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ON TRANSITIVE EXTENSIONS OF FINITE PERMUTATION GROUPS

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1. Introduction

Let G be a permutation group on a finite set Ω. A transitive group T on $\Omega\cup\{\infty\}$, where ∞ denotes an additional point, is said to be a transitive extension of G if the action on Ω of the stabilizer in T of the point ∞ is permutation isomorphic to that of G on Ω. What permutation groups have transitive extensions is a rather difficult problem. In the present paper we study this problem in the case G is simply transitive on Ω. Firstly we give some necessary condition for a simply transitive group to have a transitive extension, and secondly, making use of it, prove the non-existence of transitive extensions of some classes of simply transitive groups with particular exceptions.

Before stating our results we define some terminology. Let a permuation group G on a finite set Ω act (not necessarily faithfully) on subsets Ω_1 and Ω_2 of Ω. Then we say that

($*$) (G, Ω_1) is similar to (G, Ω_2) on Ω,

if there is an element x in the symmetric group on Ω such that
(i) x normalizes G, and
(ii) x interchanges the subsets Ω_1 and Ω_2 of Ω.
Out result is as follows.

Theorem 1. *Let G be a simply transitive group on Ω with self-paired orbitals Δ and Γ such that*
(i) *for $a\in\Omega$ $(G_a, \Delta(a))$ and $(G_a, \Gamma(a))$ are not similar on $\Omega-\{a\}$,*
(ii) *G has no orbital Π different from Δ and Γ so that $(G_a, \Pi(a))$ is similar to $(G_a, \Delta(a))$ or $(G_a, \Gamma(a))$ on $\Omega-\{a\}$, and*
(iii) *$|\Delta(a)\cap\Delta(c)|\neq 0$ for $a\in\Omega$ and $c\in\Gamma(a)$.*
Assume that G has a transitive extension. Then either of the following cases occurs:
(A). *For $a\in\Omega$ and $b\in\Delta(a)$ G_{ab} has a fixed block Λ on $\Delta(a)-\{b\}$ such that Λ is different from $\Delta(a)\cap\Delta(b)$ and (G_{ab}, Λ) is similar to $(G_{ab}, \Delta(a)\cap\Delta(b))$ on $\Omega-\{a, b\}$.*

(B). *For* $a\in\Omega$ *and* $c\in\Gamma(a)$, G_{ac} *has a fixed block* Λ *on* $\Delta(a)$ *such that* Λ *is separated from* $\Delta(a)\cap\Delta(c)$ *and* (G_{ac}, Λ) *is similar to* $(G_{ac}, \Delta(a)\cap\Delta(c))$ *on* $\Omega-\{a, c\}$.

By Theorem 1 we have, for example, the following results.

Theorem 4.3. *The symmetric group* S_m *or the alternating group* A_m *on a set* Σ, $|\Sigma|=m$, *considered as a permutation group of degree* $\binom{m}{r}$ *on r-element subsets of* Σ *with* $1<r<m-1$ *has no transitive extension except the cases* $(r, m)=(2, 4), (2, 5)$ *and* $(2, 6)$.

Theorem 4.4. *A subgroup of* $P\Gamma L(m, q)$ *containing* $PSL(m, q)$ *with* $q>2$ *and* $m>3$ *considered as a permutation group of degree* $\frac{(q^m-1)(q^m-q)\cdots(q^m-q^r)}{(q^{r+1}-1)(q^{r+1}-q)\cdots(q^{r+1}-q^r)}$ *on r-dimensional subspaces of* $PG(m-1, q)$ *with* $1\leqq r\leqq m-3$ *has no transitive extension.*

The case $r=0$ in Theorem 4.4. was treated by H. Zassenhaus [5].

Acknowledgement. Professor E. Bannai has kindly pointed out that the conclusion of Theorem 1 remains valid without the assumptions (i) and (ii) if all orbitals of G are self-paired (see the proof of Theorem 1 in section 3). The author wishes to thank him for his helpful comments. The author also wishes to thank Professor H. Nagao for his advice and encouragement.

2. Notation, definitions and prelimiany results

Let G be a permutation group on a finite set Ω. For points $a, b, c, \cdots$ of Ω we denote by $G_{a,b,c\cdots}$ and $G_{\{a,b,c\cdots\}}$ the pointwise and the global stabilizer in G of the set $\{a, b, c, \cdots\}$, respectively. A subset Δ of Ω is a fixed block of G if G fixes Δ as a set. If Δ is a fixed block of G the restriction of G to Δ and the kernel of the restriction of G to Δ are denoted by G^{Δ} and G_{Δ}, respectively. For the remainder of this section G is assumed to be simply transitive on Ω. Then an orbital of G is a mapping Δ from Ω into the subsets of Ω such that

(i) $\Delta(a)$ is an orbit of G_a for $a\in\Omega$, and

(ii) $\Delta(a)^g=\Delta(a^g)$ for all $a\in\Omega$, $g\in G$.

An orbital of G is self-paired if $b\in\Delta(a)$ implies $a\in\Delta(b)$. Now let G have a transitive extenstion T on $\Omega\cup\{\infty\}$. Then for $c\in\Omega\cup\{\infty\}$ we denote by Δ_c the orbital of T_c considered as a transitive group on $\Omega\cup\{\infty\}-\{c\}$ such that

(i). $\Delta_\infty=\Delta$, and

(ii). $\Delta_c(d)=\{\Delta_\infty(d^{g^{-1}})\}^g$ for all $g=\binom{\infty\cdots}{c\cdots}\in T$, $d\in\Omega\cup\{\infty\}-\{c\}$.

In this notation we have:

Lemma 2.

(i). $\{\Delta_a(b)\}^g=\Delta_{a^g}(b^g)$ *for all* $a, b\in\Omega\cup\{\infty\}$, $g\in T$.

(ii). *We have $\Delta_a(b)=\Delta_b(a)$ for all $a, b \in \Omega \cup \{\infty\}$ if there exists no orbital Π of G such that for $c \in \Omega$ $(G_c, \Pi(c))$ is similar to $(G_c, \Delta(c))$ on $\Omega - \{c\}$.*
(iii). *If Δ is self-paired then Δ_a is self-paired for all $a \in \Omega \cup \{\infty\}$. If further $\Delta_a(b)=\Delta_b(a)$, then for $c \in \Delta_a(b)$, $T_{\{a,b,c\}}$ acts as S_3 on $\{a, b, c\}$.*

Proof.
(i). Clear by the definition of Δ_a.
(ii). Let y be an element in T of the form $(a, b)\cdots$. Then y normalizes T_{ab} and by the assumption on Δ, y fixes the orbit $\Delta_a(b)$ of T_{ab} on $\Omega \cup \{\infty\} - \{a, b\}$. Hence by (i) we have $\Delta_a(b)=\Delta_b(a)$.
(iii). Assume that $\Delta=\Delta_\infty$ is self-paired, and let $e \in \Delta(d)$. Then T contains an element x of the form $(\infty)(e, d)\cdots$. Then for an element $y=\binom{\infty\cdots}{a\cdots}$ in T we have that $e^y \in \Delta_a(d^y)$, and $x^y=(a)(e^y, d^y)\cdots$. Hence Δ_a is seld-paired. Now let $c \in \Delta_a(b)=\Delta_b(a)$. Then T has elements of forms $(a)(b, c)\cdots$ and $(b)(a, c)\cdots$. Thus $T_{\{a,b,c\}}$ acts as S_3 on $\{a, b, c\}$.

3. Proof of Theorem 1

Let the orbitals Δ and Γ of G satisfy the assumptions of Theorem 1. In a usual way we define a graph structure on Ω as follows; a pair $\{a, b\}$ of distinct points in Ω is said to be an egde if $b \in \Delta(a)$ or equivalently if $a \in \Delta(b)$. Assume that G has a transitive extension T on $\Omega \cup \{\infty\}$, and let a be a fixed point in Ω. Then by making use of the orbital Δ_a of T_a defined in section 2 we hav as above a graph structure on $\Omega \cup \{\infty\} - \{a\}$. To distinguish the edges defined by Δ_∞ and Δ_a we say that

(*) *a pair $\{b, c\}$ of points on $\Omega \cup \{\infty\}$ is a blue edge if $b \in \Delta_\infty(c)$ and a red edge if $b \in \Delta_a(c)$.*

Note that an element $g=\binom{\infty\cdots}{a\cdots}$ in T carries blue edges to red ones. Now consider the stabilizer $T_{\infty a}$ of ∞ and a, and let b be a point in $\Delta_\infty(a)$ $(=\Delta_a(\infty))$. Then the global stabilizer $T_{\{\infty,a,b\}}$ in T of the set $\{\infty, a, b\}$ acts as S_3 on it by Lemma 2 (iii). Then an element in $T_{\{\infty,a,b\}}$ of the form $(\infty a)(b)\cdots$ carries $\Delta_\infty(a) \cap \Delta_\infty(b)$ to $\Delta_a(\infty) \cap \Delta_a(b)$ (Lemma 2 (i)). Thus, if $|\Delta_\infty(a) \cap \Delta_\infty(b)| \neq 0$, $(T_{\infty ab}, \Delta_\infty(a) \cap \Delta_\infty(b))$ and $(T_{\infty ab}, \Delta_a(\infty) \cap \Delta_a(b))$ are simliar in our sense. Assume now that Case A of Theorem 1 does not occur. Then it follows that $\Delta_\infty(a) \cap \Delta_\infty(b)=\Delta_a(\infty) \cap \Delta_a(b)$. Then taking an element x in $T_{\{\infty,a,b\}}$ of the form $(\infty b)(a)\cdots$ and considering the image of $\Delta_\infty(a) \cap \Delta_\infty(b)=\Delta_a(\infty) \cap \Delta_a(b)$ under x we conclude that $\Delta_\infty(a) \cap \Delta_\infty(b)=\Delta_a(\infty) \cap \Delta_a(b)=\Delta_\infty(b) \cap \Delta_a(b)$. In particular $\Delta_\infty(b) \cap \Delta_a(b)$ is contained in $\Delta_\infty(a)$. This implies that there is no pair $\{b, d\}$ with $d \in \Omega \cup \{\infty\} - \{\{\infty, a\} \cup \Delta_\infty(a)\}$ which is both a blue edge and a red edge. This is also true if $|\Delta_\infty(a) \cap \Delta_\infty(b)|=0$. Then for a point c in $\Gamma(a)$ with $|\Delta(a) \cap \Delta(c)| \neq 0$, $\Delta_\infty(a) \cap \Delta_\infty(c)$ and $\Delta_a(\infty) \cap \Delta_a(c)$ are fixed blocks of $T_{\infty ac}$ which have no point in common. Furthermore since $T_{\{\infty,a,c\}}$ acts as S_3 on $\{\infty, a, c\}$ it follows that $(T_{\infty,a,c}, \Delta_a(\infty)\cap\Delta_a(c))$ and $(T_{\infty,a,c}, \Delta_\infty(a)\cap\Delta_\infty(c))$ are similar on $\Omega-\{a, c\}$.

This completes the proof of Theorem 1.

4. Some applications of Theorem 1

Proposition 4.1. *Let G be a 4-fold transitive group on a set Σ, $|\Sigma|=m$. Assume that the rank 3 group G of degree $\binom{m}{2}$ on 2-element subsets of Σ has a transtivie extension T. Then one of the following holds:*

(i). $m=4$, G is S_4, and T is $PSL(3, 2)$,

(ii). $m=6$, G is A_6 or S_6, and T is $A_6\cdot E_{16}$ or $S_6\cdot E_{16}$, the semi-direct product of elementary abelian 2-group E_{16} of order 16 by A_6 or S_6,

(iii). $m\geqq 7$ and the stabilizer in G of four points in Σ has an orbit of length two on the remaining points.

In particular if G is 5-fold transitive on Σ then $m=6$, G is S_6 and T is $S_6\cdot E_{16}$.

Proof. Let Ω be the set of unordered pairs of points in Σ. For an element $\{1,2\}$ in Ω set $\Delta(\{1,2\})=\{\{i,j\}\mid |\{i,j\}\cap\{1,2\}|=1\}$ amd $\Gamma(\{1,2\})=\{\{i,j\}\mid |\{i,j\}\cap\{1,2\}|=0\}$. Then Δ and Γ are self-paired orbitals of G such that $|\Delta(\{1,2\})|=2(m-2)$, $|\Gamma(\{1,2\})|=\binom{m-2}{2}$ and $|\Delta(\{1,2\})\cap\Delta(\{i,j\})|=m-2$ or 4 according as $\{i,j\}\in\Delta(\{1,2\})$ or $\Gamma(\{1,2\})$. Since G is 4-fold transitive on Σ the stabilizer in G of $\{1,2\}$ and $\{1,3\}$ in Ω has three orbits on $\Delta(\{1,2\})-\{1,3\}$ of lenghts $1, m-3$, namely $\{2,3\}$, $\{\{1,i\}\mid 4\leqq i\leqq m\}$ and $\{\{2,i\}\mid 4\leqq i\leqq m\}$. Now assume that G has a transitive extension T on $\Omega\cup\{\infty\}$. For $\beta\in\Delta_\infty(\alpha)$ with $\alpha=\{1,2\}$, $\beta=\{1,3\}\in\Omega$ set $\nu=|\Delta_\infty(\alpha)\cap\Delta_\alpha(\beta)\cap\Delta_\beta(\infty)|$. Then since $\Delta_\infty(\alpha)\cap\Delta_\alpha(\beta)\cap\Delta_\beta(\infty)$ is a fixed block of $T_{\infty\alpha\beta}$ we have $\nu=1$ or $m-2$. Note that $\nu\neq m-3$ if $m>4$, because then both $\Delta_\infty(\alpha)\cap\Delta_\infty(\beta)$ and $\Delta_\alpha(\infty)\cap\Delta_\alpha(\beta)$ must contain $\{2,3\}$. We first prove:

Lemma 4.2. *If $\nu=1$ then $m=4$ and G is S_4.*

Proof. We assume that $m>4$ and seek a contradiction. Let $\{1,2\}$ be a fixed element in Ω and for simplicity we denote $\{1,2\}$ by α, $\{1,i\}$ and $\{2,i\}$ by β_i and $\delta_i (i=3,\cdots,m)$, respectively. We say that a pair $\{\varepsilon,\varepsilon'\}$ of points ε,ε' in $\Delta_\infty(\alpha)$ is a blue edge if $\varepsilon\in\Delta_\infty(\varepsilon')$ and a erd edge if $\varepsilon\in\Delta_\alpha(\varepsilon')$. Since $\nu=1$ and $m>4$, $\{\beta_i,\delta_i\}$'s $(i=3,\cdots m)$ are the only edges in $\Delta_\infty(\alpha)$ which are both red and blue. Now let $x=(\infty\alpha)(\beta_3)\cdots$ be an element of T. We first show that we can choose x to be an involution. Since G is 4-fold transitive on Σ, G contains an involution y having the form (1) (23)$\cdots$ on Σ. Then the action of y on Ω is of the form $(\infty)(\alpha\beta_3)\cdots$. Since $T_{\{\infty,\alpha,\beta_3\}}$ acts as S_3 on $\{\infty,\alpha,\beta_3\}$ we can take x to be conjugate to y. Now x carries red edges to blue ones and conversely.

Hence x fixes δ_3 and carries $\beta_i (i \geqq 4)$ to some δ_j $(j \geqq 4)$. Furthermore if x carries β_i to δ_j (hence δ_j to β_i), $\{\beta_i, \delta_j\}$ is an edge which are both blue and red. Hence we conclude that x is of the form $(\infty\alpha)(\beta_3)(\delta_3)(\beta_4\delta_4)(\beta_5\delta_5)\cdots(\beta_m\delta_m)$. Now let z be an involution of the form $(\infty\alpha)(\beta_4)\cdots$. Then ,similarly to the above, z has the form $(\infty\alpha)(\beta_3\delta_3)(\beta_4)(\delta_4)(\beta_5\delta_5)\cdots(\beta_m\delta_m)$. Hence it follows that $xz=(\infty)(\alpha)(\beta_3\delta_3)(\beta_4\delta_4)(\beta_5)(\delta_5)\cdots(\beta_m)(\delta_m)$. Then xz is an element of G and the action of xz on Σ must be of the form $(12)(3)(4)\cdots$. But then xz can not fix β_5, a contradiction. This complete the proof of Lemma 4.2.

We now complete the proof of proposition 4.1. we may assume that $v=m-2$. This implies that blue edges and red edges on $\Delta_\infty(\{1, 2\})$ coincide and that there is no pair $\{\delta, \gamma\}$ with $\delta \in \Delta_\infty(\{1, 2\})$ and $\gamma \in \Gamma_\infty(\{1, 2\})$ which is both a blue edge and a red edge. Now set $\alpha=\{1, 2\}$, $\gamma=\{3, 4\}$ and let $g=(\infty\alpha)(\gamma)\cdots$ be an element of T. Then $(\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma))^g=\Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma)$ is a fixed block of $T_{\infty\alpha\gamma}$ on $\Delta_\infty(\alpha)$ which is disjoint from $\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma)$. Furthermore since edges in $\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma)$ are carried by g to edges in $\Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma)$ we see that $\Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma)=\{\{1, i\}, \{1, j\} \{2, i\}, \{2, j\}\}$ for some i, j in $\Sigma-\{1, 2, 3, 4\}$. This implies that $\{i, j\}$ is a fixed block of $G_{\{1,2\}\{3,4\}}$ on $\Sigma-\{1, 2, 3, 4\}$. Then G_{1234} fixes $\{i, j\}$ pointwise or as a set. If the former case occurs G is A_6 or M_{11} by a result of H. Nagao [3]. But M_{11} considered as a rank 3 group of degree 55 has no transitive extension on 56 points. This is seen as follows. Let T be a transitive extension of M_{11}. Then since M_{11} is simple, T is also simple and has order equal to $|M_{22}|$, whence T is M_{22} by [4], contradicting a well known fact that M_{11} is not a subgroup of M_{22} (see [1]). This completes the proof of Proposition 4.1.

We now prove the following

Theorem 3.4. *The symmetric group S_m or the alternating group A_m on a set Σ, $|\Sigma|=m$ considered as permutation group of degree $\binom{m}{r}$ on r-element subsets of Σ with $1<r<m-1$ has no transitive extension except the cases* $(r, m)=(2, 4)$, $(2, 5)$ *and* $(2, 6)$.

Proof. We may assume without any loss of generality that $2r \leqq m$. Let Ω be the set of r-element subsets of Σ. For an element $\alpha=\{1, 2, \cdots, r\}$ of Ω set

$$\Delta(\alpha)=\{\{i_1, i_2, \cdots, i_r\} \mid |\{i_1, i_2, \cdots, i_r\} \cap \{1, 2, \cdots, r\}|=r-1\}, \text{ and}$$
$$\Gamma(\alpha)=\{\{i_1, i_2, \cdots, i_r\} \mid |\{i_1, i_2, \cdots, i_r\} \cap \{1, 2, \cdots, r\}|=r-2\}.$$

Then Δ and Γ are self-paired orbitals of G such that $|\Delta(\alpha)|=r(m-r)$, $|\Gamma(\alpha)|=\binom{r}{2}\binom{m-r}{2}$, and

$$|\Delta(\alpha)\cap\Delta(\beta)| = \begin{cases} m-2 & \text{if } \beta\in\Delta(\alpha), \\ 4 & \text{if } \beta\in\Gamma(\alpha). \end{cases}$$

Δ and Γ satisfy the assumptions of Theorem 1 except the case $m=2r$*. Let $\beta=\{1, 2, \cdots, r-1, r+1\}$ be an element of $\Delta(\alpha)$. Then $G_{\alpha\beta}=G_{\{1,2,\cdots,r-1\}\{r\}\{r+1\}}$ has three orbits on $\Delta(\alpha)-\{\beta\}$, namely

$$\begin{aligned} \Phi_1 &= \{\{i_1, i_2, \cdots, i_{r-1}, r+1\} \mid \{1, 2, \cdots, r-1\} \neq \{i_1, i_2, \cdots, i_{r-1}\} \subset \{1, 2, \cdots, r\}\}, \\ \Phi_2 &= \{\{1, 2, \cdots, r-1, i\} \mid r+2 \leqq i \leqq m\}, \text{ and} \\ \Phi_3 &= \{\{i_1, i_2, \cdots, i_{r-2}, r, i\} \mid \{i_1, i_2, \cdots, i_{r-2}\} \subset \{1, 2, \cdots, r-1\}, r+2 \leqq i \leqq m\}. \end{aligned}$$

Here we have that $|\Phi_1|=r-1$, $|\Phi_2|=m-r-1$ and $|\Phi_3|=(r-1)(m-r-1)$ and $\Delta(\alpha)\cap\Delta(\beta)=\Phi_1\cup\Phi_2$. Therefore if $G_{\alpha\beta}$ has a fixed block Λ on $\Delta(\alpha)-\{\beta\}$ such that $\Lambda\neq\Delta(\alpha)\cap\Delta(\beta)$ and $(G_{\alpha\beta}, \Lambda)$ is similar to $(G_{\alpha\beta}, \Delta(\alpha)\cap\Delta(\beta))$ it follows that $(r-1)(m-r-1)=r-1$ or $m-r-1$, hence $m=r+1$ or $r=2$. The former case is out of our consideration and the latter was treated in Prop 4.1. Thus we may assume that Case A of Theorem 1 does not occur.

Now let $\delta=\{1, 2, \cdots, r-1, r+2\}$ be an element of $\Gamma(\alpha)$. Then $G_{\alpha\delta}= G_{\{1,2,\cdots r-2\}\{r-1,r\}\{r+1,r+2\}}$ has four orbits on $\Delta(\alpha)$, namely

$$\begin{aligned} \Psi_1 &= \{\{1, 2, \cdots, r-2, i, j\} \mid r-1 \leqq i \leqq r, r+1 \leqq j \leqq r+2\}, \\ \Psi_2 &= \{\{i_1 i_2, \cdots, i_{r-1}, j\} \mid \{1, 2, \cdots, r-2\} \not\subset \{i_1 i_2, \cdots, i_{r-1}\} \subset \\ &\qquad \{1, 2, \cdots, r\}, r+1 \leqq j \leqq r+2\}, \\ \Psi_3 &= \{\{1, 2, \cdots, r-2, i, j\} \mid r-1 \leqq i \leqq r, r+3 \leqq j \leqq m\} \text{ and} \\ \Psi_4 &= \{\{i_1, i_2, \cdots, i_{r-1}, j\} \mid \{1, 2, \cdots, r-2\} \not\subset \{i_1, i_2, \cdots, i_{r-1}\} \subset \\ &\qquad \{1, 2, \cdots, r\}, r+3 \leqq j \leqq m\}. \end{aligned}$$

Here we have that $|\Psi_1|=4$, $|\Psi_2|=2(r-2)$, $|\Psi_3|=2(m-r-2)$ $|\Psi_4|= (m-r-2)(r-2)$ and $\Psi_1=\Delta(\alpha)\cap\Delta(\delta)$. Hence Case B of Theorem 1 may possibly hold only if $|\Psi_1|=|\Psi_2|$, $|\Psi_3|$ or $|\Psi_4|$, namely $r=4$, $r=m-4$ or $(r, m)= (3, 9)$. If $r=m-4$ then $(r, m)=(3, 7)$ or $(4, 8)$ because $2r\leqq m$. We first eliminate the cases $(r, m)=(3, 7)$ and $(3, 9)$. Assume that A_7 or S_7 of degree $\binom{7}{3}$ has an transitive extension T, and let N denote a minimal normal subgroup of T. Then N is simple and since $N\cap S_7$ is a normal subgroup of S_7 it follows that either $N=T$ or N has index two in T, contradicting a result of M. Hall [2]. Now assume that $|\Psi_1|=|\Psi_4|$ and hence $(r, m)=(3, 9)$. In this case the kernels of the restrictions of $G_{\alpha\beta}$ to Ψ_1 and Ψ_4 are $G_{12345\{6789\}}$ and $G_{1\{2,3\}\{45\}6789}$, respectively and hence are not isomorphic as abstract groups. Hence $(G_{\alpha\beta}, \Psi_1)$ and

* Even in this case the conclusion of Theorem 1 holds since all orbitals of G are self-paired (see §1).

$(G_{\alpha\beta}, \Psi_4)$ are not similar in our sense. Finally we consider the case $r=4$. Let θ and π be the orbitals of G defined as follws:

$$\theta(\{1, 2, 3, 4\}) = \{\{i_1, i_2, i_3, i_4\} \mid |\{i_1, i_2, i_3, i_4\} \cap \{1, 2, 3, 4\}| = 1\}, \text{ and}$$
$$\pi(\{1, 2, 3, 4\}) = \{\{i_1, i_2, i_3, i_4\} \mid |\{i_1, i_2, i_3, i_4\} \cap \{1, 2, 3, 4\}| = 0\}.$$

Assume first that $m \geqq 10$. Then θ and π satisfy the assumptions of Theorem 1 for Δ and Γ, respecitvely. We have that $|\theta(\alpha)| = 4\binom{m-4}{3}$, $|\pi(\alpha)| = \binom{m-4}{4}$ and

$$|\theta(\alpha) \cap \theta(\varepsilon)| = \begin{cases} \binom{m-7}{3} + 9\binom{m-7}{2} & \text{if } \varepsilon \in \theta(\alpha), \\ 16\binom{m-8}{2} & \text{if } \varepsilon \in \pi(\alpha). \end{cases}$$

For $\alpha = \{1, 2, 3, 4\}$ and $\varepsilon = \{1, 5, 6, 7\}$ of $\theta(\alpha)$, $G_{\alpha\varepsilon} = G_{\{1\}\{2,3,4\}\{5,6,7\}}$ has seven orbits on $\theta(\alpha) - \{\varepsilon\}$, namely

$$\begin{aligned}
P_1 &= \{\{1, i, j, k\} \mid \{i, j, k\} \subset \{8, 9, \cdots, m\}\}, \\
P_2 &= \{\{i, j, k, l\} \mid 5 \leqq i \leqq 7,\ 2 \leqq j \leqq 4,\ 8 \leqq k,\ l \leqq m\}, \\
P_3 &= \{\{1, i, j, k\} \mid 5 \leqq i \leqq 7,\ 8 \leqq j,\ k \leqq m\}, \\
P_4 &= \{\{i, j, k, l\} \mid 5 \leqq i, j \leqq 7,\ 2 \leqq k \leqq 4,\ 8 \leqq l \leqq m\}, \\
P_5 &= \{\{5, 6, 7, i\} \mid 2 \leqq i \leqq 4\}, \\
P_6 &= \{\{1, i, j, k\} \mid 5 \leqq i, j \leqq 7,\ 8 \leqq k\}, \text{ and} \\
P_7 &= \{\{i, j, k, l\} \mid 2 \leqq i \leqq 4,\ 8 \leqq j,\ k \leqq m\}.
\end{aligned}$$

Here $|P_1| = \binom{m-7}{3}$, $|P_2| = 9\binom{m-7}{2}$, $|P_3| = 3\binom{m-7}{2}$, $|P_4| = 9(m-7)$, $|P_5| = 3$, $|P_6| = 3(m-7)$, $|P_7| = 3\binom{m-7}{3}$, and $\theta(\alpha) \cap \theta(\varepsilon) = P_1 \cup P_2$.

Also for an element $\rho = \{5, 6, 7, 8\}$ of $\pi(\alpha)$ $G_{\alpha\rho} = G_{\{1,2,3,4\}\{5,6,7,8\}}$ has four orbits on $\theta(\alpha)$, namely

$$\begin{aligned}
O_1 &= \{\{i, j, k, l\} \mid 5 \leqq i \leqq 8,\ 1 \leqq j \leqq 4,\ 9 \leqq k,\ l \leqq m\}, \\
O_2 &= \{\{i, j, k, l\} \mid 5 \leqq i, j \leqq 8,\ 1 \leqq k \leqq 4,\ 9 \leqq l \leqq m\}, \\
O_3 &= \{\{i, j, k, l\} \mid 5 \leqq i, j, k \leqq 8,\ 1 \leqq l \leqq 4\}, \text{ and} \\
O_4 &= \{\{i, j, k, l\} \mid 1 \leqq i \leqq 4,\ 9 \leqq j,\ k,\ l \leqq m\}.
\end{aligned}$$

Here $|O_1| = 16\binom{m-8}{2}$, $|O_2| = 24(m-8)$, $|O_3| = 16$, $|O_4| = 4\binom{m-8}{3}$ and $O_1 = \theta(\alpha) \cap \theta(\rho)$. Hence we see that the conculsion of Theorem 1 may possibly hold only in the following cases.

Case 1. $|P_1|+|P_2|=|P_1|+|P_7|$.
Case 2. $|P_1|+|P_2|=|P_3|+|P_2|$.
Case 3. $|O_1|=|O_4|$.
Case 4. $|O_1|=|O_2|$.
We treat these cases separately.

Case 1. In this case $m=18$. We see that $G_{\alpha\varepsilon}$ is faithful on P_2, but not on P_7. Hence Case A of theorem 1 does not hold in this case.

Case 2. In this case $m=18$, and the kernels of the restrictions of $G_{\alpha\varepsilon}$ to P_1 and P_3 have distinct orders.

Case 3. In this case $m=28$, and $G_{\alpha\rho}$ is faithful on O_1, but not on O_4.

Case 4. In this case $m=12$. Assume that G has an transitive extension T on $\Omega\cup\{\infty\}$, where Ω denotes the set of four element subsets of Σ, $|\Sigma|=12$, Let x be an element of order three in G having the form (i, j, k) on Σ. Then x has 135 fixed points on Ω, hence 136 fixed points on $\Omega\cup\{\infty\}$. In particular, if $x^t\in G$ for $t\in T$, then $x^t=x^g$ for some $g\in G$. Then since G contains 440 conjugates of x it follows that the number of conjugates of x in T is equal to $\frac{136}{496}\times 440$, which is not an integer, a contradiction.

Finally the cases $(r, m)=(4, 8)$ and $(4, 9)$ are eliminated by a similar argument to Case 4.

Theorem 4.4 *A subgroup of $P\Gamma L(m, q)$ containing $PSL(m, q)$ with $q>2$ and $m\geqq 4$ considered as a permutation group of degree $\frac{(q^m-1)(q^m-q)\cdots(q^m-q^r)}{(q^{r+1}-1)(q^{r+1}-q)\cdots(q^{r+1}-q^r)}$ on r-dimensional subspaces of $PG(m-1, q)$ with $1\leqq r\leqq m-3$ has no transitive extension.*

Proof. Let Ω denote the set of r-dimensional subspaces of $PG(m-1, q)$. For an element α of Ω set

$$\Delta(\alpha)=\{\beta\in\Omega\,|\,\text{dim. }\alpha\cap\beta=r-1\},\text{ and}$$
$$\Gamma(\alpha)=\{\beta\in\Omega\,|\,\text{dim. }\alpha\cap\beta=r-2\}.$$

Then Δ and $\Gamma$are self-paired orbitals of G such that

$$|\Delta(\alpha)|=\frac{q(q^{r+1}-1)(q^{m-r-1}-1)}{(q-1)^2},$$
$$|\Gamma(\alpha)|=\frac{q^4(q^{r+1}-1)(q^r-1)(q^{m-r-1}-1)(q^{m-r-2}-1)}{(q-1)^2(q^2-1)^2},\text{ and}$$
$$|\Delta(\alpha)\cap\Delta(\beta)|=\begin{cases}\dfrac{q(q^{m-r-1}-1)}{q-1}-1+\dfrac{q^2(q^r-1)}{q-1} & \text{if } \beta\in\Delta(\alpha),\\[2ex] (q+1)^2 \text{ if } \beta\in\Gamma(\alpha).\end{cases}$$

It is easy to see that Δ and Γ satisfy the assumptions of Theorem 1. Let β be an element of $\Delta(\alpha)$. We may assume that

$$\alpha=\left\{\left.\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{r+1}\\ 0\\ \vdots\\ 0\end{pmatrix}\right|\ \alpha_i\in GF(q)\right\}\ \text{and}\ \beta=\left\{\left.\begin{pmatrix}\beta_1\\ \vdots\\ \beta_r\\ 0\\ \beta_{r+1}\\ 0\\ \vdots\\ 0\end{pmatrix}\right|\ \beta_i\in GF(q)\right\}$$

Then $G_{\alpha\beta}\cap PGL(m, q)$ has the following form:

$$\begin{array}{cc}
 & \begin{array}{ccc} r & 2 & \end{array}\\
\begin{array}{c} r\\ 2\\ \\ \end{array} & \begin{pmatrix}
\boxed{*} & * & *\\
0 & \boxed{\begin{matrix}* & 0\\ 0 & *\end{matrix}} & \\
 & 0 & \boxed{*}
\end{pmatrix}
\end{array}$$

It is then easy to see that $G_{\alpha\beta}$ has the following orbits on $\Delta(\alpha)-\{\beta\}$.

$\Phi_1=\{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\beta=r-1,\ \varepsilon\cap\beta\neq\varepsilon\cap\alpha\}$,

$\Phi_2=\{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\beta=r-1,\ \varepsilon\cap\beta=\varepsilon\cap\alpha,\ \varepsilon\subset\alpha\cap\beta\}$,

$\Phi_3=\{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\beta=r-1,\ \varepsilon\cap\beta=\varepsilon\cap\alpha,\ \varepsilon\not\subset\alpha\cup\beta\}$, and

$\Phi_4=\{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\beta=r-2\}$.

Here we have $\Delta(\alpha)\cap\Delta(\beta)=\Phi_1\cup\Phi_2\cup\Phi_3$, $|\Phi_1|=\dfrac{q^2(q^r-1)}{q-1}$, $|\Phi_2|=q-1$, $|\Phi_3|=\dfrac{q^2(q^{m-r-2}-1)}{q-1}$ and $|\Phi_4|=\dfrac{q^3(q^r-1)(q^{m-r-2}-1)}{(q-1)^2}$. Therefore Case A of Theorem 1 dose not hold.

Now let $\delta=\left\{\left.\begin{pmatrix}\delta_1\\ \vdots\\ \delta_{r-1}\\ 0\\ 0\\ \delta_{r+2}\\ \delta_{r+3}\\ 0\\ \vdots\\ 0\end{pmatrix}\right|\ \delta_i\in GF(q)\right\}$ be an element of $\Gamma(\alpha)$.

Then $G_{\alpha\delta}\cap PGL(m, q)$ is of the form;

$$\begin{array}{cc}
 & \begin{array}{ccc} r-1 & 4 & \end{array} \\
\begin{array}{c} r-1 \\ 4 \\ \\ \end{array} & \left(\begin{array}{c|c|c} * & & * \\ \hline & \begin{matrix} * & 0 \\ 0 & * \end{matrix} & \\ \hline 0 & & * \end{array}\right),
\end{array}$$

and $G_{\alpha\delta}$ has the following orbits on $\Delta(\alpha)$.

$$\begin{aligned}
\Psi_1 &= \{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\delta=r-1\}\ , \\
\Psi_2 &= \{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\delta=r-2,\ \varepsilon\cap\delta\neq\alpha\cap\delta\}\ , \\
\Psi_3 &= \{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\delta=r-2,\ \varepsilon\cap\delta=\alpha\cap\delta,\ \varepsilon\subset\alpha\cup\delta\}\ , \\
\Psi_4 &= \{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\delta=r-2,\ \varepsilon\cap\delta=\alpha\cap\delta,\ \varepsilon\not\subset\alpha\cup\delta\} \text{ and} \\
\Psi_5 &= \{\varepsilon\in\Delta(\alpha)\,|\,\dim.\,\varepsilon\cap\delta=r-3\}\ .
\end{aligned}$$

Here we have $\Psi_1=\Delta(\alpha)\cap\Delta(\delta)$, $|\Psi_1|=(q+1)^2$, $|\Psi_2|=\dfrac{q^3(q+1)(q^{r-1}-1)}{(q-1)}$, $|\Psi_3|=(q+1)(q^2-1)$, $|\Psi_4|=\dfrac{q^3(q+1)(q^{m-r-3}-1)}{(q-1)}$ and $|\Psi_5|=\dfrac{q^5(q^{r-1}-1)(q^{m-r-3}-1)}{(q-1)^2}$. Then since $q\neq 2$, Case B of Theorem 1 does not hold, and the proof of Theorem 4.4 is completed.

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