radiative functions). *Let Assumption* 1.1 *and* (2.9) *be satisfied and let* $N\geqq 3$. *Then there exist real-valued functions* $Z(y)=Z(y, k)$ *on* $\boldsymbol{R}^N\times(\boldsymbol{R}-\{0\})$ *and* $\psi(\omega)=\psi(\omega,k)$ *on* $S^{N-1}\times(R^N-\{0\})$ *such that* $Z(y)\in C^2(\boldsymbol{R}^N)$, $\psi(\omega)\in C^2(S^{N-1})$ *and there exists the limit*

$$(2.10)\qquad \alpha=\alpha(k,f)=\operatorname*{s-lim}_{r\to\infty} e^{-i\mu(r\cdot,k)}v(r)\qquad \text{in } X$$

for any radiative function v *for* $\{L, k, l[f]\}$ *with* $k\in\boldsymbol{R}-\{0\}$ *and* $f\in L_{2,1+\delta-\varepsilon}(I, X)$, *where* $\mu(y, k)$ *is defined by*

$$(2.11)\qquad \mu(y, k)=rk-\{\int_0^r Z(t\omega)\,dt+\xi(r)\psi(\omega)\}\qquad (r=|y|,\ \omega=y/|y|),$$

and $\xi(r)$ *denotes a real-valued smooth function on* $[0, \infty)$ *such that*

$$(2.12)\qquad \xi(r)=\begin{cases}0 & (r\leqq 1),\\ 1 & (r\geqq 2).\end{cases}$$

This theorem will be proved by making use of the next

Theorem 2.5. *Let* $Q(y)$ *be as in Theorem* 2.4 *and let* $-\frac{1}{2}<\beta\leqq 1-\delta$.

(i) *Let* v *be the radiative function for* $\{L, k, l\}$ *with* $k\in\boldsymbol{R}-\{0\}$ *and* $l\in F_{1+\beta}(I, X)$. *Then we have* $u'-iku$, $B^{1/2}u\in L_{2,\beta}(I, X)$, *where* $u=e^{i\lambda}v$ *and*

$$(2.13)\qquad \lambda(y)=\lambda(y, k)=\int_0^r Z(t\omega)\,dt+\xi(r)\psi(\omega)\,.$$

1) The condition ($B3$) of Assumption 1.1 of [10] is justified by the Rellich lemma.

Further let K be a compact set in $\boldsymbol{R}-\{0\}$. Then there exists $C=C(K, \beta)$ such that

$$\|u'-iku\|_\beta+\|B^{1/2}u\|_\beta \leqq C|||\ell|||_{1+\beta} \qquad (u=e^{i\lambda}v) \tag{2.14}$$

for any radiative function v for $\{L. k, \ell\}$ with $k\in K$ nad $\ell\in F_{1+\beta}(I, X)$.

(ii) *There exists $C=C(K)$ such that for any radiative function v for $\{L, k, \ell\}$ $(k\in K, \ell\in F_{1+\delta-\varepsilon}(I, X))$ we have*

$$|v(r)|_X \leqq C|||\ell|||_{1+\delta-\varepsilon} \qquad (r\in I). \tag{2.15}$$

In the following section we shall construct $\lambda(y)$ and prove (i) of Theorem 2.5. Theorem 2.5, (ii) and Theorem 2.4 will be shown in §4.

3. An estimate for radiative function

Let us first consider the following problem: Find a real-valued function $\lambda(y)=\lambda(y, k)$ on $\boldsymbol{R}^N\times(\boldsymbol{R}-\{0\})$ such that

$$|(L-k^2)(e^{i\mu}x)|_X = O(r^{-1-\varepsilon}) \qquad (r\to\infty) \tag{3.1}$$

for any $x\in D$, where μ is given by (2.12). If $Q_0=0$, then $\lambda(y)=0$ is a solution of this problem. We shall construct a solution of this problem which will play an important role in this and the next sections. In order to solve this problem we have to investigate some properties of the Laplace-Beltrami operator Λ_N on S^{N-1}. Let us introduce polar coordinates $(r, \theta_1, \theta_2, \cdots, \theta_{N-1})$, i.e.,

$$\begin{cases} y_1 = r\cos\theta_1, \\ y_j = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{j-1}\cos\theta_j \qquad (j=2, 3, \cdots, N-1), \\ y_N = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-1}, \end{cases} \tag{3.2}$$

where $r\geqq 0$, $0\leqq\theta_1, \theta_2, \cdots, \theta_{N-2}\leqq\pi$, $0\leqq\theta_{N-1}\leqq 2\pi$. We set

$$\begin{cases} b_j = b_j(\theta) = \sin\theta_1\sin\theta_2\cdots\sin\theta_{j-1} \qquad (j=2, 3, \cdots, N-1), \\ b_1 = b_1(\theta) = 1, \\ M_j = M_j(\theta) = b_j(\theta)^{-1}\dfrac{\partial}{\partial\theta_j}. \end{cases} \tag{3.3}$$

Then, as is well-known, we have

$$\Lambda_N = \sum_{j=1}^{N-1} b_j(\theta)^{-2}(\sin\theta_j)^{-N+j+1}\frac{\partial}{\partial\theta_j}\left\{(\sin\theta_j)^{N-j-1}\frac{\partial}{\partial\theta_j}\right\} \tag{3.4}$$

(see, for example, Erdélyi and others [1], p. 235), and hence, sesting $A=-\Lambda_N+\dfrac{(N-1)(N-3)}{4}$ as in (1.6), we obtain

(3.5) $$|A^{1/2}x|_X^2=\sum_{j=1}^{N-1}|M_jx|_X^2+|c_Nx|_X^2\ (x\in\mathcal{D}(A^{1/2}),\ c_N=\left[\frac{(N-1)(N-3)}{4}\right]^{1/2})^{2)}.$$

Moreover we set

(3.6) $$\begin{cases}\varphi=\varphi(y)=\varphi(y,\lambda)=\sum_{j=1}^{N-1}r^{-2}(M_j\lambda)^2,\\ P=P(y)=P(y,\lambda)=r^{-2}(\Lambda_N\lambda)\\ \quad=\{\sum_{j=1}^{N-1}\sum_{n,p=j}^{N}b_j(\theta)^{-2}\int_0^r\frac{\partial_2 Z}{\partial y_n\partial y_p}y_{n,j}y_{p,j}|_{y=t\omega}dt\\ \quad-(N-1)\sum_{n=1}^{N}\int_0^r\frac{\partial Z}{\partial y_n}y_n|_{y=t\omega}dt+\xi(\Lambda_N\psi(\omega))\}r^{-2}.\end{cases}$$

Here λ is given by (2.13) and $y_{n,j}=\partial y_n/\partial\theta_j$.

Lemma 3.1. *Let $x\in D$. Then we have*

(3.7) $$\begin{aligned}(e^{\pm i\lambda}B(r)-B(r)e^{\pm i\lambda})x&=e^{\pm i\lambda}(-\varphi\pm 2ir^{-2}\sum_{j=1}^{N-1}(M_j\lambda)M_j\pm iP)x\\&=(\varphi\pm 2ir^{-2}\sum_{j=1}^{N-1}(M_j\lambda)M_j\pm iP)(e^{\pm i\lambda}x),\end{aligned}$$

and

(3.8) $$\begin{aligned}(L-k^2)(e^{i\mu}x)=e^{i\mu}\{&(B(r)+2ir^{-2}\sum_{j=1}^{N-1}(M_j\lambda)M_j)x\\&+(iP+iZ'+C_1+W)x-(2kZ-C_0-Z^2-\varphi)x\},\end{aligned}$$

where λ, φ, P are as above, $W=i\xi''\psi+\xi'\psi(\xi'\psi+2Z-2k)$, and ξ, M_j are defined by (2.12) *and* (3.3), *respectively.*

Proof. $A(e^{\pm i\lambda}x)$ can be calculated as follows:

(3.9) $$\begin{aligned}A(e^{\pm i\lambda}x)=e^{\pm i\lambda}Ax&+\sum_{j=1}^{N-1}(M_j\lambda)^2e^{\pm i\lambda}x\mp 2i\sum_{j=1}^{N-1}(M_j\lambda)(M_jx)e^{\pm i\lambda}\\&\mp i\sum_{j=1}^{N-1}b_j^{-2}\left\{\frac{\partial^2\lambda}{\partial\theta_j^2}+(N-j-1)\frac{\cos\theta_j}{\sin\theta_j}\frac{\partial\lambda}{\partial\theta_j}\right\}e^{\pm i\lambda}x.\end{aligned}$$

Since the fourth term of the right-hand side of (3.9) is equal to $\mp ir^2P(y)e^{\pm i\lambda}x$ by Lemma 5.7 of [11], the first relation of (3.7) follows from (3.9). The second relation can be obtained from the first relation and

(3.10) $$M_j(e^{\pm i\lambda}x)=\pm i(M_j\lambda)e^{\pm i\lambda}x+e^{\pm i\lambda}M_jx.$$

(3.8) follows from (3.9) and

(3.11) $$(e^{i\mu}x)''=-(iZ'+k^2+Z^2-2kZ+W)e^{i\mu}x.$$

Q.E.D.

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2) $\mathcal{D}(W)$ denotes the domain of W.

Let $\lambda(y)$ satisfy the estimates

$$(3.12)\qquad \begin{cases} D^j Z(y) = O(|y|^{-j-\varepsilon}) & (|y|\to\infty, j=0, 1, 2) \\ 2kZ(y)-Q_0(y)-Z(y)^2-\varphi(y) = O(|y|^{-1-\varepsilon}) & (|y|\to\infty). \end{cases}$$

Then, since $M_j\lambda=O(|y|^{1-\varepsilon})$ and $P(y)$, $Z'(y)=O(|y|^{-1-\varepsilon})$, it follows from (3.8) that (3.1) holds good. To obtain $\lambda(y)$ which satisfies (3.12) let us consider sequence $\{Z^{(n)}(y)\}$ and $\{\psi^{(n)}(\omega)\}$ defined by

$$(3.13)\qquad \begin{cases} Z^{(0)}(y) = (2k)^{-1}Q_0(y)\,, \\ Z^{(n+1)}(y) = (2k)^{-1}\{Q_0(y)+Z^{(n)}(y)^2+\varphi(y, \lambda^{(n)})\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (n=0, 1, 2, \cdots, m-1), \\ \lambda^{(n)}(y) = \int_0^r Z^{(n)}(t\omega)\,dt+\xi(r)\psi^{(n)}(\omega)\,, \qquad (n=0, 1, 2, \cdots, n_0), \\ \psi^{(n)}(\omega) = \begin{cases} 0 \qquad (n=0, 1, 2, \cdots, n_0). \\ -\int_0^\infty (Z^{(n)}(t\omega)-Z^{(n-1)}(t\omega))\,dt+\psi^{(n-1)}(\omega) \\ \qquad\qquad\qquad\qquad (n=n_0+1, \cdots, m-1), \end{cases} \end{cases}$$

where m is given by (1.8) and n_0 denotes a positive integer which satisfies $n_0\varepsilon<1$ and $(n_0+1)\varepsilon>1$

Lemma 3.2. *Let n be an integer such that $0\leqq n\leqq m$.*

(i) *Then $Z^{(n)}(y, -k)=-Z^{(n)}(y, k)$ $(y\in \boldsymbol{R}^N, k\in \boldsymbol{R}-\{0\})$ and $Z^{(n)}(y,k)=0$ for $|y|\leqq 1$, $k\in \boldsymbol{R}-\{0\}$.*

(ii) *We have*

$$(3.14)\qquad 2kZ^{(n)}(y)-Q_0(y)-Z^{(n)}(y)^2-\varphi(y, \lambda^{(n)}) = O(|y|^{-(n+2)\varepsilon}) \qquad (|y|\to\infty).$$

(iii) *Further, $Z^{(n)}(y)\in C^{m-n}(\boldsymbol{R}^N)$ and*

$$(3.15)\qquad |D^j Z^{(n)}(y)|\leqq C_n(1+|y|)^{-j-\varepsilon} \qquad (y\in \boldsymbol{R}^N, j=0, 1, \cdots, m-n)$$

with constants C_n.

If we note (1.7) in Assumption 1.1 and (2.9), then the proof is easy by induction, and hence we shall omit it.

DEFINITION 3.3. We set

$$(3.16)\qquad \begin{cases} Z(y) = Z^{(m-2)}(y),\ \psi(\omega) = \psi^{(m-2)}(\omega)\,, \\ Y(y) = 2kZ(y)-Q_0(y)-Z(y)^2-\varphi(y)\,. \end{cases}$$

$\lambda(y, k)$ is defined by (2.13).

REMARK 3.4. (1) From Lemma 3.2 we see that $Z(y)\in C^2(\boldsymbol{R}^N)$, $\psi(\omega)\in$

$C^2(S^{N-1})$ and

$$(3.17)\qquad \begin{cases} |D^j Z(y)| \leqq C(1+|y|)^{-j-\varepsilon} & (j=0, 1, 2), \\ |Y(y)| \leqq C(1+|y|)^{-2} \end{cases}$$

for $y \in \boldsymbol{R}^N$.

(2) In order to obtain $\lambda(y)$ which satisfies (3.12) it is sufficient to set $\lambda(y)=\lambda^{(n)}(y)$, $n=[1/\varepsilon]$. But we have taken $\lambda(y)=\lambda^{(m-2)}(y)$ for the sake of convenience of showing Theorems 2.4 and 2.5.

(3) In the case that $\varepsilon>1/2$ it suffices to set $Z(y)=Z^{(0)}(y)=(2k)^{-1}Q_0(y)$ and $\psi(\omega)=0$ (cf. Ikebe [2] or Saitō [10], [11]).

Now let us enter into the proof of (i) of Theorem 2.5. Through the remainder of this section β is taken to satisfy $-1/2<\beta\leqq 1-\delta$.

Proposition 3.5. *Let $Q(y)$ be a real-valued, continuous function on $\boldsymbol{R}^N$ $(N\geqq 3)$ such that $|Q(y)|\leqq C(1+|y|)^{-2}$ $(y\in\boldsymbol{R}^N)$. Let v be the radiative function for $\{L, k, \ell[f]\}$ with $k\in\boldsymbol{C}^+$ and $f\in L_{2,1+\beta}(I, X)$. Then we have $v'-ikv$, $B^{1/2}v\in L_{2,\beta}(I, X)$.*

Proof. The proof will be divided into three steps.

(I) Let $\phi\in UC_0^\infty(\boldsymbol{R}^N)$ and set $f=(L-k^2)\phi$. Then, taking the real part of the both sides of the relation $(\alpha(1+r)^{2\beta+1}(L-k^2)\phi, \phi'-ik\phi)_0=(\alpha(1+r)^{2\beta+1}f, \phi'-ik\phi)_0$ with a real-valued smooth function α on I such that $\alpha(r)=0$ $(r\leqq 1)$, $=1$ $(r\geqq 2)$, and using partial intagration, the interior estimate (Lemma 3.1 of [10]) and (2.4), we obtain

$$(3.18)\qquad \|\phi'-ik\phi\|_\beta+\|B^{1/2}\phi\|_\beta\leqq C\|f\|_{1+\beta}$$

with $C=C(k)$ which is bounded when k moves in a compact set in $\boldsymbol{C}^+$. Here we should note the relation $(L-k^2)\phi=-(\phi'-ik\phi)'-ik(\phi'-ik\phi)+B\phi+C\phi$. Then we can proceed as in the proof of (1.7) in Lemma 1.5 of [10].

(II) Next let us assume that $f\in L_{2,1+\beta}(I, X)$ and $k\in\boldsymbol{C}^+$ with Im $k>0$[3]. Then, by translating the argument used in the proof of Lemma 1.10 of Ikebe-Saitō [3] into our case by the use of the unitary operator U, we can find a sequence $\{\phi_n\}\subset UC_0^\infty(\boldsymbol{R}^N)$ such that $f_n=(L-k^2)\phi_n$ converges to f in $L_{2,1+\beta}(I, X)$. Then it follows from the continuity of the radiative function that ϕ_n converges to the radiative function v for $\{L, k, \ell[f]\}$ in $L_{2,-\delta}(I, X)\cap H_0^{1,B}(I, X)_{loc}$, whence follows that $v'-ikv$, $B^{1/2}v\in L_{2,\beta}(I, X)$ and that (3.18) holds good with ϕ replaced by v.

(III) Finally let us assume that $k\in\boldsymbol{R}-\{0\}$ and $f\in L_{2,1+\beta}(I, X)$. Then, setting $k_n=k+i/n$ and denoting by v_n the radiative function for $\{L, k_n, \ell[f]\}$, we

3) $I_m\, z$ $(R_e z)$ means the imaginary (real) part of z.

have (3.18) with $\phi=v_n$. If we let $n\to\infty$, then we can easily see that $v'-ikv$, $B^{1/2}v\in L_{2,\beta}(I, X)$, which completes the proof. Q.E.D.

The following proposition is the key lemma to the proof of (i) of Theorem 2.5. Let us set

$$(3.19) \qquad \rho_\gamma(G) = \sup\{(1+|y|)^\gamma |G(y)| / y\in \boldsymbol{R}^N\},$$

where $G(y)$ is a function on $\boldsymbol{R}^N$ and $\gamma\in\boldsymbol{R}$.

Proposition 3.6. *Let $Q(y)$ satisfy Assumption 1.1 with $N\geqq 3$ and let $\lambda(y)$ be as in Definition 3.3. Further $Q_0(y)$ is assumed to have compact support in $\boldsymbol{R}^N$. Then there exists $C=C(k, Q)$ such that the estimate*

$$(3.20) \qquad ||u''-iku||_\beta+||B^{1/2}u||_\beta\leqq C||f||_{1+\beta}$$

holds for any radiative function v for $\{L, k, \ell[f]\}$ lwith $k\in\boldsymbol{R}-\{0\}$ and $f\in L_{2,1+\beta}(I, X)$, where $u=e^{i\lambda}v$ and λ is given by (2.13). The constant $C=C(k, Q)$ is bounded when k moves in a compact set in $\boldsymbol{R}-\{0\}$ and $\rho_2(Q_1)$, $\rho_{j+\varepsilon}(D^jQ_0)$ $(j=0, 1, \cdots, m)$ are bounded.

In order to show this proposition we need several lemmas.

Lemma 3.7. (i) *Let u be as in Proposition 3.6. Then*

$$(3.21) \qquad \begin{aligned}-(u'-iku)'-ik(u'-iku)+Bu &= e^{i\lambda}f-2i(Z+\xi'\psi)(u'-iku)\\ &\quad +(Y-C_1-iZ'-iP-W)u-2ir^{-2}Mu,\end{aligned}$$

where we set

$$(3.22) \qquad M=\sum_{j=1}^{N-1}(M_j\lambda)M_j$$

and M_j, P and Y are given by (3.3), (3.6) and (3.16), respectively.

(ii) *Set $V(y)=\sum_{j=1}^{p}g_j(y)G_j$, with C^1 functions g_j on $\boldsymbol{R}^N-\{0\}$ and operators G_j in X such that $\mathcal{D}(G_j)\subset\mathcal{D}(A^{1/2})$, A being given in (1.6). Then*

$$(3.23) \qquad \int_R^T (Vu, u'-iku)_X dr = (2ik)^{-1}\{[(Vu, u'-iku)_X]_R^T$$
$$-\int_R^T (V(u'-iku), u'-iku)_X dr-\int_R^T (V'u, u'-iku)_X dr$$
$$+2i\int_R^T ((Z+\xi'\psi)Vu, u'-iku)_X dr+\int_R^T (Vu, (Y-C_1-iZ'-iP-W)u+e^{i\lambda}f)_X dr$$
$$-\int_R^T (Vu, Bu)_X dr+2i\int_R^T (Vu, r^{-2}Mu)_X dr\} \qquad (0<R<T).$$

(iii) *Let x, $x'\in D$ and let $S(y)$ be a C^1 function on $\boldsymbol{R}^N$. Then*

$$(3.24)\qquad (SMx, x')_X+(Sx, Mx')_X = -r^2(SPx, x')_X-((MS)x, x')_X\,.$$

Proof. Multiply the both sides of $(L-k^2)v=f$ by $e^{i\lambda}$. Then, by an easy computation, we arrive at (3.21). (ii) is obtained from (i) by the use of partial integration. (iii) also follows from partial integration if we note Lemma 5.7 of [11]. Q.E.D.

Lemma 3.8. *Let $Q(y)$ be as in Proposition* 3.6 *and let v be as in Proposition* 3.6, *too. Then we have*

$$(3.25)\qquad \left(\beta+\frac{1}{2}\right)\int_R^T \alpha r^2|u'-iku|_X^2\,dr+\left(\frac{1}{2}-\beta\right)\int_R^T \alpha r^{2\beta}|B^{1/2}u|_X^2\,dr$$
$$\leqq \eta(T)+C\{\|f\|_{1+\beta}^2+\int_R^T \alpha r^{2\beta-\varepsilon}(|u'-iku|_X^2+|B^{1/2}u|_X^2)\,dr\}$$
$$+\sum_{n=0}^{m-2} k^{-n-1}\mathrm{Re}\int_R^T \alpha r^{2\beta-1}(Z^nMu, Bu)_X\,dr\,,$$

where α is a real-valued, smooth function on I such that $\alpha(r)=0$ $(r\leqq R)$, $=1$ $(r\geqq R+1)$, $\eta(T)$ is a function of T satisfying $\underset{T\to\infty}{\mathrm{ilm}}\,\eta(T)=0$, $T>R+1$, and $C=C(k, Q)$ satisfies the same properties as in Proposition 3.6. *m is as in* (1.8).

Proof. Multiply the both sides of (3.21) by $\alpha r^{2\beta+1}\overline{(u'-iku)}$, integrate over the region $\{y\in \boldsymbol{R}^N/R<|y|<T\}$ and take the real part. Then we obtain

$$(3.26)\qquad K = \mathrm{Re}\int_R^T \alpha r^{2\beta+1}\{(Bu, u'-iku)_X-((u'-iku)', u'-iku_X)\}\,dr$$
$$= \mathrm{Re}\int_R^T \alpha r^{\beta+1}\{(e^{i\lambda}f, u'-iku)_X+((Y-C_1-W)u, u'-iku)_X\}\,dr$$
$$+\mathrm{Im}\int_R^T \alpha r^{2\beta+1}((Z'+P)u, u'-iku)_X\,dr$$
$$+2\mathrm{Im}\int_R^T \alpha r^{2\beta-1}(Mu, u'-iku)_X\,dr = K_1+K_2+K_3\,,$$

where M in K_3 of the right-hand side is defined by (3.22). The left-hand side K of (3.26) is estimated from below as follows:

$$(3.27)\qquad K\geqq\left(\beta+\frac{1}{2}\right)\int_R^T \alpha r^{2\beta}|u'-iku|_X^2\,dr+\left(\frac{1}{2}-\beta\right)\int_R^T \alpha r^{2\beta}|B^{1/2}u|_X^2\,dr$$
$$-\frac{1}{2}\int_R^T \alpha' r'^{2\beta+1}|B^{1/2}u|_X^2\,dr-\frac{1}{2}T^{2\beta+1}|u'(T)-iku(T)|_X^2\,.$$

Noting that $\beta\leqq 1-\delta$, we can estimate K_1 as

$$(3.28)\qquad K_1\leqq C\|f\|_{1+\beta}[\int_R^T \alpha r^{2\beta}|u'-iku|_X^2\,dr]^{1/2}\,,$$

where we have used (2.4) and the constant which depends only on k and $Q(y)$ will be denoted by the same symbol C in the sequel. Let us estimate K_2. Set $V=\alpha r^{2\beta+1}(Z'+P)$ in (3.23) and use the Schwarz inequality and (2.4). Then

$$(3.29)\qquad K_2 \leqq -\frac{1}{2k}\operatorname{Re}\{T^{2\beta+1}((Z'(T)+P(T))u(T),\ u'(T)+iku(T))_X\} \\ +CF(T)+k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta+1}(Z(Z'+P)u,\ u'-iku)_X\,dr \\ +k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta-1}((Z'+P)u,\ Mu)_X\,dr\,,$$

where we set

$$(3.30)\qquad F(T) = \|f\|_{1+\beta}^2+\int_R^T \alpha r^{2\beta-\varepsilon}\{|u'-iku|_X^2+|B^{1/2}u|_X^2\}\,dr\,.$$

K_3 can be estimated by the use of Lemma 3.7, (ii), too. Setting $V=\alpha r^{2\beta-1}M$ in (3.23), we obtain

$$(3.31)\qquad K_3 \leqq -k^{-1}\operatorname{Re} T^{2\beta-1}(Mu(T),\ u'(T)-iku(T))_X+CF(T) \\ +2k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta-1}(ZMu,\ u'-iku)_X\,dr \\ +k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta-1}(Mu,\ (Z'+P)u)_X\,dr \\ +k^{-1}\operatorname{Re}\int_R^T \alpha r^{2\beta-1}(Mu,\ Bu)_X\,dr\,.$$

Here we have used the relation

$$(3.32)\qquad \operatorname{Re}(M(u'-iku),\ u'-iku)_X = -\frac{1}{2}r^2(P(u'-iku),\ u'-iku)_X\,,$$

which follows from (2.24) with $x=x'=u'-iku$ and $S(y)=1$. Thus it follows from (3.26), (3.27), (3.28), (3.29), (3.31) and the interior estimate (Lemma 3.1 of [10]) that we obtain

$$(3.33)\qquad J=\left(\beta+\frac{1}{2}\right)\int_R^T \alpha r^{2\beta}|u'-iku|_X^2\,dr+\left(\frac{1}{2}-\beta\right)\int_R^T \alpha r^{2\beta}|B^{1/2}u|_X^2\,dr \\ \leqq \eta(T)+CF(T) \\ +k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta-1}\{((Z'+P)u,\ Mu)_X+(Mu,\ (Z'+P)u)_X\}\,dr \\ +k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta+1}(Z(Z'+P)u,\ u'-iku)_X\,dr \\ +2k^{-1}\operatorname{Im}\int_R^T \alpha r^{2\beta-1}(ZMu,\ u'-iku)_X\,dr \\ +k^{-1}\operatorname{Re}\int_R^T \alpha r^{2\beta-1}(Mu,\ Bu)_X\,dr=\sum_{p=1}^{6} J_p\,,$$

where

$$(3.34)\qquad \eta(T)=\frac{1}{2}T^{2\beta+1}|u'(T)-iku(T)|_X^2$$
$$-\frac{1}{2k}\operatorname{Re}\{T^{2\beta+1}((Z^1(T)+P(T))u(T),\ u'(T)-iku(T))_X\}$$
$$-k^{-1}\operatorname{Re}\{T^{2\beta-1}(Mu(T),\ u'(T)-iku(T))_X\}\ .$$

Since $v'-ikv$, $B^{1/2}\in L_{2,\beta}(I, X)$ by Proposition 3.5 and the support of $Z(y)$ is compact in $\boldsymbol{R}^N$ by the compactness of the support of $Q_0(y)$, it can be easily seen that $u'-iku$, $B^{1/2}u\in L_{2,\beta}(I, X)$, which implies that $\underline{\lim}_{T\to\infty}\eta(T)=0$. By using (3.24) J_3 is shown to be zero. J_4 and J_5 can be estimated in quite the same way as in the estimation of K_2 and K_3, respectively. Thus we obtain

$$(3.35)\qquad J\leqq\eta(T)+CF(T)$$
$$+k^{-2}\operatorname{Im}\int_R^T\alpha r^{2\beta+1}(Z^2(Z'+P)u,\ u'-iku)_X\,dr$$
$$+2k^{-2}\operatorname{Im}\int_R^T\alpha r^{2\beta-1}(Z^2Mu,\ u'-iku)_X\,dr$$
$$+k^{-1}\operatorname{Re}\int_R^T\alpha r^{2\beta-1}(Mu,\ Bu)_X\,dr$$
$$+k^{-2}\operatorname{Re}\int_R^T\alpha r^{2\beta-1}(ZMu,\ Bu)_X\,dr\ .$$

Repeating the above arguments, we arrive at

$$(3.36)\qquad J\leqq\eta(T)+CF(T)+\sum_{n=1}^{m-2}k^{-n-1}\operatorname{Re}\int_R^T\alpha r^{2\beta-1}(Z^nMu,\ Bu)_X\,dr$$
$$+k^{-(m-1)}\operatorname{Im}\int_R^T\alpha r^{2\beta+1}(Z^{m-1}(Z'+P)u,\ u'-iku)_X\,dr$$
$$+2k^{-(m-1)}\operatorname{Im}\int_R^T\alpha r^{2\beta-1}(Z^{m-1}Mu,\ u'-iku)_X\,dr\ ,$$

whence (3.25) follows directly. Q.E.D.

In order to show Proposition 3.6 completely we shall estimate the term $\operatorname{Re}(Z^nMu,\ Bu)$.

Lemma 3.9. *Let $S(y)$ be a real-valued C^1 function on $\boldsymbol{R}^N$ such that $|S(y)|\leqq c$ ($y\in\boldsymbol{R}^N$) and $|DS(y)|\leqq cr^{-1}$ ($|y|>1$) with a constant $c>0$. Then we have*

$$(3.37)\qquad |\operatorname{Re}(SMx,\ Ax)_X|\leqq Cr^{1-\varepsilon}(|A^{1/2}x|_X^2+|x|_X^2)\qquad(r\geqq1,\ x\in D)$$

with $C=C(c, k)$ which is bounded when c is bounded and k moves in a compact set in $\boldsymbol{R}-\{0\}$. A is given in (1.6).

Proof. We shall divede the proof into several steps.

(I) From (3.5) we obtain

$$(3.38)\qquad J=\mathrm{Re}\,(SMx,\ Ax)_X=\mathrm{Re}\sum_{n=1}^{N-1}(M_n(SMx),\ M_nx)_X \\ +c_N^2\,\mathrm{Re}\,(SMx,\ x)_X=J_1+J_2\,.$$

Throughout this proof we shall call a term K an O.K. term when K is dominated by $Cr^{1-\varepsilon}(|A^{1/2}x|_X^2+|x|_X^2)$ for $r\geqq 1$ with $C=C(c,\ k)$. Since $|SMx|_X\leqq Cr^{1-\varepsilon}|A^{1/2}x|_X$, we can easily show that J_2 is an O.K. term. Thus we have only to consider the term J_1.

(II) Let us calculate J_1

$$(3.39)\qquad J_1=\mathrm{Re}\sum_{n=1}^{N-1}((M_nS)(Mx),\ M_nx)_X \\ +\ \mathrm{Re}\sum_{n=1}^{N-1}(S\sum_{j=1}^{N-1}(M_nM_j\lambda)(M_jx),\ M_nx)_X \\ +\ \mathrm{Re}\sum_{n=1}^{N-1}(S\sum_{j=1}^{N-1}(M_j\lambda)(M_nM_jx),\ M_nx)_X=J_{11}+J_{12}+J_{13}\,.$$

By noting that $M_nS(y)$ is bounded on $\{y\in\boldsymbol{R}^N/|y|\geqq 1\}$ J_{11} is seen to be an O.K. term. Before calculating J_{13} we mention

$$(3.40)\qquad M_nM_j-M_jM_n=\begin{cases} b_j^{-1}\dfrac{\cos\theta_j}{\sin\theta_j}M_n & (n>j)\,,\\ 0 & (n=j)\,,\\ -b_n^{-1}\dfrac{\cos\theta_n}{\sin\theta_n}M_j & (n<j)\,,\end{cases}$$

which is clear from the definition of M_j (see (3.3)). Using (3.40), we have

$$(3.41)\qquad J_{13}=\mathrm{Re}\sum_{n=1}^{N-1}\{(S\sum_{j=1}^{n-1}(M_j\lambda)b_j^{-1}\frac{\cos\theta_j}{\sin\theta_j}(M_nx),\ M_nx)_X \\ -(S\sum_{j=n+1}^{N-1}(M_j\lambda)b_n^{-1}\frac{\cos\theta_n}{\sin\theta_n}(M_jx),\ M_nx)_N\} \\ +\ \mathrm{Re}\sum_{n=1}^{N-1}(SM(M_nx),\ M_nx)_X=J_{131}+J_{132}\,.$$

Here J_{132} is an O.K. term, because, making use of (3.24) with $x=x'=M_nx$, we have

$$(3.42)\qquad J_{132}=-\frac{1}{2}\sum_{n=1}^{N=1}\{r^2(SP(M_nx),\ M_nx)_X+((MS)M_nx,\ M_nx)_X\}\,.$$

Therefore let us consider $J_{12}+J_{131}$

(III) Set

$$(3.43)\quad \begin{cases} Z_p(r\omega) = \displaystyle\int_0^r \frac{\partial(Z+\xi'\psi)}{\partial y_p}\Big|_{y=t\omega} t\,dt \qquad (p=1, 2, \cdots, N), \\ Z_{pq}(r\mu) = (b_p b_q)^{-1} \displaystyle\sum_{j=p,\, n=q}^{N} \int_0^r \frac{\partial^2(Z+\xi'\psi)}{\partial y_j \partial y_n} y_{j,p} y_{n,q}\Big|_{y=t\omega} dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (p, q=1, 2, \cdots, N), \end{cases}$$

b_j, $y_{j,p}$ being given by (3.3), (3.6), respectively. Then, setting $\cos\theta_N \equiv 1$, we obtain

$$(3.44)\quad M_j\lambda = b_j^{-1}\{-b_{j+1}Z_j + \sum_{p=j+1}^{N} Z_p b_p \frac{\cos\theta_j}{\sin\theta_j}\cos\theta_p\}\,,$$

and

$$(3.45)\quad M_n M_j \lambda = \begin{cases} Z_{jn} & (j>n), \\ Z_{nn} - b_n^{-2} \displaystyle\sum_{p=n}^{N} Z_p b_p \cos\theta_p & (j=n), \\ Z_{jn} + b_j^{-1} \dfrac{\cos\theta_j}{\sin\theta_j}(M_n\lambda) & (j<n). \end{cases}$$

Thus J_{12} takes the form

$$(3.46)\quad \begin{aligned} J_{12} = \mathrm{Re} \sum_{n=1}^{N-1} \{ & (S\sum_{j=1}^{n-1} b_j^{-1} \frac{\cos\theta_j}{\sin\theta_j}(M_n\lambda)(M_j x),\ M_n x)_X \\ & -(S b_n^{-2} \sum_{p=n}^{N} Z_p b_p \cos\theta_p (M_n x),\ M_n x)\}_X \\ & + \mathrm{Re} \sum_{n=1}^{N-1} (S \sum_{j=1}^{N-1} Z_{jn}(M_j x),\ M_n x)_X = J_{121} + J_{122}\,. \end{aligned}$$

J_{122} is an O.K. term, because $Z_{jn}(y) = O(|y|^{1-\varepsilon})$ by the first estimate of (3.17). Hence, in place of $J_{12}+J_{131}$, it is sufficient to consider

$$(3.47)\quad \begin{aligned} J' &= J_{121} + J_{131} \\ &= \sum_{n=1}^{N-1} (SF_n(M_n x),\ M_n x)_X + \mathrm{Re}\sum_{n=1}^{N-1} (SG_n,\ M_n x)_X = J_1' + J_2' \end{aligned}$$

with

$$(3.48)\quad F_n = \sum_{j=1}^{n-1} (M_j\lambda) b_j^{-1} \frac{\cos\theta_j}{\sin\theta_j} - \sum_{p=n}^{N} Z_p b_p b_n^{-2} \cos\theta_p = F_{n1} - F_{n2}\,,$$

and

$$(3.49)\quad G_n = \sum_{j=1}^{n-1} b_j^{-1} \frac{\cos\theta_j}{\sin\theta_j}(M_n\lambda)(M_j x) - \sum_{j=n+1}^{N-1} b_n^{-1} \frac{\cos\theta_n}{\sin\theta_n}(M_j\lambda)(M_j x)\,.$$

(IV) Now let us calculate F_{n1}. Using (3.44) and interchanging the order of summation, we arrive at

$$(3.50)\quad F_{n1} = -\sum_{p=2}^{N} Z_p b_p \cos\theta_p + \sum_{p=2}^{N} Z_p b_p b_n^{-2} \cos\theta_p + Z_1 \cos\theta_1\,,$$

and hence

$$F_n = -\sum_{p=2}^{N} Z_p b_p \cos\theta_p + Z_1 \cos\theta_1 , \tag{3.51}$$

which implies that J_1' is an O.K. term. As for J_2' we can interchange the order of summation to obtain

$$\begin{aligned} J_2' = \operatorname{Re}\sum_{n=1}^{N-1}\sum_{j=1}^{n-1} \{ & (Sb_j^{-1}\frac{\cos\theta_j}{\sin\theta_j}(M_n\lambda)(M_j x),\ M_n x)_X \\ & -(Sb_j^{-1}\frac{\cos\theta_j}{\sin\theta_j}(M_n\lambda)(M_n x),\ M_j x))_X\} = 0 . \end{aligned} \tag{3.52}$$

Thus we have shown that each term of $J = \operatorname{Re}(SMx, Ax)_X$ is an O.K. term, which completes the proof. Q.E.D.

Proof of Proposition 3.6. By the use of Lemma 3.9 the last term of the right-hand side of (3.25) is dominated by $CF(T)$, where $F(T)$ is defined by (3.30) and (2.4) has been used. Therefore by letting $T \to \infty$ along a suitable sequence $\{T_n\}$ in (3.25) we obtain

$$\left(\beta+\frac{1}{2}\right)\int_R^\infty \alpha r^{2\beta} |u'-iku|_X^2 dr + \left(\frac{1}{2}-\beta\right)\int_R^\infty \alpha r^{2\beta} |B^{1/2}u|_X^2 dr \leqq CF(T) . \tag{3.53}$$

Take R sufficiently large in (3.53). Then it follows that

$$\int_{R+1}^\infty r^{2\beta}(|u'-iku|_X^2 + |B^{1/2}u|_X^2) dr \leqq C\|f\|_{1+\beta}^2 \qquad (C=C(k, Q)), \tag{3.54}$$

which, together with the interior estimate (Lemma 3.1 of [10]), yields to (3.20). Q.E.D.

Now that we have shown Proposition 3.6, we can prove (i) of Theorem 2.5.

Proof of (i) of Theorem 2.5. Let v be the radiative function for $\{L, k, \ell\}$ with $k \in K$, the compact set of $\boldsymbol{R}-\{0\}$, and $\ell \in F_{1+\beta}(I, X)$. Let $k_0 \in \boldsymbol{C}^+$ such that $\operatorname{Im} k_0 < 0$. Then v can be decomposed as $v = v_0 + w$, where v_0 is the radiative function for $\{L, k_0, \ell\}$ and w is the radiative function for $\{L, k, \ell[f]\}, f = (k^2 - k_0^2) v_0$ (Lemma 1.8 of [10]). It follows follows from Lemma 1.7 of [10] that $v_0 \in L_{2,1+\beta}(I, X)$ and

$$\|u_0' - iku_0\|_\beta + \|B^{1/2}u_0\|_\beta + \|v_0\|_{1+\beta} \leqq C_0 \||\ell\||_{1+\beta} \qquad (u_0 = e^{i\lambda} v_0) \tag{3.55}$$

with $C_0 = C_0(k_0, k, \beta)$[4), and hence it suffices to show the estimate (3.20) with

4) It should be noted that Lemmas 1.7 and 1.8 are valid in our cass if the space $C_0^{2,B}(I, X)$ is replaced by $UC_0^\infty(R^N)$.

$u=e^{i\lambda}w$. To this end we shall approximate $Q_0(y)$ by a sequence $\{Q_{0n}(y)\}$, where we set

$$(3.56)\qquad Q_{0n}(y)=s_n(|y|)Q_0(y)\,,\qquad s_n(r)=s(r/n),$$

and $s(r)$ is a real-valued, smooth function on I such that $s(r)=1$ $(r\leqq 1)$, $=0$ $(r\geqq 2)$. Then it can be easily shown that for each $j=0, 1, \cdots, m\rho_{j+\varepsilon}(D^jQ_{0n})$ is bounded uniformly for $n=1, 2, \cdots$. Let us set

$$(3.57)\qquad L_n=-\frac{d^2}{dr^2}+B(r)+C_{0n}(r)+C_1(r)\qquad (C_{0n}(r)=Q_{0n}(r\omega)\times),$$

and let us denote by w_n the radiative function for $\{L_n, k, \ell[f]\}$ with $f=(k^2-k_0^2)v_0$ $(n=1, 2, \cdots)$. For each L_n the function $Z_n(y)$ and $\psi_n(\omega)$ can be constructed according to Definition 3.3 with $Q_0(y)$ replaced by $Q_{0n}(y)$ and we set

$$(3.58)\qquad u_n=e^{i\lambda_n}w_n\qquad (\lambda_n(r\omega)=\int_0^r Z_n(t\omega)\,dt+\xi(r)\psi_n(\omega))\,.$$

Now Proposition 3.6 can be applied to show

$$(3.59)\qquad \|u_n'-iku_n\|_\beta+\|B^{/12}u_n\|_\beta\leqq C\|f\|_{1+\beta}\qquad (k\in K),$$

$C=C(K, \beta)$ being independent of $n=1, 2, \cdots$. Since $D^jQ_{0n}(y)$ converges to $D^jQ_0(y)$ as $n\to\infty$ uniformly on $\boldsymbol{R}^N$ for each $j=0, 1, \cdots, m$, it follows that $\lambda_n(y)\to\lambda(y)$ $(n\to\infty)$ uniformly on every compact set in $\boldsymbol{R}^N$. Therefore, by the use of Theorem 4.1, of [10], we obtain $u_n\to u$ in $H_0^{1,B}(I, X)_{loc}$ as $n\to\infty$. Thus, letting $n\to\infty$ in the relation

$$(3.60)\qquad \|u_n'-iku_n\|_{\beta,(0,R)}+\|B^{1/2}u_n\|_{\beta,(0,R)}\leqq C\|f\|_{1+\beta}\qquad (R\in I),$$

which is a direct consequence of (3.59), we have

$$(3.61)\qquad \|u'-iku\|_{\beta,(0,R)}+\|B^{1/2}u\|_{\beta,(0,R)}\leqq C\|f\|_{1+\beta}\,.$$

Since $R>0$ is arbitrary, we have obtained (2.14). Q.E.D.

4. Proof of the main theorem

In this section we shall prove (ii) of Theorem 2.5 and Theorem 2.4 by using Theorem 2.5, (i) which has been proved in the preceding section.

(ii) of Theorem 2.5 follows from (i) of Theorem 2.5 quite similarly as in the proof of (4.16) in Theorem 4.3 of [10].

Proof of (ii) of Theorem 2.5. Let us first consider the case that v is the radiative function for $\{L. k, \ell[f]\}$ with $k\in K$ and $f\in L_{2,1+\delta-\varepsilon}(I, X)$. Using (i) of Lemma 3.7, we have

$$(4.1)\qquad \begin{cases} \dfrac{d}{dr}\{e^{2irk}(u'(r)-iku(r),\, u(r))_X\} = e^{2irk}g(r)\,, \\ g(r) = |u'-iku|_X^2+|B^{1/2}u|_X^2+(e^{i\lambda}f,\, u)_X \\ \qquad\qquad +2i((Z+\xi'\psi_r)(u'-iku),\, u)_X \\ \qquad\qquad -((Y-C_1-iZ'-iP-W)u,\, u)_X+2ir^{-2}(Mu,\, u)_X\,. \end{cases}$$

It follows from (2.14) with $\beta=\delta-\varepsilon$ that $g(r)$ is integrable over I with the estimate

$$(4.2)\qquad \begin{aligned} \int_I |g(r)|\,dt &\leqq \|u'-iku\|_0^2+\|B^{1/2}u\|_0^2+\|f\|_\delta\|v\|_{-\delta} \\ &\quad +C\{\|u'-iku\|_{\delta-\varepsilon}\|v\|_{-\delta}+\|v\|_{-\delta}^2+\|B^{1/2}u\|_{\delta-\varepsilon}\|v\|_{-\delta} \\ &\leqq C'\|f\|_{1+\delta-\varepsilon}^2\,, \end{aligned}$$

where C, C' are positive constants and we have made use of (2.4), too. By starting with (4.1) and (4.2) the estimate

$$(4.3)\qquad |v(r)|_X \leqq C\|f\|_{1+\delta-\varepsilon} \qquad (r\in I,\ C=C(K))$$

can be shown in the very same way as in the proof of Lemma 4.6, (II) of [10]. Next let us consider the general case. Let $k_0\in \boldsymbol{C}^+$ with $\operatorname{Im} k_0>0$ be fixed. Then, as in the proof of (i) of Theorem 2.5, the radiative function v for $\{L, k, \ell\}$ can be decomposed as $v=v_0+w$, where v_0 is the radiative function for $\{L, k_0, \ell\}$ with $v_0\in H_0^{1,B}(I, X)\cap L_{2,1+\delta-\varepsilon}(I, X)$ and w denotes the radiative function for $\{L, k, \ell[(k^2-k_0^2)v_0]\}$. Let us estimate v_0 and w separately. It follows from Lemma 1.7 of [10] and the inequality $|v_0(r)|_X \leqq \sqrt{2}\,\|v_0\|_B$, which is shown in the same way as in the proof of (2.34) in Lemma 2.5 of [10], that

$$(4.4)\qquad |v_0(r)|_X \leqq C|||\ell|||_{1+\delta-\varepsilon} \qquad (r\in I,\ C=C(K)).$$

On the other hand we obtain from (4.3)

$$(4.5)\qquad |w(r)|_X \leqq C\|v_0\|_{1+\delta-\varepsilon} \qquad (r\in I,\ C=C(K)),$$

which, together with (4.4) and Lemma 1.7 of [10], yields to (2.15). Q.E.D.

Proof of Theorem 2.4. Set $\beta=\delta-\varepsilon$. Theorem 2.4 will be proved by proceeding along a similar line to the one in the proof of Theorem 5.1 of [10]. First we shall show that $|v(r)|_X$ tends to a limit as $r\to\infty$. In fact, starting with (4.1) and the relation

$$(4.6)\qquad \frac{d}{dr}\{\operatorname{Im}(u'(r),\, u(r))_X\} = \operatorname{Im} g(r)\,,$$

and noing that $(u'(R_n)-iku(R_n),\, u(R_n))_X\to 0$ for some sequence $\{R_n\}$, $R_n\to\infty$, we obtain

$$(4.7)\qquad |v(r)|_X^2 = k^{-1}\{\mathrm{Im}\,(u'(r),\,u(r))_X+\mathrm{Im}\int_r^\infty e^{2ik(t-r)}g(t)\,dt\}$$
$$= k^{-1}\{\mathrm{Im}\,(u'(R),\,u(R))_X+\mathrm{Im}\int_R^r g(t)\,dt+\mathrm{Im}\int_r^\infty e^{2ik(t-r)}g(t)\,dt\}$$

with $R>0$ fixed, whence follows the existence of the limit. Now let us set

$$(4.8)\qquad \alpha_k(r) = (2ik)^{-1}e^{-i\mu(r,k)}(v'(r)+ikv(r))\,.$$

Then, by an easy computation, we have

$$(4.9)\qquad e^{-i\mu(r,k)}v(r) = \alpha^k(r)+e^{-2i\mu(r,k)}\alpha_{-k}(r)$$

and

$$(4.10)\qquad \alpha_k'(r) = (2ik)^{-1}e^{-ikr}\{Bu-e^{i\lambda}f+i(Z+\xi'\psi)(u'-iku)$$
$$+(W+C_1-Y+iP)u+2ir^{-2}\sum_{j=1}^{N-1}(M_j\lambda)(M_ju)\}\,,$$

and hence we obtain for $x\in D$

$$(4.11)\qquad \begin{cases} (\alpha_k(r),\,x)_X = (\alpha_k(1),\,x)_X+\displaystyle\int_1^r(2ik)^{-1}e^{-ikt}g(t,\,x)\,dt \\ \quad g(r,\,x) = (u,\,Bx)_X-(r^{i\lambda}f,\,x)_X+i((Z+\xi'\psi)(u'-iku),\,x)_X \\ \qquad\qquad +((C_1-Y+iP-W)u,\,x)_X-2ir^{-2}(u,\,Mu)_X-2i(Pu,\,x)_X \\ \qquad\qquad = \displaystyle\sum_{j=1}^{6}g_j(r,\,x)\,, \end{cases}$$

where we have used the relation (3.24) in Lemma 3.7, (iii). Therefore it follows from Theorem 2.4, (i) with $\beta=\delta-\varepsilon$ that $g(r,\,x)$ is integrable over $(1,\,\infty)$, which implies the convergence of $(\alpha_k(r),\,x)_X$ as $r\to\infty$. On the other hand setting $h(r)=e^{irk}(u'(r)-iku(r),\,x)_X$, $x\in D$, and proceeding as in the proof that $\lim_{r\to\infty}(\alpha_k(r),\,x)_X=\alpha_x$ exists, we can show that $\lim_{r\to\infty}h(r)=0$. In fact by the use of Lemma 3.7 and Theorem 2.5 it can be shown that $h'(r)=e^{ikr}((u'-iku)'+ik(u'-iku),\,x)_X$ is integrable over $(1,\,\infty)$, which implies the existence of $\lim_{r\to\infty}h(r)$. Moreover, since $u'-iku\in L_{2,1+\delta-\varepsilon}(I,\,X)$ we have $h(r_n)\to 0$ $(n\to\infty)$ along some sequence $\{r_n\}$, whence we obtain $\lim_{r\to\infty}h(r)=0$. Note that $\lim_{r\to\infty}e^{-2ikr}h(r)=0$ and $\lim_{r\to\infty}(Z+\xi'\psi)\,v(r)=0$, then we arrive at

$$(4.12)\qquad \begin{cases} \lim\limits_{r\to\infty}\{(e^{-i\mu}v'(r),\,x)_X-ik(e^{-i\mu}v(r),\,x)_X\} = 0\,, \\ \lim\limits_{r\to\infty}\{(e^{-i\mu}v'(r),\,x)_X+ik(e^{-i\mu}v(r),\,v)_X\} = 2ik\alpha_x\,, \end{cases}$$

and hence $\lim_{r\to\infty}(e^{-i\mu}v(r),\,x)_X=\alpha_x$ for any $x\in D$. Thus taking note of the boundedness of $|v(r)|_X$ and the denseness of D in X, we have established the weak convergence of $\{e^{-i\mu}v(r)\}$ as $n\to\infty$. Set $\alpha=w-\lim_{r\to\infty} e^{-i\mu}v(r)$. Then the proof of Theorem 2.4 will be complete if we can find a sequence $\{r_n\}$ such that

$$(4.13)\qquad \lim_{n\to\infty} |v(r_n)|_X = |\alpha|_X\,.$$

Let us take a sequence $\{r_n\}$ which satisfies

$$(4.14)\qquad \begin{cases} r_n^{\beta+(1/2)}|u'(r_n)-iku(r_n)|_X \leqq c_0\,, \\ r_n^{\beta+(1/2)}|B(r_n)^{1/2}u(r_n)|_X \leqq c_0\,, \\ |v(r_n)|_X \leqq c_0\,, \\ \lim_{n\to\infty} |v'(r_n)-ikv(r_n)|_X = 0 \qquad (\beta=\delta-\varepsilon) \end{cases}$$

with a constant $c_0>0$. Such a $\{r_n\}$ surely exists by Theorem 2.5, (i). Then from (4.9) we obtain

$$(4.15)\qquad \begin{aligned} |v(r_n)|_X^2 &= (e^{i\mu(r_n\cdot)}\alpha_k(r_n)+e^{-i\mu(r_n\cdot)}\alpha_{-k}(r_n),\ v(r_n))_X \\ &= (\alpha_k(r_n),\ e^{-i\mu(r_n\cdot)}v(r_n))_X \\ &\qquad\qquad +(\alpha_{-k}(r_n),\ e^{i\mu(r_n\cdot)}v(r_n))_X = a_n+b_n\,. \end{aligned}$$

Obviously we have $b_n\to 0$ as $n\to\infty$, and hence it is sufficient to show that $a_n\to|\alpha|_X^2$. Setting

$$(4.16)\qquad \alpha_n = e^{-i\mu(r_n\cdot)}v(r_n)\,,$$

we obtain

$$(4.17)\qquad \begin{aligned} a_n &= (\alpha_k(R),\ \alpha_n)_X-\int_{r_n}^{R}(\alpha_k'(t),\ \alpha_n)_X\,dt \\ &= (\alpha_k(R),\ \alpha_n)_X-\int_{r_k}^{R}(2ik)^{-1}e^{-ik(t-r_n)}g(t,\ u(r_n))\,dt\,, \end{aligned}$$

where $g(r, x)$ is given by (4.11). Now we shall construct a function $g_0(r)$ on $(1, \infty)$ which is integrable on $(1, \infty)$ and dominates $g(r, u(r_n))$ uniformly for n in the sense that

$$(4.18)\qquad |g(r,\ u(r_n))| \leqq g_0(r) \qquad (r\geqq r_n(\geqq 1),\ n=1, 2, \cdots,)\,.$$

In fact, the following estimates are obvious:

$$(4.19)\qquad \begin{cases} |g_2(r,\ u(r_n))| \leqq c|f(r)|_X = g_{02}(r)\,, \\ |g_4(r,\ u(r_n)|+|g_6(r,\ u(r_n))| \leqq c_0Cr^{-1-\varepsilon}|v(r)|_X = g_{04}(r) \qquad (r\geqq 1). \end{cases}$$

As for $g_1(r,\ u(r_n))=(u(r),\ B(t)u(r_n))_X$ we have from (4.14)

$$(4.20)\qquad \begin{aligned}|g_1(r, u(r_n))| &\leqq c_0 r^{-1} r_n^{(1/2)-\beta} |B(r)^{1/2} u(r)|_X \\ &\leqq c_0 r^{-(1/2)-2\beta} |r^\beta B(r)^{1/2} u(r)|_X = g_{01}(r) \qquad (r \geqq r_n,\ \beta = \delta - \varepsilon).\end{aligned}$$

Quite similarly we obtain for $r \geqq r_n$ with $\beta = \delta - \varepsilon$

$$(4.21)\qquad \begin{cases} |g_3(r, u(r_n))| \leqq c_0 r^{-\beta-\varepsilon} |r^\beta (u'(r) - iku(r))|_X = g_{03}(r)\,, \\ |g_5(r, u(r_n))| \leqq c_0^2 C r^{-(\varepsilon+\beta+(1/2))} = g_{05}(r)\,. \end{cases}$$

Combining (4.19)~(4.12), we may set $g_0(r) = \sum_{j=1}^{5} g_{0j}(r)$. Here it should be noted that $\beta + \varepsilon > 1/2$ and $\varepsilon + \beta + (1/2) > 1$. Now that the existence of $g(r)$ has been shown, (4.13) is obtained by letting $n \to \infty$ in the relation

$$(4.22)\qquad |v(r_n)|_X^2 = (\alpha, \alpha_n)_X - \int_{r_n}^{\infty} (2ik)^{-1} e^{-ik(t-r_n)} g(t, u(r_n))\, dt + b_n\,,$$

which follows from (4.15) and (4.17) with $R \to \infty$ along a sequence $\{r_m\}$ which satisfies $v'(r_m) - ikv(r_m) \to 0$ in X. Q.E.D.

5. The case of $N=2$

Now we shall consider the Schrödinger operator $S = -\Delta + Q(y)$ in $\boldsymbol{R}^2$. As in the preceding sections we set $L = USU^{-1} = -d^2/dr^2 + B(r) + C(r)$, where U is the multiplication operator by $r^{1/2}$. In this case the operator

$$(5.1)\qquad B(r) = r^{-2}(-\Lambda_2 - 1/4)$$

is not necessarily non-negative definite and, further, the element v of $UH_1(\boldsymbol{R}^2)$ does not necessarily belong to $L_2(I, X)$. We, therefore, have to modify the arguments in §2~§4.

Let us set $H_0^{1,B}(I, X) = UH_1(\boldsymbol{R}^2)$ and define the inner product and norm of $H_0^{1,B}(I, X)$ by

$$(5.2)\qquad \begin{cases} (v, w)_B = (V, W)_1\,, \\ \|v\|_B = \sqrt{(V, V)_1}\,, \end{cases}$$

where $v = UV$, $w = UW$ with $V, W \in H_1(\boldsymbol{R}^2)$ and $(\ ,\ )_1$ denotes the inner product of $H^1(\boldsymbol{R}^2)$. Obviously $H_0^{1,B}(I, X)$ is a Hilbert space.

In the case of $N \geqq 3$ we have used the estimate $|v(r)|_X \leqq \sqrt{2}\|v\|_B$ ([9], (2.6)), where the fact that $B(r) \geqq 0$ has been only employed. But in this paper, which deals with the concrete operator $S = -\Delta + Q(y)$, a sharper estimate can be shown for $v \in H_0^{1,B}(I, X) = UH_1(\boldsymbol{R}^N)$ with $N \geqq 2$.

Lemma 5.1. *Let $v \in H_0^{1,B}(I, X) = UH_1(\boldsymbol{R}^N)$ with $N \geqq 2$. Then $v(r)$ is an X-valued continuous function on $[0, \infty)$ with $v(0) = 0$, and the estimate*

(5.3) $\qquad |v(r)|_X \leqq \|v\|_B \qquad (r\in[0, \infty)).$

Proof. Starting with the relation

$$(5.4) \qquad -2\mathrm{Re}\int_{|y|\geq r}\phi\frac{\partial\phi}{\partial r}dy = |v(r)|_X^2+\int_{|y|\geq r}\frac{N-1}{|y|}|\phi(y)|^2dy \qquad (v=U\phi\in UC_0^\infty(\boldsymbol{R}^N)),$$

we obtain (5.3) for $v\in UC_0^\infty(\boldsymbol{R}^N)$. As for $v\in H_0^{1,B}(I, X)$ there exists a sequence $\{v_n\}\subset UC_0^\infty(\boldsymbol{R}^N)$ such that v_n converges to v in $H_0^{1,B}(I, X)$ as $n\to\infty$. It follows from (5.3) with $v=v_n$ that $v_n(r)$ converges to $v(r)$ in X uniformly for $r\in[0, \infty)$. Therefore $v(r)$ is an X-valued continuous function on $[0, \infty)$ with $v(0)=0$ and the estimate (5.3) holds for v, too. Q.E.D.

In this section, instead of $B(r)$ and $Q(y)$, we shall use

$$(5.5) \qquad \begin{cases} \tilde{B}(r) = r^{-2}(-\Lambda_2)\,, \\ \tilde{Q}(y) = Q_0(y)+\tilde{Q}_1(y) \qquad (\tilde{Q}_1(y) = Q_1(y)-4r^{-2})\,. \end{cases}$$

Since $\tilde{Q}_1(y)$ has a singularity at $y=0$, $\tilde{Q}_1(y)$ does not satisfy the condition (Q_1) in Assumption 1.1. But $\tilde{Q}_1(y)=O(|y|^{-2})$ as $|y|\to\infty$, and the analysis in a neighborhood of $y=\infty$ proceeds as in the preceding sections. Here we shall note that the interior estimate (Lemma 3.1 of [10]) has the following form.

Lemma 5.2. *Let $v\in H_0^{1,B}(I, X)_{loc}$ satisfy*

(5.6) $\qquad (v, (L-\bar{k}^2)\phi)_0 = \langle \ell, \phi\rangle \qquad (\phi\in UC_0(\boldsymbol{R}^2))$

with $k\in\boldsymbol{C}$ and $\ell\in F_0(I, X)$. Then for any $R>0$ there exists $C=C(R, k)$ such that

(5.7) $\qquad \|v\|_{B,(0,R)}\leqq C\{\|v\|_{0,(0,R+1)}+|||\ell|||_{0,(0,R+1)}\}\,,$

where we set for an open interval $J\subset I$

(5.8) $\qquad \|v\|_{B,J} = [\int_{y\in J}(|\nabla V|^2+|V|^2)dy]^{1/2} \qquad (v=UV,\ V\in H_1(\boldsymbol{R}^2)_{loc})\,.$

By using the relation $U^{-1}LU=S$ the proof is easy, and we shall omit it.

Now we have to modify the definition of the radiative function slightly. Let us set $I_1=(1, \infty)$. A solution $v\in H_0^{1B}(I, X)_{loc}$ of the equation (5.6) with $k\in\boldsymbol{C}^+$ and $\ell\in F_0(I, X)$ will be called the radiative function for $\{L, k, \ell\}$ if $v'-ikv\in L_{2,\delta-1}(I_1, X)$. Then it can be shown that the results of Theorems 2.2, 2.4 and 2.5 are valid in the case of $N=2$ without any serious alteration. The method of the proof is essentially the same as in the preceding sections and [10]. Therefore the proof of the theorems below will be left to the reader.

Theorem 5.3. *Let Assumption* 1.1 *with $N=2$ be satisfied. Then all the* *results of Theorem* 2.2 *hold if we have only to replace* (2.4) *by*

(5.9) $$||v||_{-\delta}+||v'-ikv||_{\delta-1,I_1}+||\tilde{B}^{1/2}v||_{\delta-1,I_1}\leqq C|||f|||_{\delta}\,.$$

Theorem 5.4. *Let Assumption* 1.1 *with* $N=2$ *and* (2.9) *be satisfied. Then all the results of Theorem* 2.5 *are valid if we have only to replace* (2.14) *by*

(5.10) $$||u'-iku||_{\beta,I_1}+||\tilde{B}^{1/2}u||_{\beta,I_1}\leqq C|||f|||_{1+\beta}\,.$$

Theorem 5.5. *Let Assumption* 1.1 *with* $N=2$ *and* (2.9) *be satisfied. Then all the results of Theorem* 2.4 *hold good.*

Finally let us apply Theorem 2.4 to the Schrödinger operator S.

Theorem 5.6. *Let Assumption* 1.1 *be satisfied. Let* $V\in H_2(\boldsymbol{R}^N)_{loc}\cap L_2(\boldsymbol{R}^N, (1+|y|)^{-2\delta}dy)$ *be a unique solution of the equation*

(5.11) $$\begin{cases}(S-k^2)V=F\,,\\ \dfrac{\partial V}{\partial|y|}-ikV\in L_2(E_1, (1+|y|)^{2\delta-2}dy)\end{cases}$$

with $k\in\boldsymbol{R}-\{0\}$ *and* $F\in L_2(\boldsymbol{R}^N, (1+|y|)^{2+2\delta-2\varepsilon}dy)$, *where we set* $E_1=\{y\in\boldsymbol{R}^N/|y|>1\}$. *Then there exists a strong limit*

(5.12) $$\alpha(k, F)=s-\lim_{r\to\infty}e^{-i\mu(r\cdot,k)}r^{(N-1)/2}V(r\cdot)$$

in $L_2(S^{N-1})$, $\mu(y, k)$ *being defined by* (2.12).

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