

A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF NON-SINGULAR G-VECTOR FIELDS ON G-MANIFOLDS

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1. Introduction

Throughout this paper G always denotes a compact Lie group and G -manifolds are smooth manifolds with smooth G -actions. In this paper we will give a necessary and sufficient condition for the existence of non-singular G -vector fields on closed G -manifolds.

Let $E \rightarrow X$ be a G -vector bundle. After choosing a G -invariant Riemannian metric on E , we denote by $S(E) \rightarrow X$ the associated G -sphere bundle of E . We abbreviate *continuous G -equivariant cross section of E to G -cross section of E* .

Let M be a compact G -manifold, and $s: M \rightarrow \tau(M)$ a G -cross section of the tangent bundle $\tau(M)$ of M . s is called a *non-singular G -vector field* on M , if s is not zero at each point of M . For a positive integer k , by a *G - k -field* on M we will mean k G -cross sections of $\tau(M)$ which are linearly independent at each point of M .

For a closed subgroup H of G we set

$$M_H = \{x \in M \mid G_x = H\},$$

where G_x is the isotropy group at x . Let $N(H)$ be the normalizer of H in G . Then M_H is an $N(H)$ -manifold and also a free $N(H)/H$ -manifold.

For a topological space X we define $|\chi|(X)$ to be the sum $\sum_Y |\chi(Y)|$, where $\chi(Y)$ is the Euler characteristic of Y and Y runs over the connected components of X .

In this paper we will obtain the following results:

Theorem 1.1. *Let M be a compact G -manifold. Let $s: \partial M \rightarrow S(\tau(\partial M))$ be a G -cross section of $S(\tau(\partial M))$. Then s is extendible to a G -cross section of $S(\tau(M))$ if and only if*

$$|\chi|(M_{G_x}/N(G_x)) = 0, \quad \text{or} \quad \dim N(G_x) - \dim G_x > 0$$

for all $x \in M$.

From this theorem we immediately obtain the main result of this paper:

Corollary 1.2. *A closed G -manifold M admits a non-singular G -vector field if and only if*

$$|\chi|(M_{G_x}/N(G_x)) = 0, \quad \text{or} \quad \dim N(G_x) - \dim G_x > 0$$

for all $x \in M$.

We also obtain the following corollary:

Corollary 1.3. *Let M be a compact G -manifold, F the stationary point set of M , and k a positive integer. We suppose that*

$$\dim N(G_x) - \dim G_x \geq k$$

for all $x \in M - F$. Then M admits a G - k -field if and only if F admits a k -field.

Note. The existence of a non-singular continuous G -vector field on a compact G -manifold implies the existence of a non-singular smooth G -vector field. This fact is assured by the differentiable approximation theorem [3; (6.7)] and the usual process of averaging cross sections.

2. Preliminaries

(2-1) For a closed subgroup H of G let (H) be the conjugacy class of H in G . If H is an isotropy group occurring on a G -manifold M , (H) is called an isotropy type on M .

Set

$$M_H = \{x \in M \mid G_x = H\}$$

$$M_{(H)} = \{x \in M \mid (G_x) = (H)\}.$$

Then M_H and $M_{(H)}$ are submanifolds of M , but, in general, not compact. If (H) is a maximal isotropy type on a compact G -manifold M , M_H and $M_{(H)}$ are compact.

(2-2) Let $\pi: E \rightarrow X$, $\pi': E' \rightarrow X'$ be G -fibre bundles, and $f: E \rightarrow E'$ a G -bundle map covering $\bar{f}: X \rightarrow X'$. Let $s': X' \rightarrow E'$ be a G -cross section of E' . s' induces a G -cross section

$$s: X \rightarrow \bar{f}^*E' = \{(x, e) \in X \times E' \mid \bar{f}(x) = \pi'(e)\}$$

of the induced G -fibre bundle \bar{f}^*E' which sends $x \in X$ to $(x, s'\bar{f}(x)) \in X \times E'$. E is canonically isomorphic to \bar{f}^*E' . So s induces a G -cross section of E . We denote this G -cross section by f^*s' , and call induced G -cross section from s' by f .

(2-3) Recall known results:

Proposition 2.1 (Segal [2; Proposition 1.3]). *Let X, Y be G -spaces, and X compact. If $f_0, f_1: X \rightarrow Y$ are G -homotopic G -maps, and $E \rightarrow Y$ is a G -vector bundle, then there is a G -bundle isomorphism*

$$\varphi: f_0^*E \cong f_1^*E.$$

From this proposition we easily obtain the following. I denotes the interval $[0,1]$ with trivial G -action.

Proposition 2.2. *If X is a compact G -space, then any G -vector bundle E over $X \times I$ is isomorphic to $(E|X \times \{0\}) \times I$ as G -vector bundles.*

Proposition 2.1 may be stated in a more precise form as the following: *If f_0, f_1 are G -homotopic relative to a closed G -invariant subspace A of X , and if we consider f_0^*E, f_1^*E subspaces of $X \times E$, then the G -bundle isomorphism φ satisfies $\varphi(x, e) = (x, e)$ for all $x \in A$ and $e \in E$.*

From this fact we obtain

Proposition 2.3. *Let $E_i \rightarrow X_i, i=1,2$, be G -vector bundles with X_1 compact. Let $f: X_1 \times I \rightarrow X_2$ be a G -homotopy which is constant on a closed G -invariant subspace A of X_1 . Let $f_0: E_1 \rightarrow E_2$ be a G -bundle map over $f_0 = f|X_1 \times \{0\}$. Then there is a G -bundle map $\tilde{f}: E_1 \times I \rightarrow E_2$ over f which is a homotopy of f_0 and is constant on $E_1|A$.*

Corollary 2.4. *Let $E \rightarrow X$ be a G -vector bundle with X compact, A a closed G -invariant subspace of X , and $i: A \rightarrow X$ the inclusion map. Let $f: X \rightarrow A$ be a G -map such that f is G -homotopic to the identity of X relative to A . Then there is a G -bundle map $\tilde{f}: E \rightarrow E|A$ over f which is the identity on $E|A$.*

(2-4) The following result has been obtained by U. Koschorke in his paper [1; §1].

Proposition 2.5. *Let M be a compact manifold, and $s: \partial M \rightarrow S(\tau(\partial M))$ a cross section of $S(\tau(\partial M))$. Then s is extendible to a cross section of $S(\tau(M))$ if and only if $|\chi|(M) = 0$.*

3. Proof of Theorem 1.1

The proof will proceed by an induction for the number of isotropy types on M .

(3-1) In the first place, let M be of one isotropy type, and let (H) be the isotropy type on M .

“if” part: M_H is a compact $N(H)$ -manifold with boundary $M_H \cap \partial M$. Consider the sphere bundle

$$S(\tau(M_H))/N(H) \rightarrow M_H/N(H).$$

The G -cross section s induces a cross section

$$s_1: \partial(M_H/N(H)) \rightarrow S(\tau(M_H))/N(H) | \partial(M_H/N(H)).$$

This is assured by the G -equivariancy of s and the fact

$$\partial(M_H/N(H)) = \partial M_H/N(H) = M_H \cap \partial M/N(H).$$

We may extend s_1 to a cross section of $S(\tau(M_H))/N(H)$ as follows. If

$$\dim N(H) - \dim H > 0,$$

then the dimension as a cell complex of $M_H/N(H) = M_H/(N(H)/H)$ is less than or equal to the dimension of fibre of $S(\tau(M_H))/N(H)$. Therefore the obstruction to extending s_1 over $M_H/N(H)$ vanishes. If

$$\dim N(H) - \dim H = 0,$$

then

$$|\chi|(M_H/N(H)) = 0$$

by the assumption, and then

$$S(\tau(M_H))/N(H) \cong S(\tau(M_H/N(H)))$$

for $N(H)/H$ is a finite group and $M_H/N(H) = M_H/(N(H)/H)$. The image of s_1 is in $S(\tau(\partial M_H))/N(H)$. Then s_1 can be extended over $M_H/N(H)$ by Proposition 2.5.

Let

$$s_2: M_H/N(H) \rightarrow S(\tau(M_H))/N(H)$$

be an extension of s_1 . Let

$$\pi: S(\tau(M_H)) \rightarrow S(\tau(M_H))/N(H)$$

be the canonical projection, and let

$$\pi^*s_2: M_H \rightarrow S(\tau(M_H))$$

be the induced $N(H)$ -cross section. π^*s_2 may be considered an $N(H)$ -cross section of $S(\tau(M))|M_H$, since $S(\tau(M_H))$ is a subbundle of $S(\tau(M))|M_H$. Moreover π^*s_2 is an extension of $s| \partial M_H$. Since M is of one isotropy type, the G -action

$$G \times (S(\tau(M))|M_H) \rightarrow S(\tau(M))$$

induces a G -bundle isomorphism

$$G \times_{N(H)} (S(\tau(M))|M_H) \cong S(\tau(M)).$$

Then π^*s_2 induces a G -cross section

$$s_3: M \rightarrow S(\tau(M))$$

which is an extension of s .

“only if” part: Let

$$t: M \rightarrow S(\tau(M))$$

be an extension of s . By the G -equivariancy of t , t is restricted to an $N(H)$ -cross section

$$t_1: M_H \rightarrow S(\tau(M_H)).$$

t_1 induces a cross section

$$t_2: M_H/N(H) \rightarrow S(\tau(M_H))/N(H).$$

If

$$\dim N(H) - \dim H = 0,$$

then

$$S(\tau(M_H))/N(H) \cong S(\tau(M_H/N(H))),$$

and

$$t_2(\partial(M_H/N(H))) \subset S(\tau(\partial(M_H/N(H)))).$$

Therefore

$$\dim N(H) - \dim H = 0$$

implies

$$|\chi|(M_H/N(H)) = 0$$

by Proposition 2.5.

This completes the proof for the case in which M is of one isotropy type.

(3-2) Let the theorem be true for the case in which the number of isotropy types is $k-1$. Let M be a compact G -manifold with k isotropy types.

“if” part: Let (H) be a maximal isotropy type on M . Then $M_{(H)}$ is a compact G -submanifold of M with one isotropy type. From the preceding argument we obtain a G -cross section

$$s_1: M_{(H)} \rightarrow S(\tau(M))|M_{(H)}$$

such that the image of s_1 is in $S(\tau(M_{(H)}))$ and $s_1|_{\partial M_{(H)}} = s|_{\partial M_{(H)}}$.

Let $T(M_{(H)})$ be a closed G -invariant tubular neighborhood of $M_{(H)}$ in M .

By Corollary 2.4 we obtain a G -bundle map

$$\pi: S(\tau(M))|T(M_{(H)}) \rightarrow S(\tau(M))|M_{(H)}$$

such that π covers the canonical projection of $T(M_{(H)})$ to $M_{(H)}$ and π is the identity on $S(\tau(M))|M_{(H)}$. π induces a G -cross section

$$\pi^*s_1: T(M_{(H)}) \rightarrow S(\tau(M))|T(M_{(H)})$$

from s_1 such that $\pi^*s_1|M_{(H)} = s_1$.

To obtain a G -cross section

$$L_1 = \partial M \cup T(M_{(H)}) \rightarrow S(\tau(M))|L_1$$

extending s , we may apply Lemma 5.1 (stated in the last section) as follows. Apply $E \rightarrow X$, A , B , D , s_A and s_B in Lemma 5.1 to $\tau(M) \rightarrow M$, $T(M_{(H)})$, ∂M , $\partial M_{(H)}$, π^*s_1 and s , respectively. So, in this case,

$$C = A \cap B = T(M_{(H)}) \cap \partial M = T(M_{(H)})|\partial M_{(H)},$$

and this is compact. Moreover this is equivariantly deformable to $\partial M_{(H)}$, and

$$\pi^*s_1|\partial M_{(H)} = s|\partial M_{(H)}.$$

Let K be a closed G -invariant collar of $\partial M_{(H)}$ in $M_{(H)}$. By Proposition 2.2, $T(M_{(H)})|K$ has the desired property as U in Lemma 5.1. Therefore we can apply Lemma 5.1 to this case. So we obtain a G -cross section

$$s_2: L_1 \rightarrow S(\tau(M))|L_1$$

extending s .

Let $T^0(M_{(H)})$ be the part of $T(M_{(H)})$ corresponding to the open disc bundle. Set

$$L = M - T^0(M_{(H)}).$$

Then L is a compact G -manifold with *corner*. Smoothing the corner of L , let L' be the resulting smooth G -manifold. Note that L and L' are the same topological space. Let

$$\varphi: \partial L' \times [0, 1] \rightarrow L'$$

be a G -invariant collar of $\partial L'$ in L' such that

$$\varphi(\partial L' \times \{0\}) = \partial L'.$$

Identify $\partial L' \times [0, 1]$ with the image of φ . By Proposition 2.2 there is a G -bundle isomorphism

$$S(\tau(M))|\partial L' \times [0, 1] \cong (S(\tau(M))|\partial L') \times [0, 1].$$

$S(\tau(M))|_{\partial L'}$ admits a G -cross section s_3 which is a restriction of s_2 . From s_3 the above isomorphism induces a G -cross section

$$s_4: \partial L' \times [0, 1] \rightarrow S(\tau(M))|_{\partial L' \times [0, 1]}.$$

Set

$$L_2 = L' - \partial L' \times [0, 1].$$

L_2 is a compact G -manifold with $k-1$ isotropy types. $S(\tau(M))|_{L_2}$ admits a G -cross section s_5 on $\partial L_2 = \partial L' \times \{1\}$ which is the restriction of s_4 . $S(\tau(M))|_{L_2}$ and $S(\tau(L_2))$ are identical, since the smoothing process of the corner of L does not change the differentiable structure outside the corner. By careful consideration we see that the image of s_5 is in $S(\tau(\partial L_2))$. So, by the hypothesis of the induction, s_5 is extendible to a G -cross section

$$s_6: L_2 \rightarrow S(\tau(L_2)) = S(\tau(M))|_{L_2}.$$

Then s_2, s_4 and s_6 give a G -cross section of $S(\tau(M))$ extending s .

“only if” part: It suffices to show that, for each isotropy type (H) on M , if

$$\dim N(H) - \dim H = 0$$

then

$$|\mathcal{X}|(M_H/N(H)) = 0.$$

Let (H) be a maximal isotropy type on M . As in the *“only if”* part for the case of one isotropy type, we see that

$$\dim N(H) - \dim H = 0$$

implies

$$|\mathcal{X}|(M_H/N(H)) = 0.$$

Let (H') be another isotropy type on M . Consider $L'_{H'}$ where L' is the compact G -manifold in the previous *“if”* part. Since L' is of $k-1$ isotropy types,

$$\dim N(H') - \dim H' = 0$$

implies

$$|\mathcal{X}|(L'_{H'}/N(H')) = 0$$

by the hypothesis of the induction. Furthermore $L'_{H'}$ is equivariantly homotopy equivalent to $M_{H'}$. Then

$$\dim N(H') - \dim H' = 0$$

implies

$$|\mathcal{X}|(M_{H'}/N(H')) = 0.$$

Thus Theorem 1.1 is completely proved.

4. Proof of Corollary 1.3

If M admits a G - k -field, then the k -field is tangent to F by equivariancy. So the "only if" part is trivial.

Now we assume that F admits a k -field. Then there are k cross sections s_1, s_2, \dots, s_k of $\tau(F)$ which are linearly independent at each point of F . Since $\tau(F)$ is a subvector bundle of $\tau(M)|F$, we may regard s_1, \dots, s_k as G -cross sections of $\tau(M)|F$. As in the proof of Theorem 1.1 we may extend s_1 to a nowhere vanishing G -cross section s_1' of $\tau(M)$. Choosing a G -invariant Riemannian metric on $\tau(M)$, let E be the G -invariant subvector bundle of $\tau(M)$ which is orthogonal to the image of s_1' . The assumption

$$\dim N(G_x) - \dim G_x \geq k$$

enables us to extend s_2 to a nowhere vanishing G -cross section s_2' of E by the same method in the proof of Theorem 1.1. By repeating this process we may extend s_3, \dots, s_k to a G -cross sections s_3', \dots, s_k' of $\tau(M)$ such that s_1', \dots, s_k' are linearly independent at each point of M . So we obtain a G - k -field on M .

5. Concluding lemma

We will conclude this paper by proving the following lemma which was used in the proof of Theorem 1.1. G acts trivially on intervals considered.

Lemma 5.1. *Let $E \rightarrow X$ be a G -vector bundle. Let A, B be G -invariant subspaces with $C = A \cap B$ compact. We assume that there is a G -invariant subspace D of C such that C is equivariantly deformable to D , i.e., there is a G -homotopy*

$$F: C \times [0, 1] \rightarrow C$$

such that $F(x, 0) = x$ and $F(x, 1) \in D$ for all $x \in C$. We also assume that there is a G -invariant neighborhood U of C in A such that there is a G -homeomorphism

$$\varphi: C \times [0, 3] \approx U$$

with $\varphi(x, 0) = x$ for all $x \in C$. Let

$$s_A: A \rightarrow S(E)|A$$

and

$$s_B: B \rightarrow S(E)|B$$

be G -cross sections which agree on D . Then there is a G -cross section

$$s: A \cup B \rightarrow S(E)|A \cup B$$

which agrees with s_A on $A-U$ and s_B on B .

Proof. By Proposition 2.3 there is a G -bundle map

$$\bar{F}: (S(E)|C) \times [0, 1] \rightarrow S(E)|C$$

which covers F and is the identity on $(S(E)|C) \times \{0\}$, and also there is a G -bundle isomorphism

$$\bar{\varphi}: (S(E)|C) \times [0, 3] \cong S(E)|U$$

which covers φ and is the identity on $(S(E)|C) \times \{0\}$. We define a G -bundle map

$$K: (S(E)|C) \times [0, 2] \rightarrow S(E)|C$$

by

$$K(v, t) = \begin{cases} \bar{F}(v, t) & \text{if } 0 \leq t \leq 1 \\ \bar{F}(v, 2-t) & \text{if } 1 \leq t \leq 2 \end{cases}$$

for $v \in S(E)|C$ and $t \in [0, 2]$. Then we may define a G -cross section

$$s_1: C \times [0, 2] \rightarrow (S(E)|C) \times [0, 2]$$

of the G -bundle $(S(E)|C) \times [0, 2]$ by

$$s_1(x, t) = \begin{cases} K^*s_B(x, t) & \text{if } 0 \leq t \leq 1 \\ K^*s_A(x, t) & \text{if } 1 \leq t \leq 2. \end{cases}$$

This is well-defined since K^*s_A and K^*s_B agree on $C \times \{1\}$. s_1 satisfies the following equations for all $x \in C$

$$s_1(x, 0) = (s_B(x), 0)$$

and

$$s_1(x, 2) = (s_A(x), 2).$$

We define a map

$$\lambda: [2, 3] \rightarrow [0, 3]$$

by $\lambda(t) = 3t - 6$ for $t \in [2, 3]$. We denote by s_2 the G -cross section of $(S(E)|C) \times [2, 3]$ which is induced from the G -cross section $s_A|U$ by the composition

$$(S(E)|C) \times [2, 3] \xrightarrow{(id, \lambda)} (S(E)|C) \times [0, 3] \xrightarrow{\bar{\varphi}} S(E)|U.$$

s_2 satisfies the following equations for all $x \in C$

$$s_2(x, 2) = (s_A(x), 2)$$

and

$$s_2(x, 3) = \bar{\varphi}^{-1}s_A\varphi(x, 3).$$

Since s_1 and s_2 agree on $C \times \{2\}$, we obtain a G -cross section

$$s_3: C \times [0, 3] \rightarrow (S(E)|C) \times [0, 3]$$

from s_1 and s_2 . Then the induced G -cross section

$$(\bar{\varphi}^{-1})^*s_3: U \rightarrow S(E)|U$$

satisfies the following equations

$$\begin{aligned} (\bar{\varphi}^{-1})^*s_3(x) &= s_B(x) && \text{for all } x \in C \\ (\bar{\varphi}^{-1})^*s_3(x) &= s_A(x) && \text{for all } x \in \varphi^{-1}(C \times \{3\}). \end{aligned}$$

So we obtain a G -cross section

$$s: A \cup B \rightarrow S(E)|A \cup B$$

defined by

$$s(x) = \begin{cases} s_A(x) & \text{if } x \in A - U \\ (\bar{\varphi}^{-1})^*s_3(x) & \text{if } x \in U \\ s_B(x) & \text{if } x \in B. \end{cases}$$

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