Kawauch, A.
Osaka J. Math.
13 (1976), 567–590

$\tilde{H}$-COBORDISM, I; THE GROUPS AMONG THREE DIMENSIONAL HOMOLOGY HANDLES

AKIO KAWAUCHI

(Received September 25, 1975)

This paper will introduce a concept of a cobordism theory, called $\tilde{H}$-cobordism, between 3-dimensional homology handles. The set of the types of distinguished homology orientable handles modulo $\tilde{H}$-cobordism relation will form an abelian group $\Omega(S^1\times S^2)$, called *the $\tilde{H}$-cobordism group of homology orientable handles*. As a basic property of the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$ the following commutative triangle will be established:

$$\begin{tikzcd} C^1 \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \Omega(S^1\times S^2) \arrow[dl, "\psi"] \\ & G_- & \end{tikzcd}$$

Here, C^1 is the Fox-Milnor's 1-knot cobordism group (See Fox-Milnor [3].), G_- is the Levine's integral matrix cobordism group (See Levine [9].), e is a homomorphism and ϕ, ψ are epimorphisms. In particular the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$ will have an infinite rank. Analogously *the $\tilde{H}$-cobordism group $\Omega(S^1\times_\tau S^2)$ of homology non-orientable handles* will be also constructed. We shall show that the $\tilde{H}$-cobordism group $\Omega(S^1\times_\tau S^2)$ is isomorphic to the direct sum of infinitely many copies of the cyclic group of order two. Furthermore, it will be shown that the assignment τ: $m\to m'$ of the type m of any distinguished homology non-orientable handle to the type m' of its 2-fold orientation-cover (which is a distinguished homology orientable handle) induces a well-defined homomorphism τ^*: $\Omega(S^1\times_\tau S^2)\to T_2\subset\Omega(S^1\times S^2)$ from $\Omega(S^1\times_\tau S^2)$ to the subgroup T_2 of $\Omega(S^1\times S^2)$ consisting of elements of order two. As one consequence T_2 will be infinitely generated.

Section 1 will construct the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$ of homology orientable handles. In Section 2 we will discuss the properties of the invariants of $\Omega(S^1\times S^2)$ and compare $\Omega(S^1\times S^2)$ with Fox-Milnor's 1-knot cobordism group C^1 and with the Levine's integral matrix cobordism group G_-. Section 3 will concern the zero element and the order-two-elements of the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$. It will be shown that *the type m of a distinguished homology orientable* *handle* $M(\alpha, \iota)$ *represents the zero element of* $\Omega(S^1\times S^2)$ *(that is, m is null-$\tilde{H}$-cobordant) if* $M(\alpha, \iota)$ *is embeddable to a homology 4-sphere.* To consider the order-two-elements of $\Omega(S^1\times S^2)$, we will introduce the $\tilde{H}$-cobordism group $\Omega(S^1\times_\tau S^2)$ of homology non-orientable handles and determine its group structure and discuss the homomorphism $\tau^*: \Omega(S^1\times_\tau S^2)\rightarrow T_2\subset\Omega(S^1\times S^2)$ in this section.

Throughout this paper, spaces and maps will be considered from the piecewise linear point of view.

1. A construction of the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$

A 3-dimensional *homology orientable handle* M is a compact 3-manifold having the integral homology group of the orientable handle $S^1\times S^2$: $H_*(M; Z)\approx H_*(S^1\times S^2; Z)$. A homology orientable handle M is said to be *distinguished* if generators $\alpha\in H_1(M; Z)(\approx Z)$ and $\iota\in H_3(M; Z)(\approx Z)$ are specified. In that case the notation $M(\alpha, \iota)$ will be used. Two distinguished homology orientable handles $M(\alpha, \iota)$, $M'(\alpha', \iota')$ are said to *have the same type* if there is a piecewise-linear homeomorphism $h: M(\alpha, \iota)\cong M'(\alpha', \iota')$ which induces an isomorphism $h_*: H_*(M(\alpha, \iota); Z)\approx H_*(M'(\alpha', \iota'); Z)$ with $h_*(\alpha)=\alpha'$ and $h_*(\iota)=\iota'$. The class of distinguished homology orientable handles having the same type as $M(\alpha, \iota)$ is called *the type* of $M(\alpha, \iota)$. The set of all types is denoted by $\mathfrak{C}_+(S^1\times S^2)$. Let m be a type of $M(\alpha, \iota)$. By $-m$ we denote the type of $M(\alpha, -\iota)$. It is easily checked that the four distinguished handles $S^1\times S^2(\alpha, \iota)$, $S^1\times S^2(\alpha, -\iota)$, $S^1\times S^2(-\alpha, -\iota)$ and $S^1\times S^2(-\alpha, \iota)$ of the orientable handle $S^1\times S^2$ have the same type. We denote this type by 0.

DEFINITION 1.1. Two types m_1, m_2 in $\mathfrak{C}_+(S^1\times S^2)$ are *$\tilde{H}$-cobordant* and denoted by $m_1\sim m_2$, if for some representatives $M_1(\alpha_1, \iota_1)\in m_1$, $M_2(\alpha_2, \iota_2)\in m_2$ there exists a pair (W, φ) where W is a compact connected oriented 4-manifold with $\partial W=M_1(\alpha_1, \iota_1)+M_2(\alpha_2, -\iota_2)$ (disjoint union) and φ is a cohomology class in $H^1(W; Z)$ whose restrictions $\varphi|M_i(\alpha_i, \iota_i)\in H^1(M_i(\alpha_i, \iota_i); Z)$ are dual to α_i for $i=1, 2$, and such that the infinite cyclic cover $\tilde{W}_\varphi$ associated with φ has a finitely generated rational homology group $H_*(\tilde{W}_\varphi; Q)$ [that is, for each i, $H_i(\tilde{W}_\varphi; Q)$ is a finite dimensional vector space over Q.].

As usual the triad $(W, M_1(\alpha_1, \iota_1), M_2(\alpha_2, \iota_2))$ is called *an $\tilde{H}$-cobordism.*

It is easily seen that $m\sim 0$ if and only if for some representative $M(\alpha, \iota)\in m$, there exists a pair (W^+, φ) where W^+ is a compact connected oriented 4-manifold with $\partial W^+=M(\alpha, \iota)$ and $\varphi\in H^1(W^+; Z)$ with $\varphi|M(\alpha, \iota)\in H^1(M(\alpha, \iota); Z)$ dual to α, and such that the infinite cyclic ocver $\tilde{W}^+_\varphi$ associated with φ has a finitely generated rational homology group $H_*(\tilde{W}^+_\varphi; Q)$. In this case the notation $(W^+, M(\alpha, \iota), \phi)$ may be adopted as an $\tilde{H}$-cobordism.

Lemma 1.2. *The $\tilde{H}$-cobordism relation $\sim$ is an equivalence relation.*

Proof. The relation $\sim$ is reflexive, since the infinite cyclic cover $\tilde{M}$ of any homology orientable handle M has a finitely generated rational homology group $H_*(\tilde{M}; Q)$. [To see this, notice that for any i, $i \neq 2$, $H_i(\tilde{M}; Q)$ is finitely generated (See for example Kawauchi [6, Proposition 3.4] for $i=1$.). The partial Poincaré duality theorem (See Kawauchi [6].) then asserts a duality $H^0(\tilde{M}; Q) \approx H_2(\tilde{M}; Q)$. So $H_2(\tilde{M}; Q) \approx Q$.] The relation is obviously symmetric. Further the use of the Mayer-Vietoris sequence easily yields that the relation is transitive. This completes the proof.

DEFINITION 1.3. The set $\Omega(S^1 \times S^2)$ is defined to be the set of $\mathfrak{C}_+(S^1 \times S^2)$ modulo the $\tilde{H}$-cobordism relation $\sim$.

For any $m \in \mathfrak{C}_+(S^1 \times S^2)$ the symbol $[m]$ denotes the element of $\Omega(S^1 \times S^2)$ having m as the representative.

Now we shall introduce a sum oparation, called a circle union, in the set $\Omega(S^1 \times S^2)$.

Let m_0, $m_1 \in \mathfrak{C}_+(S^1 \times S^2)$ and $M_i(\alpha_i, \iota_i) \in m_i$, $i=0, 1$. Choose for each i a polygonal oriented simple closed curve ω_i in $M_i(\alpha_i, \iota_i)$ which represents the homology class α_i. Then for each i there exists a closed connected orientable surface F_i in $M_i(\alpha_i, \iota_i)$ which intersects ω_i in a single point. [To see this, first note that the identity map $\omega_i \subset \omega_i$ can be extended to a piecewise-linear map f_i: $M_i(\alpha_i, \iota_i) \to \omega_i$ by means of the elementary obstruction theory. Second, note that there is a point $p_i \in \omega_i$ such that the preimage $f_i^{-1}(p_i)$ is a closed (not necessarily connected) orientable surface. Now choose as F_i the component of $f_i^{-1}(p_i)$ containing p_i.]

Consider the solid torus $S^1 \times B^2$ and choose piecewise-linear embeddings

$$h_0\colon\ S^1 \times B^2 \times 0 \to M_0(\alpha_0, \iota_0)$$
$$h_1\colon\ S^1 \times B^2 \times 1 \to M_1(\alpha_1, \iota_1)$$

such that

(1) there exist points $s \in S^1$, $b \in \operatorname{Int} B^2$ with $h_0(s \times B^2 \times 0) \subset F_0$, $h_0(S^1 \times b \times 0) = \omega_0$, $h_1(s \times B^2 \times 1) \subset F_1$ and $h_1(S^1 \times b \times 1) = \omega_1$,

(2) both h_0 and h_1 are orientation-reversing with respect to the orientations of $S^1 \times B^2 \times 0$ and $S^1 \times B^2 \times 1$ induced from some orientation of $S^1 \times B^2 \times [0, 1]$,

(3) ω_0 and ω_1 are homologous in the adjunction space $M_0(\alpha_0, \iota_0) \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1(\alpha_1, \iota_1)$.

Then the manifold $M = M_0(\alpha_0, \iota_0) \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1(\alpha_1, \iota_1) - S^1 \times \operatorname{Int} B^2 \times [0, 1]$ is a homology handle. [*Proof.* Let $i=0$ or 1. Consider the manifold $M_i' = M_i - h_i(S^1 \times \operatorname{Int} B^2 \times i)$. Let $b' \in \partial B^2$ and the simple closed curve $\omega_i' = h_i(S^1 \times b' \times i) \subset \partial M_i'$ be oriented so that ω_i' is homologous to ω_i in M_i. Let $\eta_i = h_i(s \times \partial B^2 \times i) \subset \partial M_i'$ be oriented suitably. It is easily checked that ω_i'

represents a generator of $H_1(M'_i; Z)(\approx Z)$ and η_i represents the zero element of $H_1(M'_i; Z)$ (since η_i bounds an orientable surface $F_i - h_i(s\times \operatorname{Int} B^2\times i)$ in M'_i) and that ω'_i, η_i represent a basis for $H_1(\partial M'_i; Z)$. Then from consideration of the Mayer-Vietoris sequence we obtain that $H_1(M; Z)\approx Z$. Since M is orientable, $H_*(M; Z)\approx H_*(S^1\times S^2; Z)$ by Poincaré duality.]

From construction it can be seen that the homology classes $\alpha_i\in H_1(M_i(\alpha_i, \iota_i); Z)$, $i=0, 1$, specify a unique homology class $\alpha_1\in H_1(M; Z)$ and that the fundamental classes $\iota_i\in H_3(M_i(\alpha_i, \iota_i); Z)$, $i=0, 1$, specify a unique fundamental class $\iota\in H_3(M; Z)$.

DEFINITION 1.4. The distinguished homology orientable handle $M(\alpha, \iota)$ is called a *circle union* of $M_0(\alpha_0, \iota_0)$ and $M_1(\alpha_1, \iota_1)$ and denoted by $M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, \iota_1)$. Also, the type of $M(\alpha, \iota)$ is called a circle union of the types m_0 and m_1 and denoted by $m_0\bigcirc m_1$.

Clearly the type of $M_0(\alpha_0, -\iota_0)\bigcirc M_1(\alpha_1, -\iota_1)$ is $-(m_0\bigcirc m_1)=(-m_0)\bigcirc(-m_1)$.

1.5. Remark to Definition 1.4. It should be remarked that the circle union $m_0\bigcirc m_1$ depends upon the choices of ω_0, ω_1, h_0 and h_1. Consider for example a distinguished orientable handle $S^1\times S^2(\alpha, \iota)$. Let $\omega\subset S^1\times S^2(\alpha, \iota)$ be an oriented simple closed curve representing α of geometrical index*) 1 and $T(\omega)$ be the regular neighborhood of ω in $S^1\times S^2(\alpha, \iota)$. If the circle union $S^1\times S^2(\alpha, \iota)\bigcirc S^1\times S^2(\alpha, -\iota)$ is defined to be the double of $cl(S^1\times S^2)(\alpha, \iota)-T(\omega))$, then $S^1\times S^2(\alpha, \iota)\bigcirc S^1\times S^2(\alpha, -\iota)$ has the same type as $S^1\times S^2(\alpha, \iota)$. On the other hand, consider for example an oriented simple closed curve $\omega'\subset S^1\times S^2(\alpha, \iota)$ representing α of geometrical index 3 and algebraic index 1 (See figure 1.) and let $T(\omega')$ be the regular neighborhood of ω' in $S^1\times S^2(\alpha, \iota)$.

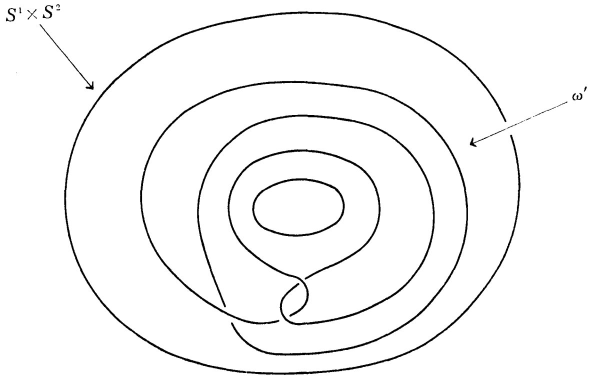


figure 1.

*) A simple closed curve ω in $S^1\times S^2$ has *geometric index* λ, if λ is the least number of intersections that a curve ambient isotopic to ω can have with $s_1\times S^2$ and has *algebraic index* λ', if λ' is the unique integer such that ω is homologous to λ' times $S^1\times s_2$ for a point $(s_1, s_2)\in S^1\times S^2$.

If the circle union $S^1\times S^2(\alpha, \iota)\bigcirc' S^1\times S^2(\alpha, -\iota)$ is defined to be the double of $cl(S^1\times S^2(\alpha, \iota)-T(\omega'))$, then $S^1\times S^2(\alpha, \iota)\bigcirc' S^1\times S^2(\alpha, -\iota)$ does not have the same type as $S^1\times S^2(\alpha, \iota)\bigcirc S^1\times S^2(\alpha, -\iota)$, because $\pi_1(S^1\times S^2(\alpha, \iota)\bigcirc S^1\times S^2(\alpha, -\iota))\approx Z$, but $\pi_1(S^1\times S^2(\alpha, \iota)\bigcirc' S^1\times S^2(\alpha, -\iota))$ is non-abelian. [In fact, the natural injection $\partial T(\omega')\to S^1\times S^2(\alpha, \iota)\bigcirc' S^1\times S^2(\alpha, -\iota)$ induces a monomorphism $\pi_1(\partial T(\omega'))\to\pi_1(S^1\times S^2(\alpha, \iota)\bigcirc' S^1\times S^2(\alpha, -\iota_0))$ by the loop theorem.]

In spite of Remark 1.5 we can prove the following for arbitrary two circle unions $m_0\bigcirc m_1$, $m_0\bigcirc' m_1$ of given two types m_0, m_1:

Lemma 1.6. $m_0\bigcirc m_1\sim m_0\bigcirc' m_1$.

Proof. Let $M_0(\alpha_0, \iota_0)\in m_0$ and $M_1(\alpha_1, \iota_1)\in m_1$. Assume $M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, \iota_1)\in m_0\bigcirc m_1$ and $M_0(\alpha_0, \iota_0)\bigcirc' M_1(\alpha_1, \iota_1)\in m_0\bigcirc' m_1$ are given by the following:

$$\begin{aligned}&M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, \iota_1)\\ &\quad= M_0(\alpha_0, \iota_0)\times 0\cup_{h_0} S^1\times B^2\times[0, 1]\cup_{h_1} M_1(\alpha_1, \iota_1)\times 0-S^1\times \operatorname{Int} B^2\times[0, 1]\\ &M_0(\alpha_0, \iota_0)\bigcirc' M_1(\alpha_1, \iota_1)\\ &\quad= M_0(\alpha_0, \iota_0)\times 1\cup_{h_0'} S^1\times B^2\times[0, 1]_{h_1'} M_1(\alpha_1, \iota_1)\times 1-S^1\times \operatorname{Int} B^2\times[0, 1]\,.\end{aligned}$$

Then we let

$$W= M_0(\alpha_0, \iota_0)\times[0, 1]\begin{array}{c}\cup_{h_0} S^1\times B^2\times[0, 1]\cup_{h_1}\\ \\ \cup_{h_0'} S^1\times B^2\times[0, 1]\cup_{h_1'}\end{array} M_1(\alpha_1, \iota_1)\times[0, 1]\,.$$

(See figure 2.)

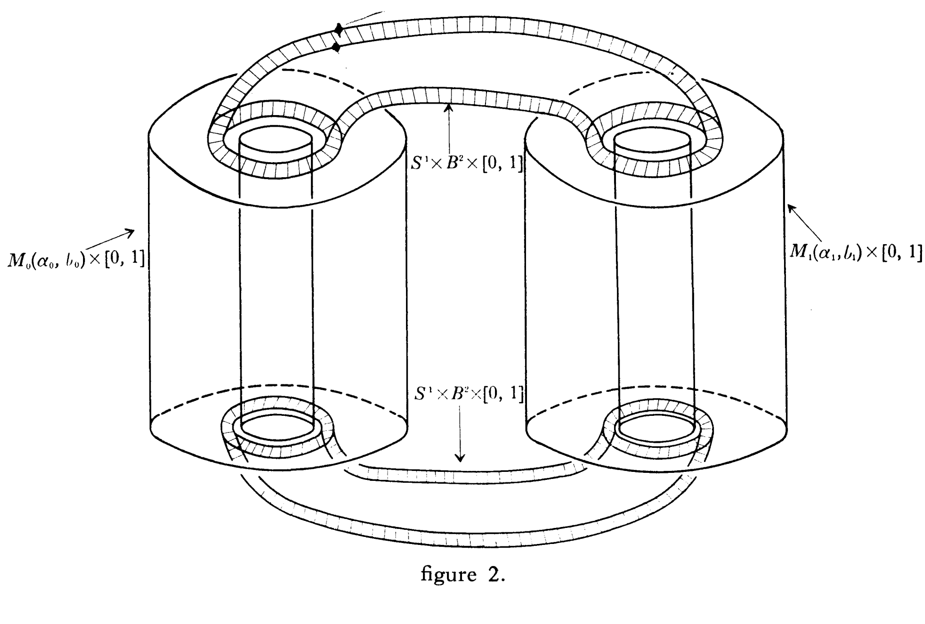


figure 2.

Clearly we have $\partial W = M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, \iota_1) + M_0(\alpha_0, -\iota_0) \bigcirc' M_1(\alpha_1, -\iota_1)$. Note that α_0, α_1 represent the same element α in $H_1(W; Z)$. Let $\varphi \in H^1(W; Z)$ be dual to α and $\tilde{W}_\varphi$ be the infinite cyclic cover of W associated with φ. Since $\tilde{W}_\varphi$ is the union of $\tilde{M}_0(\alpha_0, \iota_0) \times [0, 1]$, $R^1 \times B^2 \times [0, 1]$, $R^1 \times B^2 \times [0, 1]$ and $\tilde{M}_1(\alpha_1, \iota_1) \times [0, 1]$, each two intersections of which is empty or homeomorphic to $R^1 \times B^2$, it follows from the Mayer-Vietoris sequence that $H_*(\tilde{W}_\varphi; Q)$ is finitely generated over Q, where $\tilde{M}_i(\alpha_i, \iota_i)$ are the infinite cyclic covers of $M_i(\alpha_i, \iota_i)$, $i=0, 1$. Thus, the triad $(W, M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, \iota_1), M_0(\alpha_0, \iota_0) \bigcirc' M_1(\alpha_1, \iota_1))$ gives an $\tilde{H}$-cobordism and hence $m_0 \bigcirc m_1 \sim m_0 \bigcirc' m$. This completes the proof.

Lemma 1.7. *$m_0 \sim m_1$ is equivalent to $m_0 \bigcirc -m_1 \sim 0$.*

Proof. Assume $m_0 \sim m_1$. Then for some representatives $M_0(\alpha_0, \iota_0) \in m_0$, $M_1(\alpha_1, \iota_1) \in m_1$ there is an $\tilde{H}$-cobordism $(W, M_0(\alpha_0, \iota_0), M_1(\alpha_1, \iota_1))$. Note that there is a cohomology class $\varphi \in H^1(W; Z)$ such that for each i $\varphi | M_i(\alpha_i, \iota_i) \in H^1(M(\alpha_i, \iota_i); Z)$ is dual to α_i. Let $M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1) = M_0(\alpha_0, \iota_0) \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1(\alpha_1, -\iota_1) - S^1 \times \operatorname{Int} B^2 \times [0, 1]$ and $W' = W \cup_{h_0, h_1} S^1 \times B^2 \times [0, 1]$ (See figure 3.). Clearly $\partial W' = M_0(\alpha_0, -\iota_0) \bigcirc M_1(\alpha_1, -\iota_1)$. The cohomology class $\varphi \in H^1(W; Z)$ is easily extended to a cohomology class $\varphi' \in H^1(W'; Z)$ such that the restriction $\varphi' | M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1) \in H^1(M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1); Z)$ is dual to the specified generator of $H_1(M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1); Z)$. By applying the Mayer-Vietoris sequence, it is not difficult to see that the infinite cyclic cover $\tilde{W}'_{\varphi'}$ of W' associated with φ' has a finitely generated rational homology group $H_*(\tilde{W}'_{\varphi'}; Q)$. [Use that $H_*(\tilde{W}_\varphi; Q)$ is finitely generated over Q.] So, $m_0 \bigcirc -m_1 \sim 0$.

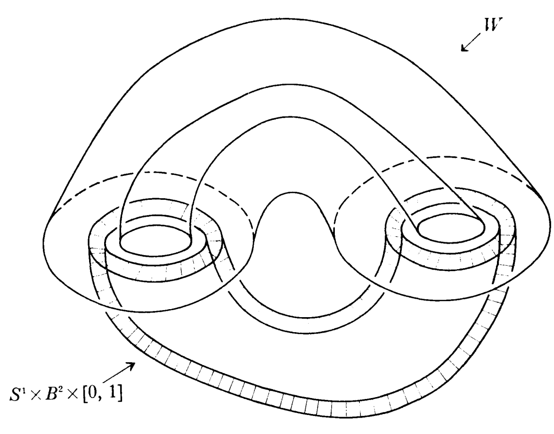


figure 3.

Conversely assume $m_0 \bigcirc -m_1 \sim 0$. For $M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1) \in m_0 \bigcirc -m_1$ there is an $\tilde{H}$-cobordism $(W'', M_0(\alpha_0, \iota_0) \bigcirc M_1(\alpha_1, -\iota_1), \phi)$. By the definition of the circle union there is a natural injection j: $S^1\times\partial B^2\times[0, 1]\to M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, -\iota_1)$. Let $W'''=W''\cup_j S^2\times B^1\times[0, 1]$. It is easy to see that the boundary $\partial W'''$ is equal to the disjoint union $M_0(\alpha_0, \iota_0)+M_1(\alpha_1, -\iota_1)$ and that the triad $(W''', M_0(\alpha_0, \iota_0), M_1(\alpha_1, \iota_1))$ gives an $\tilde{H}$-cobordism between $M_0(\alpha_0, \iota_0)$ and $M_1(\alpha_1, \iota_1)$. This completes the proof.

Lemma 1.8. *If $m_0\sim 0$ and $m_1\sim 0$, then $m_0\bigcirc m_1\sim 0$.*

Proof. For $M_0(\alpha_0, \iota_0)\in m_0$, $M_1(\alpha_1, \iota_1)\in m_1$, there are $\tilde{H}$-cobordisms $(W_0, M_0(\alpha_0, \iota_0), \phi)$ and $(W_1, M_1(\alpha_1, \iota_1), \phi)$. Let $M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, \iota_1)=M_0(\alpha_0, \iota_0)\cup_{h_0}S^1\times B^2\times[0, 1]\cup_{h_1}M_1(\alpha_1, \iota_1)-S^1\times\operatorname{Int}B^2\times[0, 1]$. If we let $W=W_0\cup_{h_0}S^1\times B^2\times[0, 1]\cup_{h_1}W_1$ (See figure 4.), then the triad $(W, M_0(\alpha_0, \iota_0)\bigcirc M_1(\alpha_1, \iota_1), \phi)$ gives an $\tilde{H}$-cobordism. So, $m_0\bigcirc m_1\sim 0$, which completes the proof.

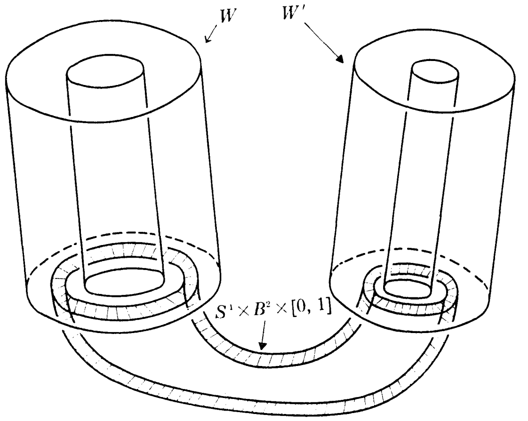


figure 4.

Now we can derive the following theorem which is a main purpose of this section.

Theorem 1.9. *The set $\Omega(S^1\times S^2)$ forms an abelian group under the sum $[m_0]+[m_1]=[m_0\bigcirc m_1]$. The zero element of this group is $[0]$. The inverse of any element $[m]$ is the element $[-m]$.*

Proof. To show that the sum $[m_0]+[m_1]=[m_0\bigcirc m_1]$ is well-defined, let $m_0\sim m_0'$ and $m_1\sim m_1'$. By Lemma 1.7 $m_0\bigcirc -m_0'\sim 0$ and $m_1\bigcirc -m_1'\sim 0$. Then by Lemma 1.8 $(m_0\bigcirc -m_0')\bigcirc(m_1\bigcirc -m_1')\sim 0$. Since $(m_0\bigcirc m_1)\bigcirc m_2\sim m_0\bigcirc(m_1\bigcirc m_2)$ and $m_0\bigcirc m_1=m_1\bigcirc m_0$ for all m_0, m_1 and m_2, we obtain $(m_0\bigcirc m_1)\bigcirc -(m_0'\bigcirc m_1')\sim(m_0\bigcirc -m_0')\bigcirc(m_1\bigcirc -m_1')$. Hence again by Lemma 1.7 $m_0\bigcirc m_1\sim m_0'\bigcirc m_1'$. Thus, $[m_0]=[m_0']$ and $[m_1]=[m_1']$ imply $[m_0]+[m_1]=[m_0']+[m_1']$. It is clear that $([m_0]+[m_1])+[m_2]=[m_0]+([m_1]+[m_2])$ and $[m_0]+[m_1]=[m_1]+[m_0]$. Also, we have $[m]+[0]=[m\bigcirc 0]=[m]$ and, by Lemma 1.7, $[m]+[-m]=[0]$. This completes the proof.

The group $\Omega(S^1\times S^2)$ is called *the $\tilde{H}$-cobordism group of 3-dimensional homology orientable handles*. The zero element is denoted by 0 and the inverse of $[m]$ is $-[m]$.

2. Relating the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$ to the Fox-Milnor's group C and the Levine's group G_-

The purpose of this section is to prove the following theorem.

Theorem 2.1. *There is a commutative triangle*

$$\begin{tikzcd} C^1 \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \Omega(S^1\times S^2) \arrow[dl, "\psi"] \\ & G_- & \end{tikzcd}$$

of groups and homomorphisms, where the homomorphisms ϕ: $C^1\to G_-$ and ψ: $\Omega(S^1\times S^2)\to G_-$ are onto.

A knot $k\subset S^3$ is a polygonal oriented 1-sphere k in the oriented piecewise-linear 3-sphere S^3. Two knots $k_1\subset S^3$, $k_2\subset S^3$ have *the same knot type* if there is a piecewise-linear homeomorphism $(S^3, k_1)\to(S^3, k_2)$ which is orientation-preserving as both the maps $S^3\to S^3$ and $k_1\to k_2$. *The knot type* of a knot $k\subset S^3$ will mean the class of knots with the same knot type as $k\subset S^3$. The set of knot types is denoted by $\mathcal{K}$. Let $\mathscr{k}$ be a knot type and $(k\subset S^3)\in\mathscr{k}$ be a representative knot. By $-\mathscr{k}$, we denote the knot type of the knot $(-k\subset -S^3)$, where $-k$ and $-S^3$ are the same as k and S^3 but have the opposite orientations, respectively.

Now we shall construct a function e: $\mathcal{K}\to\mathfrak{C}_+(S^1\times S^2)$. Let $\mathscr{k}$ be a knot type and $(k\subset S^3)\in\mathscr{k}$ be a knot. Consider the regular neighborhood $T(k)\subset S^3$ of the knot $k\subset S^3$. Then $T(k)$ is clearly piecewise-linear homeomorphic to the solid torus $S^1\times B^2$. We note that the solid tours $T(k)$ in S^3 has unique meridian and longitude curves*) (up to isotopies of $\partial T(k)$ and the orientations of curves).

*) *A meridian curve* of a solid (knotted) torus T in S^3 is a simple closed curve ω in ∂T such that ω is homologous to 0 in T but not in ∂T. *A longitude curve* of T in S^3 is a simple closed curve ω in ∂T such that ω is homologous to 0 in S^3-Int T but not in ∂T. The uniqueness of the meridian and longitude curves follows from a more general principle: *Let X be a homology orientable circle* i.e. *X is a compact 3-manifold with $H_*(X; Z)\approx H_*(S^1; Z)$ and $H_*(\partial X; Z)\approx H_*(S^1\times S^1; Z)$. If ω, $\omega'\subset\partial X$ are homologous to 0 in X but not in ∂X, then with suitable orientations of ω, ω', ω is isotopic to ω' in ∂X.* [*Proof.* Take a simple closed curve ω^* in ∂X intersecting ω in single point. Using that ω represents the zero element of $H_1(X; Z)$ and that the natural homomorphism $H_1(\partial X; Z)\to H_1(X; Z)$ is onto, it follows that ω^* represents a generator of $H_1(X; Z)$. Let f: $\partial X\to\omega^*$ be a natural projection such that for some point $p^*\in\omega^*$, $f^{-1}(p^*)=\omega$. Then we may find an extension f': $X\to\omega^*$ of f such that $(f')^{-1}(p^*)=F$ is a connected surface with $\partial F=\omega$. Since the infinite cyclic covering

The orientation of the longitude curve should be chosen so that the longitude curve is homologous to k in $T(k)$. The orientation of the meridian curve should be chosen so that the linking number of the meridian curve and the knot k in S^3 is $+1$. Let h: $S^1 \times S^1 \to \partial T(k)$ be a piecewise-linear homeomorphism such that for some point (s_1, s_2) in $S^1 \times S^1$ the curves $h(s_1 \times S^1)$ and $h(S^1 \times s_2)$ are the meridian curve and the longitude curve of $T(k)$, respectively. Define M to be the adjunction space S^3-Int $(T(k)) \cup_h B^2 \times S^1$. ($\partial B^2$ is identified with S^1.) By applying the Mayer-Vietoris sequence, we have $H_1(M; Z) \approx Z$. Hence M is a homology orientable handle by Poincaré duality. Note that the oriented meridian curve of $T(k)$ represents a generator α of $H_1(M; Z)$. We specify the orientation of M compatible with the orientation of $S^3 - T(k)$ induced from that of S^3. So, a generator $\iota \in H_3(M; Z)$ is specified.

DEFINITION 2.2. The distinguished homology orientable handle $M(\alpha, \iota)$ is called the distinguished homology orientable handle *obtained from* S^3 *by the elementary surgery along the knot* $k \subset S^3$.

By using the uniqueness of the meridian curve, the longitude curve and the regular neighborhood, it is easily checked that the type of $M(\alpha, \iota)$ is uniquely determined by the knot type $\mathscr{k}$ of $k \subset S^3$. So we denote this type by $e(\mathscr{k})$.

Thus, we have the following:

Lemma 2.3. *There is a function* e: $\mathcal{K} \to \mathfrak{C}_+(S^1 \times S^2)$.

For any two knot types $\mathscr{k}_1$, $\mathscr{k}_2$, one can construct a unique knot type $\mathscr{k}_1 \# \mathscr{k}_2$ well-known as *the knot sum*. Two knot types $\mathscr{k}_1$, $\mathscr{k}_2$ are *cobordant* if for a representative knot $k \subset S^3$ of the knot sum $\mathscr{k}_1 \# -\mathscr{k}_2$ k bounds a locally flat 2-cell in the 4-cell B^4. The set $\mathcal{K}$ modulo this knot cobordism relation forms an abelian group C^1, called *the knot cobordism group*. (See Fox-Milnor [3] for details.) The sum operation of C^1 is the usual knot sum operation.

Lemma 2.4. *The function* e: $\mathcal{K} \to \mathfrak{C}_+(S^1 \times S^2)$ *induces a homomorphism* $C^1 \to \Omega(S^1 \times S^2)$ *also denoted by* e.

Proof. For two knot types $\mathscr{k}_1$, $\mathscr{k}_2$, it is directly checked that $e(\mathscr{k}_1 \# \mathscr{k}_2)$ is a circle union of $e(\mathscr{k}_1)$ and $e(\mathscr{k}_2)$ *i.e.* $e(\mathscr{k}_1 \# \mathscr{k}_2) = e(\mathscr{k}_1) \bigcirc e(\mathscr{k}_2)$. [Note that for $(K_i \subset S^3) \in \mathscr{k}_i$, $i=1, 2$, the exterior of the knot sum $(K_1 \subset S^3) \# (K_2 \subset S^3)$ is the adjunction

p: $\tilde{X} \to X$ associated with the Hurewicz homomorphism can be constructed by using f', we may regard $F \subset \tilde{X}$. Note that $[F] \in H_2(X, \partial X; Z)$ is a generator (See [7, Lemma 2.5].). By using the isomorphism p_*: $H_2(\tilde{X}, \partial \tilde{X}; Z) \approx H_2(X, \partial X; Z)$ (See [7, Remark 2.4].), $[F] \in H_2(\tilde{X}, \partial \tilde{X}; Z)$ is a generator *i.e.* a finite fundamental class (See [6].). Similarly, we can find a surface $F' \subset \tilde{X}$ with $\partial F' = \omega'$ and such that $[F']$ is a finite fundamental class of $\tilde{X}$. The boundary-isomorphism ∂: $H_2(\tilde{X}, \partial \tilde{X}; Z) \approx H_1(\partial \tilde{X}; Z)$, then, implies that $[\omega]$ and $[\omega']$ are equal up to sign. That is, with suitable orientations of ω, ω', ω is homologous to ω' and hence homotopic to ω' in $\partial \tilde{X} = S^1 \times R^1$. Accordingly, ω is homotopic to ω' in ∂X which implies that ω is isotopic to ω' in ∂X.]

space of the exteriors of $K_i \subset S^3$ along uniquely specified annuli on the boundaries] Hence it suffices to show that if a knot type $\mathscr{k}$ is cobordant to the trivial knot type, then $e(\mathscr{k}) \sim 0$. According to Fox-Milnor [3], this knot type $\mathscr{k}$ can be realized as a local knot type of a piecewise linear 2-sphere $S(\mathscr{k})$ in S^4 with just one locally knotted point. Let $N = N(S(\mathscr{k}), S^4)$ be the regular neighborhood of $S(\mathscr{k})$ in S^4. Let $W = S^4 - \operatorname{Int} N$ and $M = \partial W$. Notice that $H_*(\partial W; Z) \approx H_*(S^1; Z)$ by the Alex ander duality. By using the Mayer-Vietoris sequence of the triple $(S^4; W, N)$, we obtain that $H_1(M; Z) \approx Z$. Hence M is a homology orientable handle. M may be a distinguished homology orientable handle obtained from S^3 by the elementary surgery along a representative knot $(k \subset S^3) \in \mathscr{k}$: $M = M(\alpha, \iota)$. [For N is obtained from a 4-cell by attaching a 2-handle along a solid torus $T \subset S^3$ representing $\mathscr{k}$. Using $H_1(M; Z) = Z$ and the unique longitude curve of $T \subset S^3$, M with suitably chosen $\alpha \in H_1(M; Z)$ and $\iota \in H_3(M; Z)$ belongs to $e(\mathscr{k})$]. Since W has the homology of a circle, it follows from Milnor [11, Assertion 5] that the rational homology group $H_*(\tilde{W}; Q)$ of any infinite cyclic cover $\tilde{W}$ is finitely generated over Q. This shows that the triad $(W; M(\alpha, \iota), \phi)$ gives an $\tilde{H}$-cobordism. Therefore $e(\mathscr{k}) \sim 0$. This completes the proof.

Usually any knot type cobordant to the trivial knot type is called *a slice knot type.*

In the proof of Lemma 2.4, we have also proved the following:

Corollary 2.5 (Kato [5]). *If a knot type $\mathscr{k}$ is a slice knot type, then any representative homology orientable handle of $e(\mathscr{k})$ is embeddable to the 4-sphere S^4.*

A Seifert matrix A (with sign -1) is an integral square matrix with $\det(A - A') = \pm 1$. (A' is the transpose of A.) Two Seifert matrices A_1, A_2 are said to be *cobordant* if the block sum $A_1 \oplus -A_2$ is congruent (over Z) to a matrix of the form $\begin{pmatrix} O & B \\ C & D \end{pmatrix}$ (B, C, D are square matrices of the same size.) The set of Seifert matrices modulo this cobordism relation forms an abelian group G_-, called *the matrix cobordism group.* (See Levine [9] for details. Note that only Seifert matrices with sign -1 are considered here.) In [10] Levine calculated that G_- is isomorphic to the direst sum $\sum_{i=1}^{\infty} Z^i + \sum_{i=1}^{\infty} (Z/2Z)^i + \sum_{i=1}^{\infty} (Z/4Z)^i$.

For a while we would like to spare time for describing familiar algebraic invariants of a polygonal oriented 1-sphere in a piecewise linear oriented homology 3-sphere, called *a homological knot.* The arguments may proceed in the same way as the usual knot theory. Let $k \subset \bar{S}^3$ be a homological knot. k bounds an oriented connected surface F, called *a Seifert surface for k,* by using a notion of the transverse regularity. We define a pairing θ: $H_1(F; Z) \otimes H_1(F; Z) \to Z$ such that $\theta(\alpha \otimes \beta) = L(\alpha, i_*(\beta))$, where L denotes the homological linking number in $\bar{S}^3$ and $i_*(\beta)$ denotes the translate of the cycle β off F in the positive normal direction. With a basis for $H_1(F; Z)$, θ represents an integral square matrix A, called *a Seifert matrix for* $k\subset\bar{S}^3$ *associated with surface* F. Using a formula $\theta(\alpha\otimes\beta)-\theta(\beta\otimes\alpha)=\alpha\cdot\beta$, where $\alpha\cdot\beta$ is the intersection number, we obtain $\det(A-A')=\pm 1$. (See for example Levine [8].) So, A is in fact a Seifert matrix. The integral polynomial $A(t)=\det(tA-A')$ is called *the Alexander polynomial* of $k\subset\bar{S}^3$. Let $X=\bar{S}^3-\operatorname{Int} T(k)$ for the regular neighborhood $T(k)$ of k in $\bar{S}^3$ and $\tilde{X}$ be the infinite cyclic cover of X associated with the Hurewicz homomorphism $\pi_1(X)\to H_1(X; Z)$. We choose an orientation of $\tilde{X}$ induced by that of X and a generator t of the covering transformation group of $\tilde{X}$ associated with a generator α of $H_1(X; Z)$ with linking number $L(\alpha, k)=+1$. By using the Mayer-Vietoris sequence, the matrix $tA-A'$ is a relation matrix of $H_1(\tilde{X}; Z)$ as a $Z[t]$-module. The Seifert surface F induces a generaotr μ of $H_2(\tilde{X}, \partial\tilde{X}; Z)$ $(\approx Z)$, called *a finite fundametnal class* of $\tilde{X}$. (See Kawauchi [6, Theorem 2.3] and also Erle [1].) By Kawauchi [6, Theorem 2.3] (See also Milnor [11, p 127].) there is a duality $\cap\mu$: $H^q(\tilde{X}; Q)\approx H_{2-q}(\tilde{X}, \partial\tilde{X}; Q)$ for all q, since $H_*(\tilde{X}; Q)$ is finitely generated over Q. Hence using a canonical isomorphism $H^1(\tilde{X}, \partial\tilde{X}; Q)\approx H^1(\tilde{X}; Q)$, the cup product $H^1(\tilde{X}, \partial\tilde{X}; Q)\times H^1(\tilde{X}, \partial\tilde{X}; Q)\to H^2(\tilde{X}, \partial\tilde{X}; Q)$ is a non-singular skew-symmetric bilinear form. Define a symmetric bilinear form

$$\langle\ ,\ \rangle\colon H^1(\tilde{X}, \partial\tilde{X}; Q)\times H^1(\tilde{X}, \partial\tilde{X}; Q)\to H^2(\tilde{X}, \partial\tilde{X}; Q)\overset{\cap\mu}{\approx} H_0(\tilde{X}; Q)=Q$$

by the equality $\langle x, y\rangle=(x\cup ty)\cap\mu+(y\cup tx)\cap\mu$. This bilinear form is isometric on t: $\langle tx, ty\rangle=\langle x, y\rangle$ and non-singular.

DEFINITION 2.6. The pair $(\langle\ ,\ \rangle, t)$ is called *the quadratic form* of the homological knot $k\subset S^3$. (See Erle [1] and Milnor [11].)

The signature of $k\subset\bar{S}^3$ is the signature of this form $\langle\ ,\ \rangle$.

The following proposition is essentially proved by Erle [1].

Proposition 2.7. *Let A be any Seifert matrix for a homological knot $k\subset\bar{S}^3$ associated with a Seifert surface. A is S-equivalent to a non-singular Seifert matrix A_* such that, with a suitable basis for $H^1(\tilde{X}, \partial\tilde{X}; Q)$, the linear isomorphism t: $H^1(\tilde{X}, \partial\tilde{X}; Q)\to H^1(\tilde{X}, \partial\tilde{X}; Q)$ and the form $\langle\ ,\ \rangle\colon H^1(\tilde{X}, \partial\tilde{X}; Q)\times H^1(\tilde{X}, \partial\tilde{X}; Q)\to Q$ represent the matrices $A_*'^{-1}A_*$ and A_*+A_*', respectively.* (In fact, Erle [1] proved this proposition for any usual knot $k\subset\bar{S}^3$. Without difficulty, Erle's proof may be applied for homological knot $k\subset\bar{S}^3$. See Trotter [13] for a concept of S-equivalences.)

By Proposition 2.7, the signature of $k\subset\bar{S}^3$ is equal to the signature $\sigma(A_*+A_*')=\sigma(A+A')$.

Let $m\in\mathfrak{C}_+(S^1\times S^2)$ and $M(\alpha, \iota)\in m$. We choose a polygonal oriented simple closed curve ω in $M(\alpha, \iota)$ representing α and let $T(\omega)$ be the regular neighborhood of ω in $M(\alpha, \iota)$. Also we choose polygonal oriented simple closed curves k and l in $\partial T(\omega)$ intersecting in a single point such that k is oriented so as to be

$L(k, \omega)=+1$ and bounds a 2-cell in $T(\omega)$ and such that l is homologous to ω in $T(\omega)$. (Note that in any case the choice of l is not unique.) Let $(s_1, s_2)\in S^1\times S^1$ and define a piecewise-linear homeomorphism $h\colon S^1\times S^1\to\partial T(\omega)$ such that $h(s_1\times S^1)=k$ and $h(S^1\times s_2)=l$. Let $\bar{S}^3=M(\alpha, \iota)-\mathrm{Int}\, T(\omega)\cup_h B^2\times S^1$. It is easy to see that $\bar{S}^3$ is a homology 3-sphere. (Notice that k is homologous to 0 in $M(\alpha, \iota)-\mathrm{Int}\, T(\omega)$.) The orientation of $\bar{S}^3$ is chosen so as to coincide with that of $M(\alpha, \iota)-\mathrm{Int}\, T(\omega)$. Thus, we obtain a homological knot $k\subset\bar{S}^3$ from $M(\alpha, \iota)$ (, although the homeomorphism type of the pair $(\bar{S}^3, k)$ is never uniquely determined by the type of $M(\alpha, \iota)$).

DEFINITION 2.8. A Seifert matrix for the homological knot $k\subset\bar{S}^3$ associated with a Seifert surface is called *a Seifert matrix for* $M(\alpha, \iota)$ (*or the type* m).

Accordingly if A is a Seifert matrix for a knot type κ, then A is also a Seifert matrix for the type $e(\kappa)$.

DEFINITION 2.9. The Alexander polynomial $A(t)=\det(tA-A')$ of $k\subset\bar{S}^3$ is called *the Alexander polynomial of* $M(\alpha, \iota)$ (*or the type* m).

This definition coincides with that of Kawauchi [7, Definition 1.3], because the matrix $tA-A'$ is a relation matrix of $H_1(\tilde{M}(\alpha, \iota); Z)$ by the canonical isomorphism $H_1(\tilde{X}; Z)\approx H_1(\tilde{M}(\alpha, \iota); Z)$. Here $\tilde{X}$ denotes the infinite cyclic cover of $X=M(\alpha, \iota)-\mathrm{Int}\, T(\omega)$ with the uniquely specified generator t of the covering transformation group and with the associated orientation. $\tilde{M}(\alpha, \iota)$ denotes the infinite cyclic cover of $M(\alpha, \iota)$ such that the covering projection $\tilde{M}(\alpha, \iota)\to M(\alpha, \iota)$ is an extension of the covering projection $\tilde{X}\to X$. $\tilde{M}(\alpha, \iota)$ has an orientation compatible with that of $\tilde{X}$. The generator of the covering transformation group of $\tilde{M}(\alpha, \iota)$ is an extension of $t\colon \tilde{X}\to\tilde{X}$, also denoted by t. Note that the finite fundamental class $\mu\in H_2(\tilde{X}, \partial\tilde{X}; Z)$ determined by a Seifert surface specifies a unique generator of $H_2(\tilde{M}(\alpha, \iota); Z)$, also denoted by μ by the canonical isomorphism $H_2(\tilde{X}, \partial\tilde{X}; Z)\approx H_2(\tilde{M}(\alpha, \iota); Z)$. This $\mu\in H_2(\tilde{M}(\alpha, \iota); Z)$ is called *the finite fundamental class* of $\tilde{M}(\alpha, \iota)$. By using the canonical isomorphisms $H^i(\tilde{X}, \partial\tilde{X}; Q)\approx H^i(\tilde{M}(\alpha, \iota); Q)$, $i=1, 2$, the bilinear form $\langle\ ,\ \rangle\colon H^1(\tilde{X}, \partial\tilde{X}; Q)\times H^1(\tilde{X}, \partial\tilde{X}; Q)\to Q$ passes to the form $(\ ,\ )\colon H^1(\tilde{M}(\alpha, \iota); Q)\times H^1(\tilde{M}(\alpha, \iota); Q)\to Q$ defined by the equality $(x, y)=(x\cup ty)\cap\mu+(y\cup tx)\cap\mu$ for all x, y in $H^1(\tilde{M}(\alpha, l); Q)$.

DEFINITION 2.10. The pair $((\ ,\ ), t)$ is called *the quadratic form of* $M(\alpha, \iota)$ (*or the type* m).

The signature of $M(\alpha, \iota)$ (*or the type* m), denoted by $\sigma(M(\alpha, \iota))$ (or $\sigma(m)$) is the signature of the homological knot $k\subset\bar{S}^3$. So, the signature of $M(\alpha, \iota)$ coincides with the signature of the bilinear form $(\ ,\ )$. Easily $\sigma(M(\alpha, \iota))=\sigma(M(-\alpha, \iota))$ and $\sigma(M(\alpha, -\iota))=-\sigma(M(\alpha, \iota))$.

From Proposition 2.7, the following is immediately obtained:

Lemma 2.11. *Let A be a Seifert matrix for $M(\alpha, \iota)$. A is S-equivalent to a non-singular Seifert matrix A_* such that, with a suitable basis for $H^1(\tilde{M}(\alpha, \iota); Q)$, the linear isomorphism $t\colon H^1(\tilde{M}(\alpha, \iota); Q) \to H^1(\tilde{M}(\alpha, \iota); Q)$ and the form $(\ ,\ )\colon H^1(\tilde{M}(\alpha, \iota); Q) \times H^1(\tilde{M}(\alpha, \iota); Q) \to Q$ represent the matrices $A_*'^{-1}A_*$ and A_*+A_*', respectively.*

Note that by Lemma 2.11 $\sigma(M(\alpha, \iota))=\sigma(A_*+A_*')=\sigma(A+A')$.

For the quadratic form $((\ ,\ ), t)$ of the type m of $M(\alpha, \iota)$, if $H^1(\tilde{M}(\alpha, \iota); Q)$ contains a half-dimensional vector subspace V with $tV=V$ and such that $(x, y)=0$ for all x, y in V, then the quadratic form $((\ ,\ ), t)$ is said to be *null-cobordant* (See Levine [10].).

The following theorem is a basically important result.

Theorem 2.12. *If $m \sim 0$, then the quadratic form $((\ ,\ ), t)$ of m is null-cobordant.*

Proof. Since $m \sim 0$, for $M(\alpha, \iota) \in m$ there exists an $\tilde{H}$-cobrodism $(W, M(\alpha, \iota), \phi)$. Hence for some $\varphi \in H^1(W; Z)$ with $\varphi \,|\, M(\alpha, \iota) \in H^1(M(\alpha, \iota); Z)$ dual to α, the infinite cyclic cover $\tilde{W}_\varphi$ associated with φ has a finitely generated rational homology group $H_*(\tilde{W}_\varphi; Q)$. Note that by Kawauchi [6, Theorem 2.3], the Poincaré dualities $\cap \bar{\mu}\colon H^*(\tilde{W}_\varphi\colon Q) \approx H_{3-*}(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Q)$ and $\cap \bar{\mu}\colon H^*(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Q) \approx H_{3-*}(\tilde{W}_\varphi; Q)$ hold, where $\bar{\mu} \in H_3(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Z)$ is a finite fundamental class determined from μ by the boundary-isomorphism $\partial\colon H_3(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Z) \approx H_2(\tilde{M}(\alpha, \iota); Z)$.

Now we consider the following commutative (up to sign) diagram:

$$\begin{array}{ccccccc}
\longrightarrow H^1(\tilde{W}_\varphi; Q) & \xrightarrow{i^*} & H^1(\tilde{M}(\alpha, \iota); Q) & \xrightarrow{\delta} & H^2(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Q) \longrightarrow \\
\downarrow \cap \bar{\mu} & & \downarrow \cap \mu & & \downarrow \cap \bar{\mu} \\
\longrightarrow H_2(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); Q) & \xrightarrow{\partial} & H_1(\tilde{M}(\alpha, \iota); Q) & \xrightarrow{i_*} & H_1(\tilde{W}_\varphi; Q) \longrightarrow .
\end{array}$$

Here the top and bottom sequences are exact and the vertical homomorphisms are isomorphisms.

For all $u \in H^1(\tilde{W}_\varphi; Q)$, suppose $(i^*(u), y)=0$. This situation is equivalent to $\delta(t-t^{-1})y=0$ *i.e.* $(t-t^{-1})y \in \operatorname{Im} i^*$, because $(i^*u, y)=[i^*(u) \cap (t-t^{-1})y] \cap \mu = [u \cup \delta(t-t^{-1})y] \cap \bar{\mu}$. Using $(t-t^{-1})\operatorname{Im} i^* \subset \operatorname{Im} i^*$ and the isomorphism*) $t-t^{-1}\colon H^1(\tilde{M}(\alpha, \iota); Q) \approx H^1(\tilde{M}(\alpha, \iota); Q)$, $(t-t^{-1})y \in \operatorname{Im} i^*$ is equivalent to $y \in \operatorname{Im} i^*$. Thus we showed that the orthogonal complement of $\operatorname{Im} i^*$ is $\operatorname{Im} i^*$ itself. In particular, $\dim_Q \operatorname{Im} i^* = \frac{1}{2} \dim_Q H^1(\tilde{M}(\alpha, \iota); Q)$. Since $t \operatorname{Im} i^* \subset \operatorname{Im} i^*$, the quad-

---

*) To prove this isomorphism, it suffices to check that the characteristic polynomial $A'(t)$ of $t\colon H^1(\tilde{M}(\alpha, \iota); Q) \to H^1(\tilde{M}(\alpha, \iota); Q)$ satisfies $A'(\pm 1) \neq 0$, because $t-t^{-1}=t^{-1}(t-1)(t+1)$. For the Alexander polynomial $A(t)$ of $M(\alpha, \iota)$, $A'(t)$ equals to $A(t)$ up to units of $Q[t]$: $A'(t) \doteq A(t)$. (See [7, Lemma 2.6].) Since $A(\pm 1) \neq 0$, the result follows.

ratic form $((\ ,\ ), t)$ is null-cobordant. This completes the proof.

Lemma 2.13. *There is a homomorphism ψ: $\Omega(S^1\times S^2)\to G_-$.*

Proof. Let $m\in\mathfrak{C}_+(S^1\times S^2)$ and A a Seifert matrix for m. We define $\psi[m]=[A]$. To prove the well-definedness, first we shall show that if $m\sim 0$, then A is null-cobordant. By Lemma 2.11, A is S-equivalent to a non-singular Seifert matrix A_* such that t represents $A_*'^{-1}A_*$ and the form $(\ ,\ )$ represents A_*+A_*'. Since by Thoerem 2.13 the quadratic form $((\ ,\ ), t)$ is null-cobordant, there exists a *symplectic* basis $e_1, e_2, \cdots, e_s, e_1^*, e_2^*, \cdots, e_s^*$ of $H^1(\tilde{M}(\alpha, \iota); Q)$: $(e_i, e_j)=(e_i^*, e_j^*)=0$, $(e_i, e_j^*)=\delta_{ij}$ such that the vector subspace V spanned by $e_1, e_2, \cdots, e_s$ is invariant under t. (See for example Milnor-Husemoller [12, p 13].) Then there is a non-singular *rational* matrix P such that the matrix $P^{-1}A_*'^{-1}A_*P$ is of the form $\begin{pmatrix} Q & R\\ O & S\end{pmatrix}$ (, since $tV=V$), where Q, R, S are rational square matrices of the same size, and such that $P'(A_*+A_*')P=\begin{pmatrix} O & I\\ I & O\end{pmatrix}$, $I=\begin{pmatrix} O & & 1\\ & 1 & \\ 1 & \cdot^{\cdot^{\cdot}} & O\end{pmatrix}$.

Using the equality $P'A_*P=[P'(A_*+A_*')P(E+P^{-1}A_*'^{-1}A_*P)^{-1}]'$ (E is the unit matrix.), it is not difficult to see that the matrix $P'A_*P$ is of the form $\begin{pmatrix} O & B\\ C & D\end{pmatrix}$. ($B, C, D$ are rational square matrices of the same size.) [Note that $\det(E+P^{-1}A_*'^{-1}A_*P)\neq 0$, since the Alexander polynomial $A(t)$ satisfies $A(-1)\neq 0$.] Then by Levine [9, Lemma 8] A_* is null-cobordant. Since A is S-equivalent to A_*, it follows that A is cobordant to A_*. Hence A is null-cobordant. Let $m_1, m_2\in\mathfrak{C}_+(S^1\times S^2)$. Notice that if A_1, A_2 are Seifert matrices for m_1, m_2, respectively, then the block sum $A_1\oplus A_2$ is a Seifert matirx for a circle union $m_1\bigcirc m_2$. [To see this, let $M_i(\alpha_i, \iota_i)\in m$, $i=1, 2$, and consider homological knots $k_i\subset\bar{S}_i^3$ obtained from $M(\alpha_i, \iota_i)$, $i=1, 2$. Then one can verify that the homological knot sum $(k_1\subset\bar{S}_1^3)\#(k_2\subset\bar{S}_2^3)$, defined to be analogous to the usual knot sum, is a homological knot obtained from some circle union $M_1(\alpha_1, \iota_1)\bigcirc M_2(\alpha_2, \iota_2)$. Now the desired result easily follows.] If $m_1\sim m_2$, then $m_1\bigcirc -m_2\sim 0$. Hence the block sum $A_1\oplus -A_2$ is null-cobordant, since $A_1\oplus -A_2$ is a Seifert matrix for $m_1\bigcirc -m_2$. Thus, $[m_1]=[m_2]$ implies $[A_1]=[A_2]$; that is, $\psi[m]=[A]$ is well-defined. Further, ψ is a homomorphism, since for any $m_1, m_2\in\mathfrak{C}_+(S^1\times S^2)$

$$\begin{aligned}\psi([m_1]+[m_2]) &= \psi[m_1\bigcirc m_2]\\ &= [A_1\oplus A_2]\\ &= [A_1]+[A_2]\\ &= \psi[m_1]+\psi[m_2]\,.\end{aligned}$$

This completes the proof.

2.14. Proof of Theorem 2.1. Levine [9] defined the homolorphism $\phi: C^1 \to G_-$ sending any knot cobordism class to the matrix cobordism class of the corresponding Seifert matices. By Lemma 2.4, the homomorphism $e: C^1 \to \Omega(S^1 \times S^2)$ is obtained and by Lemma 2.13, the homomorphism $\psi: \Omega(S^1 \times S^2) \to G_-$ is obtained. From construction, we have $\psi e = \phi$. Since ϕ is onto (See for example Levine [9].), ψ is onto. This proves Theorem 2.1.

Here are four corollaries to Theorem 2.1.

Corollarly 2.15. *The $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$ has the free part of infinite rank.*

This follows from the facts that G_- has the free part of infinite rank and that the homomorphism ψ is onto.

The reduced Alexander polynomial $\tilde{A}(t)$ of a type $m \in \mathfrak{C}_+(S^1 \times S^2)$ is the integral polynomial obtained from the Alexander polynomial $A(t)$ of m by cancelling the factors of the type $f(t)f(t^{-1})$.

Corollary 2.16. *If $m \sim 0$, then the Alexander polynomial $A(t)$ splits as follows: $A(t) \doteq f(t)f(t^{-1})$ for some integral polynomial $f(t)$ and the signature $\sigma(m)$ is 0. More generally, if $m_1 \sim m_2$, then the reduced Alexander polynomils $\tilde{A}_1(t)$, $\tilde{A}_2(t)$ are the same polynomial (up to $\pm t^i$): $\tilde{A}_1(t) \doteq \tilde{A}_2(t)$ and the signatures $\sigma(m_1)$, $\sigma(m_2)$ are equal: $\sigma(m_1) = \sigma(m_2)$.*

Corollary 2.17. *For any $[k] \in C^1$, the equalities $\sigma[k] = \sigma([e(k)])$ and $\tilde{A}_{[k]}(t) = \tilde{A}_{[e(k)]}(t)$ hold.*

Corollary 2.18. *For any $m \in \mathfrak{C}_+(S^1 \times S^2)$, the signature $\sigma(m)$ is even. For any integer i, there exists $m \in \mathfrak{C}_+(S^1 \times S^2)$ with $\sigma(m) = 2i$.*

2.19. Addendum. *Re*-examination of the Seifert matrices. Let $m \in \mathfrak{C}_+(S^1 \times S^2)$ and $M(\alpha, \iota) \in m$. A Seifert matrix for $M(\alpha, \iota)$ (or m) may be also defined as follows: Let $f: M(\alpha, \iota) \to S^1$ be a piecewise-linear map with $f_*: H_1(M(\alpha, \iota); Z) \approx H_1(S^1; Z)$ and such that for some point $0 \in S^1$, $F = f^{-1}(0)$ is a closed orientable connected surface (See Kawauchi [6, Corollary 1.3].). Using that $[F] \in H_2(M(\alpha, \iota); Z)$ is a generator, we may orient F so that $[F] = \varphi \cap \iota$, where $\varphi \in H^1(M(\alpha, \iota); Z)$ is a dual element of $\alpha \in H_1(M(\alpha, \iota); Z)$. Let M^* be the oriented manifold (with orientation induced by that of $M(\alpha, \iota)$) obtained from $M(\alpha, \iota)$ by splitting along F. Let $\partial M^* = F \cup F'$. Here the component of ∂M^* with orientation coinciding with that of F is identified with F. F' denotes the copy of F but with the oposite orientation. Let $i': F \to F' \subset \partial M^* \subset M^*$ be the natural injection. If $a \in H_1(F; Z)$, let $a' \in H_2(M(\alpha, \iota), M(\alpha, \iota) - F; Z)$ be the image of a under the composite

$$H_1(F;Z)\xrightarrow{i'_*}H_1(M^*;Z)\underset{\approx}{\longrightarrow}H_1(M(\alpha,\iota)-F;Z)\underset{\approx}{\xrightarrow{\partial^{-1}}}H_2(M(\alpha,\iota),M(\alpha,\iota)F;Z).$$

By using a duality γ_U: $H_2(M(\alpha, \iota), M(\alpha, \iota)-F; Z)\approx H^1(F; Z)$, relating a slant product, where U is the Thom class of $M(\alpha, \iota)$ corresponding to the fundamental class ι, define a pairing

$$\theta'\colon\ H_1(F;Z)\otimes H_1(F;Z)\to Z$$

by the equality $\theta'(a\otimes b)=\gamma_U(a')\cap b\in H_0(F; Z)=Z$.

It is checked that with a basis for $H_1(F; Z)$ *θ' represents a Seifert matrix for* $M(\alpha, \iota)$. The formula $\theta'(a\otimes b)-\theta'(b\otimes a)=a\cdot b$ is also obtained.

3. Elements of $\Omega(S^1\times S^2)$ of order zero and two and the $\tilde{H}$-cobordism group $\Omega(S^1\times_\tau S^3)$ of homology non-orientable handles

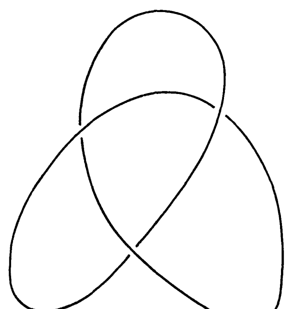
A general problem of bringing about a better understanding of $\tilde{H}$-cobordism between the types of distinguished homology orientable handles seems still difficult, but a partial answer is presented here.

Theorem 3.1. *If a representative homology orientable handle $M(\alpha, \iota)$ of a type $m\in\mathfrak{C}_+(S^1\times S^2)$ is embeddable in a homology 4-sphere $\bar{S}^4$, then $m\sim 0$.*

Proof. Assume $M(\alpha, \iota)\subset\bar{S}^4$. Then $M(\alpha, \iota)$ separates $\bar{S}^4$ into two manifolds, say, W_1, W_2 and, by easy computation of the homology, one of W_1, W_2 has the homology of a circle, say, $H_*(W_1; Z)\approx H_*(S^1; Z)$. Then the triad $(W_1, M(\alpha, \iota), \phi)$ gives an $\tilde{H}$-cobordism. This completes the proof of Theorem 3.1.

Here are a few examples, whose somewhat analogous properties were also noticed by Kato [5, Theorems 5.1 and 5.5] in higher dimensions.

EXAMPLES 3.2. First we consider a (suitably oriented) trefoil 3_1. (See fiugre 5.)

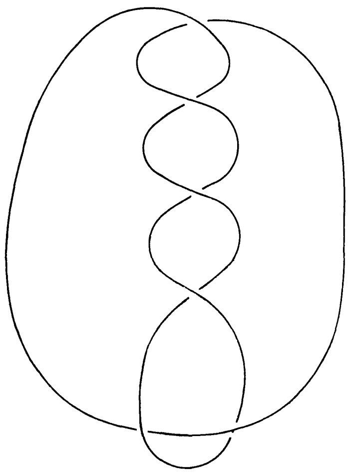


3_1

figure 5.

Using that $\sigma(e(3_1))=\sigma(3_1)=\pm 2\neq 0$ or that $A(t)=t^2-t+1$ is irreducible, $e(3_1)\not\sim 0$. Hence by Theorem 3.1, $e(3_1)$ is not embeddable to the 4-sphere S^4. Note that $e(3_1)$ is locally-flatly embeddable to the 5-sphere S^5, since according to Hirsch [4] every compact orientable 3-manifold is locally-flatly embeddable to S^5.

On the other hand, consider the stevedore's knot 6_1. (See figure 6.)

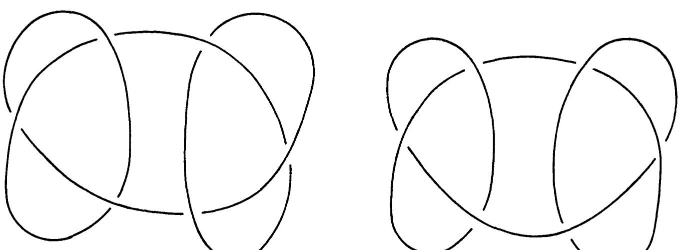


6_1

figure 6.

Since this knot is a slice knot, by Corollary 2.5, $e(6_1)$ is embeddable to S^4.

Similar arguments also apply for the granny knot $3_1 \# 3_1$ and the square knot $3_1 \# -3_1^*$. (See figure 7.)



$3_1 \# 3_1$ $3_1 \# -3_1^*$

figure 7.

In fact, $e(3_1 \# 3_1)$ is not embeddable to S^4, although $e(3_1 \# -3_1^*)$ is embeddable to S^4, since $\sigma(e(3_1 \# 3_1))=2\sigma(3_1)=\pm 4\neq 0$ and $3_1 \# -3_1^*$ is a slice knot.

Next we would like to discuss order-two-elements of $\Omega(S^1\times S^2)$. To do this, we shall introduce the $\tilde{H}$-cobordism group of homology non-orientable handles.

A homology non-orientable handle M is a compact 3-manifold having the homology of the non-orientable handle $S^1\times_\tau S^2$: $H_*(M; Z)\approx H_*(S^1\times_\tau S^2; Z)$, and is said to be *distinguished* if a generator $\alpha\in H_1(M; Z)$ is specified. If a homology non-orientable handle M is distinguished, then the notation $M(\alpha)$ will be used. Two distinguished homology non-orientable handles $M_1(\alpha_1)$, $M_2(\alpha_2)$ have *the same type* if there is a piecewise-linear homeomorphism h: $M_1(\alpha_1)\to M_2(\alpha_2)$ such that $h_*(\alpha_1)=\alpha_2$. *The type* of $M(\alpha)$ is the class of distinguished homology non-orientable handles with the same type as $M(\alpha)$. The set of the types is denoted by $\mathfrak{C}_+(S^1\times_\tau S^2)$.

In $\mathfrak{C}_+(S^1\times_\tau S^2)$ an $\tilde{H}$-cobordism relation is defined as an analogy of Definition 1.1.

DEFINITION 3.3. Two types m_1, m_2 in $\mathfrak{C}_+(S^1\times_\tau S^2)$ are *$\tilde{H}$-cobordant* and denoted by $m_1\sim m_2$ if for $M_1(\alpha_1)\in m_1$, $M_2(\alpha_2)\in m_2$ there exists a pair (W, φ), where W is a compact connected 4-manifold with $\partial W=M_1(\alpha_1)+M_2(\alpha_2)$ (disjoint union) and $\varphi\in H^1(W; Z)$ whose restrictions $\varphi|M_i(\alpha_i)\in H^1(M_i(\alpha_i); Z)$ are dual to α_i, $i=1, 2$, such that the infinite cyclic cover $\tilde{W}_\varphi$ associated with φ is orientable and has a finitely generated rational homology group $H_*(\tilde{W}_\varphi; Q)$. [Note that any infinite cyclic cover $\tilde{M}(\alpha)$ is always orientable (See Kawauchi [7].).]

Let m_0, $m_1\in\mathfrak{C}_+(S^1\times_\tau S^2)$ and $M_0(\alpha_0)\in m_0$, $M_1(\alpha_1)\in m_1$. Choose polygonal oriented simple closed curves $\omega_0\subset M_0(\alpha_0)$, $\omega_1\subset M_1(\alpha_1)$ which represent α_0, α_1, respectively. It is not difficult to see that the regular neighborhoods $T(\omega_0)\subset M_0(\alpha_0)$ of ω_0 and $T(\omega_1)\subset M_1(\alpha_1)$ of ω_1 are both peicewise-linearly homeomorphic to the solid Kliein bottle $S^1\times_\tau B^2$. Note that there exists closed connected *orientable* surfaces $F_0\subset M_0(\alpha_0)$, $F_1\subset M_1(\alpha_1)$ transversally intersecting ω_0, ω_1, in single points, respectively.

Consider two piecewise-linear embeddings

$$h_0\colon\ S^1\times_\tau B^2\times 0\to M_0(\alpha_0)$$
$$h_1\colon\ S^1\times_\tau B^2\times 1\to M_1(\alpha_1)$$

such that there exist points $s\in S^1$, $b\in \mathrm{Int}\, B^2$ with $h_0(S^1\times_\tau b\times 0)=\omega_0$, $h_0(s\times_\tau B^2\times 0)\subset F_0$, $h_1(S^1\times_\tau b\times 1)=\omega_1$ and $h_1(s\times_\tau B^2\times 1)\subset F_1$ and such that ω_0 and ω_1 are homologous in the adjunction space $M_0(\alpha_0)\cup_{h_0}S^1\times_\tau B^2\times[0, 1]\cup_{h_1}M_1(\alpha_1)$.

As an analogy of Definition 1.4, we may have Definition 3.4.

DEFINITION 3.4. The homology non-orientable handle

$$M_0(\alpha_0)\bigcirc M_1(\alpha_1)=M_0(\alpha_0)\cup_{h_0}S^1\times_\tau B^2\times[0, 1]\cup_{h_1}M_1(\alpha_1)-S^1\times_\tau \mathrm{Int}\, B^2\times[0, 1]$$

distinguished naturally is called *a circle union* of $M_0(\alpha_0)$ and $M_1(\alpha_1)$. The type of $M_0(\alpha_0)\bigcirc M_1(\alpha_1)$ is denoted by $m_0\bigcirc m_1$.

It is not difficult to check that for two circle unions $m_0\bigcirc m_1$, $m_0\bigcirc' m_1$, $m_0\bigcirc m_1\sim m_0\bigcirc' m_1$. Further, we can prove that $m_0\sim m_1$ if and only if $m_0\bigcirc m_1\sim 0$ as an analogy of Lemma 1.7, where 0 is the type of $S^1\times_\tau S^2$. [Note that $S^1\times_\tau S^2(\alpha)$ has the same type as $S^1\times_\tau S^2(-\alpha)$.] As a result, the set $\Omega(S^1\times_\tau S^2)=\mathfrak{C}_+(S^1\times_\tau S^2)/\sim$ forms an abelian group under the sum $[m_0]+[m_1]=[m_0\bigcirc m_1]$, called *the $\tilde{H}$-cobordism group of homology non-orientable handles*. Every non-zero element of $\Omega(S^1\times_\tau S^2)$ has order 2, since $m\sim m$ implies $m\bigcirc m\sim 1$ The zero element of $\Omega(S^1\times_\tau S^2)$ is the $\tilde{H}$-cobordism class containing the type 0 of $S^1\times_\tau S^2$.

Theorem 3.5. *$\Omega(S^1\times_\tau S^2)$ is the direct sum of infinite copies of the cyclic group of order* 2.

To prove Theorem 3.5, the Alexander polynomial seems to be usefull.

The Alexander polynomial $A(t)$ of $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$ is the integral polynomial which is a generator of the smallest principal ideal containing the ideal associated with a relation matrix of $H_1(\tilde{M}(\alpha);Z)$as a $Z[t]$-module (See Kawauchi [7] for details.). Here, $\tilde{M}(\alpha)$ denotes the infinite cyclic cover of $M(\alpha)\in m$ and t denotes a generator of the covering transformation group of $\tilde{M}(\alpha)$, *related to the generator* $\alpha\in H_1(M(\alpha);Z)$. $A(t)$ is the complete invariant of $M(\alpha)$ or the type m up to units$\pm t^s\in Z(t)\cdot A(t)$ satisfies the properties that $A(t)\doteq A(-t^{-1})$ and $A|(1)|=1$; and, conversely, any integral polynomial with these properties is the Alexander polynomial of some $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$. (See [7].) For characteristic polynomial $A'(t)$ of the linear isomorphism t: $H_1(\tilde{M}(\alpha);Q)\to H_1(\tilde{M}(\alpha);Q)$ we have $A(t)\doteq A'(t)$, that is, $A(t)$, $A'(t)$ are equal up to units $qt^s\in Q[t]$.

The following is an analogous result to Corollary 2.16.

Lemma 3.6. *Let $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$. If $m\sim 0$, then the Alexander polynomial $A(t)$ of m has a type of $f(t)f(-t^{-1})$ for some integral polynomial $f(t)$.*

Before showing Lemma 3.6 we shall show Theorem 3.5.

3.7. Proof of Theorem 3.5. Consider for example the irreducible integral polynomials $A_n(t)=nt^2+t-n$, $n=1, 2, 3, \cdots$. These $A_n(t)$ are realized as the Alexander polynomials of some $m_n\in\mathfrak{C}_+(S^1\times_\tau S^2)$, $n=1, 2, 3,\cdots$. Then it is easy to see that m_1, m_2, m_3, $\cdots$ represent a set of linearly independent elements of $\Omega(S^1\times_\tau S^2)$. [Notice that if $A_1(t)$, $A_2(t)$ are the Alexander polynomials of m_1, m_2, respectively, then the product $A_1(t)A_2(t)$ is the Alxeander polynomial of any circle union $m\bigcirc m_2$.] This completes the proof.

3.8. Proof of Lemma 3.6. Since $m\sim 0$, for $M(\alpha)\in m$ there exists a pair (W, φ), where W is a compact connected 4-manifold with $\partial W=M(\alpha)$ and $\varphi\in H^1(W; Z)$ with $\varphi|M(\alpha)\in H^1(M(\alpha); Z)$ dual to α, such that the infinite cyclic cover $\tilde{W}_\varphi$ is orientable and has a finitely generated rational homology group $H_*(\tilde{W}_\varphi; Q)$. Then from the exact sequence $H^1(\tilde{W}_\varphi; Q)\xrightarrow{i^*} H^1(\tilde{M}(\alpha); Q) \xrightarrow{\delta} H^2(\tilde{W}_\varphi, \tilde{M}(\alpha); Q)$ we obtain the short exact sequence $0\to \operatorname{Im} i^* \to H^1(\tilde{M}(\alpha); Q) \to \operatorname{Im}\delta \to 0$. Then we have $A(t)\doteq B(t)C(t)$, where $B(t)$, $C(t)$ are the characteristic poylnomials of t: $\operatorname{Im} i^*\to \operatorname{Im} i^*$, t: $\operatorname{Im}\delta\to\operatorname{Im}\delta$, respectively. Since the square

$$\begin{array}{ccc} H^1(\tilde{M}(\alpha); Q) & \xrightarrow{\quad\delta\quad} & H^2(\tilde{W}_\varphi, \tilde{M}(\alpha); Q) \\ \approx\downarrow\cap\mu & & \approx\downarrow\cap\bar{\mu} \\ H_1(\tilde{M}(\alpha); Q) & \xrightarrow{\quad i_*\quad} & H_1(\tilde{W}_\varphi; Q) \end{array}$$

is commutative, we obtain the Poincaré dual isomorphism $\cap\bar{\mu}$: $\operatorname{Im}\delta\approx\operatorname{Im} i_*$, where $\mu\in H_2(\tilde{M}(\alpha); Z)$ and $\bar{\mu}\in H_3(\tilde{W}_\varphi, \tilde{M}(\alpha); Z)$ are the finite fundamental classes such that $\bar{\mu}$ is mapped to μ by the boundary isomorphism ∂: $H_3(\tilde{W}_\varphi, \tilde{M}(\alpha); Z)\approx H_2(\tilde{M}(\alpha); Z)$ $(\approx Z)$. (See Kawauchi [6, Theorem 2.3].) Notice that $t\bar{\mu}=-\bar{\mu}$. Using the identity $\operatorname{Im} i^*=\operatorname{Hom}(\operatorname{Im} i_*, Q)$ and the equality $(tu)\cap\bar{\mu}=-t^{-1}(u\cap\bar{\mu})$, the Poincaré dual isomorphism $\cap\bar{\mu}$: $\operatorname{Im}\delta\approx\operatorname{Im} i_*$ gives the equality$C(-t^{-1})\doteq B(t)$. This proves Lemma 3.6.

Lemma 3.9. *This is a well-defined function*

$$\tau\colon \mathfrak{C}_+(S^1\times_\tau S^2)\to\mathfrak{C}_+(S^1\times S^2)$$

induced by the 2-fold orientation covering.

Proof. Let $m\ni\mathfrak{C}_+(S^1\times_\tau S^2)$ and $M(\alpha)\in m$. Consider the infinite cyclic covering p: $\tilde{M}(\alpha)\to M(\alpha)$ associated with the Hurewicz homomorphism. Let t be the generator of the covering transformation group of $\tilde{M}(\alpha)$ related to α. The 2-fold covering τ': $M'\to M(\alpha)$ from the orbits space $M'=\tilde{M}(\alpha)/t^2$ to $M(\alpha)$ induced by the projection p: $\tilde{M}(\alpha)\to M(\alpha)$ is the 2-fold orientation covering, since $\tilde{M}(\alpha)$ is orientable.

We must prove that M' is a homology orientable handle. Let p': $\tilde{M}(\alpha)\to M'$ be the natural projection. The short exact sequence $0\to C_\sharp(\tilde{M}(\alpha))\xrightarrow{t^2-1} C_\sharp(\tilde{M}(\alpha)) \xrightarrow{p'} C_\sharp(M')\to 0$ of simplicial chain $Z[t^2]$-modules induces the following exact sequence

$$\begin{array}{c} H_1(\tilde{M}(\alpha); Z) \xrightarrow{p'_*} H_1(M'; Z) \to H_0(\tilde{M}(\alpha); Z) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Z \end{array}$$

of $Z[t^2]$-modules, where $H_1(M'; Z)$ and $H_0(\tilde{M}(\alpha); Z)$ are regarded as trivial $Z[t^2]$-modules. Let ε: $Z[t^2] \to Z$ be the augmentation homomorphism such that $\varepsilon(t^2)=1$. By taking a tensor product, we obtain an exact sequence

$$\begin{array}{ccccccc} H_1(\tilde{M}(\alpha); Z)\otimes_\varepsilon Z & \xrightarrow{p'_*\otimes 1} & H_1(M'; Z)\otimes_\varepsilon Z & \to & H_0(\tilde{M}(\alpha); Z)\otimes_\varepsilon Z & \to & 0\,. \\ & & \| & & \| & & \\ & & H_1(M'; Z) & & Z & & \end{array}$$

Sublemma 3.9.1. $H_1(\tilde{M}(\alpha); Z)\otimes_\varepsilon Z=0$.

By assuming this sublemma, we obtain that $H_1(M'; Z)\approx Z$. By the Poincaré duality, M' is a homology orientable handle. Let $\alpha' \in H_1(M'; Z)$ be a generator determined by α under the 2-fold orientation covering τ: $M' \to M(\alpha)$. Let $\iota \in H_3(M'; Z)$ be any generator. The distinguished homology orientable handles $M'(\alpha', \iota)$, $M'(\alpha', -\iota)$ have the same type, because t of $\tilde{M}(\alpha)$ induces a homeomorphism t': $M' \to M'$ with $t'_*(\alpha')=\alpha'$ and $t'_*(\iota)=-\iota$. This type is denoted by $\tau(m)$. Thus the function τ: $\mathfrak{C}_+(S^1\times_\tau S^2) \to \mathfrak{C}_+(S^1\times S^2)$ is obtained. This completes the proof.

3.10. Proof of Sublemma 3.9.1. Note that there exists a presentation square matrix $S(t)$ of $H_1(\tilde{M}(\alpha); Z)$ as a $Z[t]$-module *i.e.* $Z[t]^{2g} \xrightarrow{S(t)} Z[t]^{2g} \to H_1(\tilde{M}(\alpha); Z) \to 0$ is exact for some integer $g \geq 0$. [To see this, let $F \subset M(\alpha)$ be a closed orientable connected 2-sided surface in $M(\alpha)$ intersecting a simple closed curve representing α in a single ponit, and M^* be the manifold obtained from $M(\alpha)$ by splitting along F. Since M^* is orientable, we have an isomorphism $H_1(M^*; Z) \approx H_1(F; Z)$. Let i_1, i_2: $F \to F_1 \cup F_2 = \partial M^* \subset M^*$ be two natural identifications. With suitable bases of $H_1(F; Z)$, $H_1(M^*; Z)$, i_{1*}, i_{2*}: $H_1(F; Z) \to H_1(M^*; Z)$ represent square integral matrices S_1, S_2, respectively. By applying the Mayer-Vietoris sequence, we obtain an exact sequence

$$H_1(F; Z)\otimes Z[t] \xrightarrow{i_*} H_1(M^*; Z)\otimes Z[t] \to H_1(\tilde{M}(\alpha); Z) \to 0\,,$$

where $i_*(x)=ti_{1*}(x)-i_{2*}(x)$. Thus, we can obtain an exact sequence

$$Z[t]^{2g} \xrightarrow{S(t)} Z[t]^{2g} \to H_1(\tilde{M}(\alpha): Z) \to 0\,,$$

where $S(t)=tS_1-S_2$.] By taking a tensor product, we obtain an exact sequence

$$\begin{array}{ccccccc} Z[t]^{2g}\otimes_\varepsilon Z & \xrightarrow{S^\varepsilon(t)} & Z[t]^{2g}\otimes_\varepsilon Z & \to & H_1(\tilde{M}(\alpha); Z)\otimes_\varepsilon Z & \to & 0\,. \\ \| & & \| & & & & \\ [Z[t]/(t^2-1)]^{2g} & & [Z[t]/(t^2-1)]^{2g} & & & & \end{array}$$

We shall show that $A^{\varepsilon}(t)=\det S^{\varepsilon}(t)$ is a unit in the quotient ring $Z[t]/(t^2-1)$. Note that $A(t)=\det S(t)$ is the Alexander polynomial of $M(\alpha)$. So, $A(t)$ satisfies $A(t)\doteq A(-t^{-1})$ and $|A(1)|=1$. We can write $t^{-s}A(t)=\sum_{i=-s}^{s} a_i t^i$, $a_i=(-1)^i a_{-i}$ $(s>0)$. Then $t^{\eta(s)}A^{\varepsilon}(t)=A^{\varepsilon}(1)$ and $A^{\varepsilon}(t)=t^{\eta(s)}A^{\varepsilon}(1)$ is a unit in $Z[t]/(t^2-1)$, where $\eta(s)=0$ if s is even, 1 if s is odd. This implies that the homomorphism $S^{\varepsilon}(t)$: $Z[t]^{2g}\otimes_{\varepsilon}Z\to Z[t]^{2g}\otimes_{\varepsilon}Z$ is an isomorphism. Therefore $H_1(\tilde{M}(\alpha);Z)\otimes_{\varepsilon}Z=0$. This proves Sublemma 3.9.1.

Lemma 3.11. *The function* τ: $\mathfrak{C}_+(S^1\times_\tau S^2)\to\mathfrak{C}_+(S^1\times S^2)$ *carries the Alexander polynomial* $A(t)$ *of any* $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$ *to the Alexander polynomial* $A^\tau(t)$ *of* $\tau(m)\in\mathfrak{C}_+(S^1\times S^2)$ *such that* $A^\tau(t^2)\doteq A(t)A(-t)$.

Proof. Let $M(\alpha)\in m$. With a basis for $H_1(\tilde{M}(\alpha); Q)$, t: $H_1(\tilde{M}(\alpha); Q)\to H_1(\tilde{M}(\alpha); Q)$ represents a matrix B. Then $A(t)\doteq\det(tE-B)$. For the linear isomorphism $t'=t^2$: $H_1(\tilde{M}(\alpha); Q)\to H_1(\tilde{M}(\alpha); Q)$ representing B^2, we have $A^\tau(t')\doteq\det(t'E-B^2)$. Hence,

$$\begin{aligned}A^\tau(t^2)&\doteq\det(t^2E-B^2)\\&\doteq\det(tE-B)\det(tE+B)\\&\doteq A(t)A(-t)\,.\end{aligned}$$

This completes the proof.

The reduced Alexander polynomial $\tilde{A}(t)$ of $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$ is the integral polynomial obtained from the Alexander polynomial $A(t)$ of m by cancelling the factors of the type $f(t)f(-t^{-1})$.

Theorem 3.12. *The function* τ: $\mathfrak{C}_+(S^1\times_\tau S^2)\to\mathfrak{C}_+(S^1\times S^2)$ *induces a homomorphism* τ^*: $\Omega(S^1\times_\tau S^2)\to T_2\subset\Omega(S^1\times S^2)$ *carrying the reduced Alexander polynomial* $\tilde{A}(t)$ *to the reduced Alexander polynomial* $\tilde{A}^\tau(t)$ *such that* $A^\tau(t^2)\doteq A(t)A(-t)$, *where* T_2 *is the subgroup of* $\Omega(S^1\times S^2)$ *consisting of elements of order* 2.

Proof. For $m_1, m_2\in\mathfrak{C}_+(S^1\times_\tau S^2)$, the equality $\tau(m_1\bigcirc m_2)=\tau(m_1)\bigcirc\tau(m_2)$ is easily obtained. For $m\in\mathfrak{C}_+(S^1\times_\tau S^2)$, assume $m\sim 0$. Then for $M(\alpha)\in m$ there exists an $\tilde{H}$-cobordism $(W, M(\alpha), S^1\times_\tau S^2)$. The 2-fold orientation cover $(W', M', S^1\times S^2)$ of $(W, M(\alpha), S^1\times_\tau S^2)$ gives an $\tilde{H}$-cobordism. So, $\tau(m)\sim 0$. Therefore τ^* is a homomorphism to T_2. The remainder follows from Corollary 2.16 and Lemmas 3.6 and 3.11. This completes the proof.

Corollary 3.13. T_2 *is infinitely generated.*

Proof. Consider for example $m_n\in\mathfrak{C}_+(S^1\times_\tau S^2)$ with Alexander polynomial $A_n(t)=nt^2+t-n$, $n=1, 2, 3, \cdots$, as in 3.7. Then the Alexander polynomial of the 2-fold orientation cover $\tau(m_n)$ is $A_n(t)=n^2t^2-(2n^2+1)t+n^2$. Since for

$n=1, 2, 3, \cdots$ these Alexander polynomials $A_n(t)$ are irreducible and mutually distinct, the set $\{\tau(m_1), \tau(m_2), \tau(m_3), \cdots\}$ gives a linearly independent subset of T_2, which completes the proof.

One may ask whether the subgroup T_2' of order-2-elements of the Fox-Milnor's knot cobordism group C^1 is infinitely generated.

As a matter of fact, T_2' is also infinitely generated, although it seems to be difficult to set up a general argument.

Claim. *T_2' is infinitely generated.*

In fact, consider the knot $k_n \subset S^3$ with the numbers of crossings $2n$, $2n$, illustrated in figure 8^n. In the case $n=1$, this knot k_1 is called the figure eight knot: $k_1=4_1$ (See figure 8^1.).

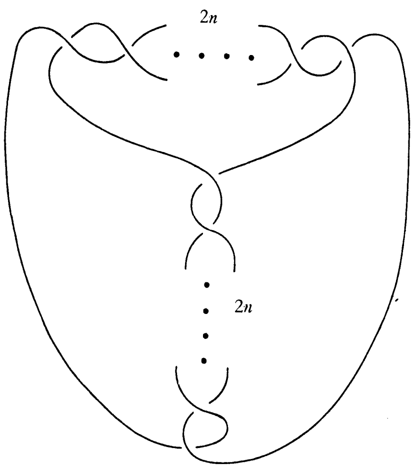


figure 8^n.

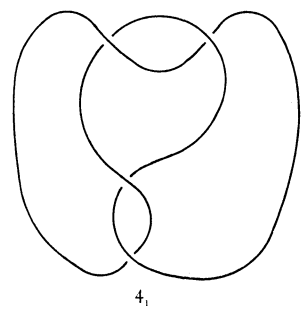


figure 8^1.

One can easily shown*) that each knot $k_n \subset S^3$ is -amphicheiral**) by an analogy of the method which is used for showing that the figure eight knot is -amphicheiral. Since the Alexander polynomial of $k_n \subset S^3$ is $A_n(t)=n^2t^2-(2n^2+1)t+n^2$, which is irreducible, it follows that T_2' is infinitely generated.

One can also derive the conclusion of Corollary 3.13 by using these knots.

In concluding this paper, the author would like to propose a few questions and one interesting conjecture.

Question. *Is* $\mathrm{Im}\,\tau^* = T_2$?

---

*) See, for example, S. Kinoshita and T. Yajima: *On the graphs of knots,* Osaka Math. J. 9 (1957), 155–163.

**) An oriented knot $k \subset S^3$ is said to be *-amphicheiral,* if $-k \subset S^3$ and $-k \subset -S^3$ belong to the same knot type. (See Fox [2, pl 143] for details.)

This question seems closely related to a question due to Fox and Milnor: Is an element of order 2 of C^1 necessarily determined by a -amphicheiral knot?

One may also ask whether τ^* is injective, although the author expects a negative answer.

The following conjecture seems to be justified by Lemma 3.11.

Conjecture. *The Alexander polynomial $A(t)$ of a -amphicheiral knot necessarily satisfies $A(t^2)\doteq f(t)f(-t)$ for some integral polynomial $f(t)$ with $f(t)\doteq f(-t^{-1})$.*

One can easily checked that any -amphicheiral knot in the Alexander and Briggs knot table satisfies this assertion.

For example, the Alexander polynomial of the knot 8_{12} which is known to be -amphicheiral is $A(t)=t^4-7t^3+13t^2-7t+1$. Then,

$$A(t^2)=(t^4+t^3-3t^2-t+1)(t^4-t^3-3t^2+t+1)\,.$$

OSAKA CITY UNIVERSITY

References

[1] D. Erle: *Die quadratische Form eines Knotens und ein Satz über Knotenmannigfaltigkeiten*, J. Reine Angew. Math. **236** (1969), 174–218.

[2] R.H. Fox: *A quick trip through knot theory*, Topology of 3-Manifolds and Related Topics, M.K. Fort, Jr., Perntice-Hall 1962.

[3] ——— and J.W. Milnor: *Singularties of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267.

[4] M.W. Hirsch: *The imbedding of bounding manifolds in Euclidean space*, Ann. of Math. **74** (1961), 494–497.

[5] M. Kato: *Higher dimensional PL knotes and knot manifolds*, J. Math. Soc. Japan **21** (1969), 458-480.

[6] A. Kawauchi: *A partial Poincaré duality theorem for infinite cyclic coverings*, Quart. J. Math. **26** (1975), 437–458.

[7] ———: *Three dimensional homology handles and circles*, Osaka J. Math. **12** (1975), 565-581.

[8] J. Levine: *Polynomial invariants of knots of codimension two*, Ann. of Math. **84** (1966), 537–554.

[9] ———: *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[10] ———: *Invariants of knot cobordism*, Invent. math. **8** (1969), 98–110.

[11] J.W. Milnor: *Infinite cyclic coverings*, Conference on the Topology of Manifolds, Prindle, Weber and Schmidt, Boston Mass. 1968.

[12] ——— and D. Husemoller: Symmetric Bilinear Forms, Springer, 1973.

[13] H.F. Trotter: *On S-eqnivalences of Seifert matrices*, Invent. math. **20** (1973) 173–207.