Matsuda, M.
Osaka J. Math.
9 (1972), 535-545

ON THE POTENTIAL TAKEN WITH RESPECT TO COMPLEX-VALUED KERNELS

MINORU MATSUDA

(Received December 24, 1971)

In the potential theory, we have two theorems called the existence theorem concerning the potential taken with respect to real-valued and symmetric kernels. They are stated as follows. Let $K(X, Y)$ be a real-valued function defined in a locally compact Hausdorff space Ω, lower semi-continuous for any points X and Y, may be $+\infty$ for $X=Y$, always finite for $X\neq Y$ and bounded from above for X and Y belonging to disjoint compact sets of Ω respectively. For a given positive measure μ, the potential is defined by

$$K\mu(X) = \int K(X, Y)d\mu(Y),$$

and the K-energy of μ is defined by $\int K\mu(X)d\mu(X)$. A subset of Ω is said to be of positive K-transfinite diameter, when it charges a positive measure μ of finite K-energy with compact support, otherwise said to be of K-transfinite diameter zero. Let $K(X, Y)$ be symmetric : $K(X, Y)=K(Y, X)$ for any points X and Y. Then we have two following theorems.

Theorem A. *Let F be a compact subset of positive K-transfinite diameter, and $f(X)$ be a real-valued upper semi-continuous function with lower bound on F. Then, given any positive number a, there exist a positive measure μ supported by F and a real constant γ such that*
(1) $\mu(F)=a$,
(2) $K\mu(X)\geqq f(X)+\gamma$ *on F with a possible exception of a set of K-transfinite diameter zero, and*
(3) $K\mu(X)\leqq f(X)+\gamma$ *on the support of μ.*

Theorem B. *In the above theorem, suppose the further conditions : $K(X, Y)>0$ and $\inf f(X)>0$ for any points X and Y of F. Then, given any compact subset F of positive K-transfinite diameter, there exists a positive measure μ supported by F such that*
(1) $K\mu(X)\geqq f(X)$ *on F with a possible exception of a set of K-transfinite diameter zero, and*

(2) $K\mu(X) \leqq f(X)$ *on the support of* μ.

Recently, N. Ninomiya ([5]) proved the existence theorems for the potential taken with respect to complex-valued and *symmetric* kernels and to complex-valued measures, which are the extension of the above theorems in the case of the real-valued kernels. We state them as follows. Let $K(X, Y)$ be a complex-valued function defined in a locally compact Hausdorff space Ω. Let $k(X, Y) = \Re K(X, Y)$ be a function lower semi-continuous, symmetric, may be $+\infty$ for $X = Y$, always finite for $X \neq Y$ and bounded from above for X and Y belonging to disjoint compact sets of Ω respectively, and $n(X, Y) = \Im K(X, Y)$ be a finite continuous function satisfying that $n(X, Y) = -n(Y, X)$ for any points X and Y of Ω. For any compact subset F and any positive numbers a and b, denote by $\mathfrak{M}(a, F, b)$ the family of all the complex-valued measures supported by F whose real parts and imaginary parts are positive measures with total mass a and b respectively, by $\mathfrak{M}(a, F)$ the family of all the complex-valued measures supported by F whose real parts are positive measures with total mass a and imaginary parts are any positive measures, by $\mathfrak{M}(F, b)$ the family of all the complex-valued measures supported by F whose real parts are any positive measures and imaginary parts are positive measures with total mass b, and by $\mathfrak{M}(F)$ the family of all the complex-valued measures supported by F whose real parts and imaginary parts are any positive measures. For any such measure α, the potential is defined by

$$K\alpha(X) = \int K(X, Y) d\alpha(Y).$$

Then we have two following theorems.

Theorem A′. *Let F be a compact subset of positive k-transfinite diameter, and $f(X)$ be a complex-valued function whose real part $\Re f(X)$ and imaginary part $\Im f(X)$ are upper semi-continuous functions with lower bound on F. Then, given any positive numbers a and b, there exist a measure α of $\mathfrak{M}(a, F, b)$ and a complex constant γ such that*

(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero,*

(2) $\Re K\alpha(X) \leqq \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,

(3) $\Im K\alpha(X) \geqq \Im\{f(X)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero, and*

(4) $\Im K\alpha(X) \leqq \Im\{f(X)+\gamma\}$ *on the support of* $\Im\alpha$.

Theorem B′. *In the above theorem, suppose the further conditions* : $k(X, Y) > 0$, $\inf \Re f(X) > 0$ *and* $\inf \Im f(X) > 0$ *for any points X and Y of F. Then, given any positive number a such that $a|n(X, Y)| < \Im f(X)$ for points X and Y of F, there exist a measure α of $\mathfrak{M}(a, F)$ and a real constant γ such that*

(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on* F *with a possible exception of a set of k-transfinite diameter zero,*

(2) $\Re K\alpha(X) \leqq \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,

(3) $\Im K\alpha(X) \geqq \Im f(X)$ *on* F *with a possible exception of a set of k-transfinite diameter zero, and*

(4) $\Im K\alpha(X) \leqq \Im f(X)$ *on the support of* $\Im\alpha$.

Similarly, given any positive number b *such that* $b|n(X, Y)| < \Re f(X)$ *for points* X *and* Y *of* F, *there exist a measure* α *of* $\mathfrak{M}(F, b)$ *and a complex constant* γ *such that*

(1′) $\Re K\alpha(X) \geqq \Re f(X)$ *on* F *with a possible exception of a set of k-transfinite diameter zero,*

(2′) $\Re K\alpha(X) \leqq \Re f(X)$ *on the support of* $\Re\alpha$,

(3′) $\Im K\alpha(X) \geqq \Im\{f(X)+\gamma\}$ *on* F *with a possible exception of a set of k-transfinite diameter zero, and*

(4′) $\Im K\alpha(X) \leqq \Im\{f(X)+\gamma\}$ *on the support of* $\Im\alpha$.

In this paper we are going to extend these existence theorems to the potential taken with respect to complex-valued kernels and to complex-valued measures, under an additional condition of the continuity principle for the adjoint kernel.

Let $K(X, Y)$ be a complex-valued function, not always symmetric, defined in a locally compact Hausdorff space Ω. Let $k(X, Y) = K\Re(X, Y)$ be a function lower semi-continuous, may be $+\infty$ for $X=Y$, always finite for $X \neq Y$ and $n(X, Y) = \Im K(X, Y)$ be a finite continuous function. For any positive measure μ, consider the adjoint potential defined by

$$\check{k}\mu(X) = \int \check{k}(X, Y)d\mu(Y) = \int k(Y, X)d\mu(Y).$$

Then, we have two following theorems.

Theorem 1. *Let* F *be a compact subset of positive k-transfinite diameter, and* $f(X)$ *be a complex-valued function whose real part* $\Re f(X)$ *and imaginary part* $\Im f(X)$ *are upper semi-continuous functions with lower bound on* F, *and* a *and* b *be two positive numbers. If the adjoint kernel* $\check{k}(X, Y) = k(Y, X)$ *satisfies the continuity principle*[1], *there exist a measure* α *of* $\mathfrak{M}(a, F, b)$ *and a complex constant* γ *such that*

(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on* F *with a possible exception of a set of k-transfinite diameter zero,*

1) We say that $k(X, Y)$ satisfies the continuity principle when for any positive measure μ with compact support, the following implication holds: (the restriction of $k\mu(X)$ to the support of μ is finite and continuous) $=$ ($k\mu(X)$ is finite and continuous in the whole space Ω).

(2) $\Re K\alpha(X) \leqq \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,
(3) $\Im K\alpha(X) \geqq \Im\{f(X)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero, and*
(4) $\Im K\alpha(X) \leqq \Im\{f(X)+\gamma\}$ *on the support of* $\Im\alpha$.

Theorem 2. *In the above theorem, suppose the further conditions* : $k(X, Y) > 0$, $\inf \Re f(X) > 0$, *and* $\inf \Im f(X) > 0$ *for any points X and Y of F. Then, there exist a measure* α *of* $\mathfrak{M}(F)$ *and a real constant* γ *such that*
(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero,*
(2) $\Re K\alpha(X) \leqq \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$.
(3) $\Im K\alpha(X) \geqq \Im f(X)$ *on F with a possible exception of a set of k-transfinite diameter zero, and*
(4) $\Im K\alpha(X) \leqq \Im f(X)$ *on the support of* $\Im\alpha$.
Similarly, there exist a measure α *of* $\mathfrak{M}(F)$ *and a pure imaginary constant* γ *such that*
(1′) $\Re K\alpha(X) \geqq \Re f(X)$ *on F with a possible exception of a set of k-transfinite diameter zero,*
(2′) $\Re K\alpha(X) \leqq \Re f(X)$ *on the support of* $\Re\alpha$,
(3′) $\Im K\alpha(X) \geqq \Im\{f(X)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero, and*
(4′) $\Im K\alpha(X) \leqq \Im\{f(X)+\gamma\}$ *on the support of* $\Im\alpha$.

Before we prove the theorems, we prepare some lemmas.

Lemma 1. *Let* μ *be a positive measure with compact support. If the adjoint kernel* $\check{k}(X, Y)$ *satisfies the continuity principle, the set* $E=\{X \mid k\mu(X)=+\infty\}$ *of* Ω *is of k-transfinite diameter zero.*

Lemma 2. *Let F be a compact subset, and f(X) be a complex-valued function whose real part* $\Re f(X)$ *and imaginary part* $\Im f(X)$ *are upper semi-continuous functions with lower bound on F respectively, and a and b be two positive numbers. If the real part k(X, Y) of K(X, Y) is a finite continuous function defined in* Ω, *there exist a measure* α *of* $\mathfrak{M}(a, F, b)$ *and a complex constant* γ *such that*
(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on F*,
(2) $\Re K\alpha(X) = \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,
(3) $\Im K\alpha(X) \geqq \Im\{f(X)+\gamma\}$ *on F, and*
(4) $\Im K\alpha(X) = \Im\{f(X)+\gamma\}$ *on the support of* $\Im\alpha$.

Lemma 3. *In above* Lemma 2, *suppose the further conditions* : $k(X, Y) > 0$, $\inf \Re f(X) > 0$, *and* $\inf \Im f(X) > 0$ *for any points X and Y of F and both* $\Re f(X)$ *and* $\Im f(x)$ *are finite and continuous. Then, there exist a measure* α *of* $\mathfrak{M}(F)$ *and a real constant* γ *such that*

(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on* F,
(2) $\Re K\alpha(X) = \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,
(3) $\Im K\alpha(X) \geqq \Im f(X)$ *on* F, *and*
(4) $\Im K\alpha(X) = \Im f(X)$ *on the support of* $\Im\alpha$.

Lemma 4. *In above Theorem 2, suppose the further conditions: both* $\Re f(X)$ *and* $\Im f(X)$ *are finite and continuous. Then, there exist a measure* α *of* $\mathfrak{M}(F)$ *and a real constant* γ *such that*
(1) $\Re K\alpha(X) \geqq \Re\{f(X)+\gamma\}$ *on* F *with a possible exception of a set of k-transfinite diameter zero,*
(2) $\Re K\alpha(X) \leqq \Re\{f(X)+\gamma\}$ *on the support of* $\Re\alpha$,
(3) $\Im K\alpha(X) \geqq \Im f(X)$ *on* F *with a possible exception of a set of k-transfinite diameter zero, and*
(4) $\Im K\alpha(X) \leqq \Im f(X)$ *on the support of* $\Im\alpha$.

Proof of Lemma 1. Let the set E be of positive k-transfinite diameter. $\check{k}(X, Y)$ satisfying the continuity principle, there exists a positive measure σ such that
(a) the compact support of σ is contained in the set E, and
(b) $\check{k}\sigma(X)$ is finite and continuous in the whole space Ω.
Hence we have

$\int k\mu(X)d\sigma(X) = +\infty$, that is, $\int \check{k}\sigma(X)d\mu(X) = +\infty$, which is a contradiction.

Proof of Lemma 2. For any positive number c, denote by $m(c, F)$ the set of all positive measures supported by F with total mass c. We define the point-to-set mapping φ on the product space $m(a, F)\times m(b, F)$ into $\mathfrak{F}(m(a, F) \times m(b, F))$ which is the family of all closed convex subsets in $m(a, F)\times m(b, F)$. For any $\alpha=\mu+i\nu$, that is, $\alpha=(\mu, \nu)$ of $m(a, F)\times m(b, F)$, φ is defined as follows.

$$\begin{aligned}\varphi((\mu, \nu)) = \{&(\lambda, \tau)\in m(a, F)\times m(b, F) \mid \\ &\int(k\mu(X)-n\nu(X)-\Re f(X))d\lambda(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\tau(X) \\ &= \inf\,(\int(k\mu(X)-n\nu(X)-\Re f(X))d\xi(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\eta(X) \mid \\ &(\xi, \eta)\in m(a, F)\times m(b, F))\}.\end{aligned}$$

Obviously $\varphi((\mu, \nu)) \neq \phi$. For, putting

$$\begin{aligned}d = \inf\,(&\int(k\mu(X)-n\nu(X)-\Re f(X))d\xi(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\eta(X) \mid \\ &(\xi, \eta)\in m(a, F)\times m(b, F)),\end{aligned}$$ there exist sequences of

$\xi_n \in m(a, F)$ and $\eta_n \in m(b, F)$ such that

$\int(k\mu(X)-n\nu(X)-\Re f(X))d\xi_n(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\eta_n(X)\to d$. As we have vaguely convergent subnets $\xi_{n_k}\in m(a, F)$ and $\eta_{n_k}\in m(b, F)$ such that $\xi_{n_k}\to\xi_0$ and $\eta_{n_k}\to\eta_0$, there holds $\varphi((\mu, \nu))\ni(\xi_0, \eta_0)$. Moreover $\varphi((\mu, \nu))$ is upper semi-continuous in the following sense : if nets $\{\delta_\alpha \mid \alpha\in D$, a directed set$\}$ and $\{\zeta_\alpha \mid \alpha\in D\}$ converge to δ and ζ with respect to the product topology respectively, and if $\delta_\alpha\in\varphi(\zeta_\alpha)$ for any $\alpha\in D$, then $\delta\in\varphi(\zeta)$. In fact, if we put $\delta_\alpha=(\lambda_\alpha, \tau_\alpha)$, $\zeta_\alpha=(\sigma_\alpha, \gamma_\alpha)$, $\delta=(\lambda_0, \tau_0)$, and $\zeta=(\sigma_0, \gamma_0)$, we have

$$\begin{aligned}&\int(k\sigma_\alpha(X)-n\gamma_\alpha(X)-\Re f(X))d\lambda_\alpha(X)+\int(k\gamma_\alpha(X)+n\sigma_\alpha(X)-\Im f(X))d\tau_\alpha(X)\\ &\leqq\int(k\sigma_\alpha(X)-n\gamma_\alpha(X)-\Re f(X))d\xi(X)+\int(k\gamma_\alpha(X)+n\sigma_\alpha(X)-\Im f(X))d\eta(X)\end{aligned}$$

for any $(\xi, \eta)\in m(a, F)\times m(b, F)$. By the limit process, we have

$$\begin{aligned}&\int(k\sigma_0(X)-n\gamma_0(X)-\Re f(X))d\lambda_0(X)+\int(k\gamma_0(X)+n\sigma_0(X)-\Im f(X))d\tau_0(X)\\ &\leqq\int(k\sigma_0(X)-n\gamma_0(X)-\Re f(X))d\xi(X)+\int(k\gamma_0(X)+n\sigma_0(X)-\Im f(X))d\eta(X)\end{aligned}$$

for any $(\xi, \eta)\in m(a, F)\times m(b, F)$. Then we have $\delta\in\varphi(\zeta)$. Consequently, by the fixed point theorem of Fan and Glicksberg ([1]), there exists an element $\alpha=(\mu, \nu)\in m(a, F)\times m(b, F)$ such that $\varphi((\mu, \nu))\ni(\mu, \nu)$. Hence we have

$$\begin{aligned}&\int(k\mu(X)-n\nu(X)-\Re f(X))d\mu(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\nu(X)\\ &\leqq\int(k\mu(X)-n\nu(X)-\Re f(X))d\xi(X)+\int(k\nu(X)+n\mu(X)-\Im f(X))d\eta(X)\end{aligned}$$

for any $(\xi, \eta)\in m(a, F)\times m(b, F)$. If we put

$$\gamma_1=\frac{1}{a}\int(k\mu(X)-n\nu(X)-\Re f(X))d\mu(X),$$

and

$\gamma_2=\dfrac{1}{b}\displaystyle\int(k\nu(X)+n\mu(X)-\Im f(X))d\nu(X)$, we have

$$\int(k\mu(X)-n\nu(X)-\Re f(X)-\gamma_1)d\xi(X)+\int(k\nu(X)+n\mu(X)-\Im f(X)-\gamma_2)d\eta(X)\geqq 0$$

for any $(\xi, \eta)\in m(a, F)\times m(b, F)$. The existence of a positive measure $\xi_0\in m(a, F)$ with $\int(k\mu(X)-n\nu(X)-\Re f(X)-\gamma_1)d\xi_0(X)<0$ leads us to a contradiction as follows. For any signed measure τ_0 supported by F with total mass zero such that $\eta=\nu+\varepsilon\tau_0$ is a positive measure for any positive number $\varepsilon(<1)$, we have

$$\begin{aligned}&\int(k\mu(X)-n\nu(X)-\Re f(X)-\gamma_1)d\xi_0(X)\\ &+\varepsilon\int(k\nu(X)+n\mu(X)-\Im f(X)-\gamma_2)d\tau_0(X)\geqq 0.\end{aligned}$$

Making $\varepsilon\to 0$, we have a contradiction. So we have

$$\int(k\mu(X)-n\nu(X)-\Re f(X)-\gamma_1)d\xi(X)\geqq 0 \text{ for any } \xi\in m(a, F).$$

By the same way as above, we have

$$\int(k\nu(X)+n\mu(X)-\Im f(X)-\gamma_2)d\eta(X)\geqq 0 \qquad \text{for any } \eta\in m(b, F).$$

By these inequalities, we have

(1) $k\mu(X)-n\nu(X)\geqq\Re f(X)+\gamma_1$ on F,

(2) $k\mu(X)-n\nu(X)=\Re f(X)+\gamma_1$ on the support of μ,

(3) $k\nu(X)+n\mu(X)\geqq\Im f(X)+\gamma_2$ on F, and

(4) $k\nu(X)+n\mu(X)=\Im f(X)+\gamma_2$ on the support of ν.

Consequently for a complex-valued measure $\alpha=\mu+i\nu$ of $\mathfrak{M}$ (a, F, b) and a complex constant $\gamma=\gamma_1+i\gamma_2$, we have

(1) $\Re K\alpha(X)\geqq\Re\{f(X)+\gamma\}$ on F,

(2) $\Re K\alpha(X)=\Re\{f(X)+\gamma\}$ on the support of $\Re\alpha$,

(3) $\Im K\alpha(X)\geqq\Im\{f(X)+\gamma\}$ on F, and

(4) $\Im K\alpha(X)=\Im\{f(X)+\gamma\}$ on the support of $\Im\alpha$.

Thus the proof is completed.

Proof of Lemma 3. Putting $k'(X, Y)=k(X, Y)/\Im f(X)$ and $n'(X, Y)=n(X, Y)/\Im f(X)$, $k'(X, Y)$ snd $n'(X, Y)$ are finite continuous functions, and $k'(X, Y)>0$ for any points X and Y of F. Taking a positive number a which is less than

$$\frac{\min\{k(X, Y)\mid X\in F,\ Y\in F\}\cdot\min\{\Im f(X)\mid X\in F\}}{\max\{|n(X, Y)|\mid X\in F,\ Y\in F\}\cdot\max\{\Im f(X)\mid X\in F\}},$$

we have $\int(k'\nu(X)+n'\mu(X))d\nu(X)>0$ for any $(\mu, \nu)\in m(a, F)\times m(1, F)$. For this positive number a we consider the point-to-set mapping φ defined on $m(a, F)\times m(1, F)$ into $\mathfrak{F}(m(a, F)\times m(1, F))$ which is the family of all closed convex subsets in $m(a, F)\times m(1, F)$. For any $(\mu, \nu)\in m(a, F)\times m(1, F)$, φ is defined as follows.

$$\begin{aligned}\varphi((\mu, \nu)) = \{&(\lambda, \tau)\in m(a, F)\times m(1, F)\mid\\ &\int(k\mu(X)-n\nu(X)-\int(k'\nu(X)+n'\mu(X))d\nu(X)\cdot\Re f(X))d\lambda(X)+\\ &\int(k'\nu(X)+n'\mu(X))d\tau(X) = \inf(\int(k\mu(X)-n\nu(X)-\\ &\int(k'\nu(X)+n'\mu(X))d\nu(X)\cdot\Re f(X))d\xi(X)+\\ &\int(k'\nu(X)+n'\mu(X))d\eta(X)\mid(\xi, \eta)\in m(a, F)\times m(1, F))\}\end{aligned}$$

Obviously $\varphi((\mu, \nu))$ is a non-empty closed convex subset and φ is upper semi-continuous as in Lemma 2. Hence, by the fixed point theorem of Fan and Glicksberg, there exists an element $(\mu_0, \nu_0)\in m(a, F)\times m(1, F)$ such that $\varphi((\mu_0, \nu_0))\ni(\mu_0, \nu_0)$. Then we have

$$\begin{aligned}&\int(k\mu_0(X)-n\nu_0(X)-\int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X)\cdot\Re f(X))d\mu_0(X)+\\ &\int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X)\leqq\int(k\mu_0(X)-n\nu_0(X)-\\ &\int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X)\cdot\Re f(X))d\xi(X)+\int(k'\nu_0(X)+n'\mu_0(X))d\eta(X)\end{aligned}$$

for any $(\xi, \eta)\in m(a, F)\times m(1, F)$. Putting

$$\gamma_1 = \frac{1}{a}\cdot\int(k\mu_0(X)-n\nu_0(X)-\int(k'\nu_0(X)+n'\mu_0(X))\,d\nu_0(X)\cdot\Re f(X))d\mu_0(X),$$

and

$$\begin{aligned}&\gamma_2 = \int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X), \text{ we have}\\ &\int(k\mu_0(X)-n\nu_0(X)-\int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X)\cdot\Re f(X)-\gamma_1)d\xi(X)+\\ &\int(k'\nu_0(X)+n'\mu_0(X)-\gamma_2)d\eta(X)\geqq 0\end{aligned}$$

for any $(\xi, \eta)\in m(a, F)\times m(1, F)$. By the same way as Lemma 2, we have two following inequalities.

(1) $\int(k\mu_0(X)-n\nu_0(X)-\int(k'\nu_0(X)+n'\mu_0(X))d\nu_0(X)\cdot\Re f(X)-\gamma_1)d\xi(X)\geqq 0$ for any $\xi\in m(a, F)$, and

(2) $\int(k'\nu_0(X)+n'\mu_0(X)-\gamma_2)d\eta(X)\geqq 0$ for any $\eta\in m(1, F)$.

From these inequalities we have

(1) $k\mu_0(X)-n\nu_0(X)-\gamma_2\cdot\Re f(X)\geqq\gamma_1$ on F,

(2) $k\mu_0(X)-n\nu_0(X)-\gamma_2\cdot\Re f(X)=\gamma_1$ on the support of μ_0,

(3) $k'\nu_0(X)+n'\mu_0(X)\geqq\gamma_2$ on F, and

(4) $k'\nu_0(X)+n'\mu_0(X)=\gamma_2$ on the support of ν_0.

By the property of the number a, γ_2 is strictly positive. Putting $\mu=\frac{\mu_0}{\gamma_2}$, $\nu=\frac{\nu_0}{\gamma_2}$ and $\gamma=\frac{\gamma_1}{\gamma_2}$, we have

(1) $k\mu(X)-n\nu(X)\geqq\Re f(X)+\gamma$ on F,

(2) $k\mu(X)-n\nu(X)=\Re f(X)+\gamma$ on the support of μ,

(3) $k\nu(X)+n\mu(X)\geqq\Im f(X)$ on F, and

(4) $k\nu(X)+n\mu(X)=\Im f(X)$ on the support of ν.

Thus, the measure $\alpha=\mu+i\nu$, and the real constant γ are what Lemma 3 needs.

Proof of Lemma 4. As $k(X, Y)$ is a lower semi-continuous function such that $\inf\{k(X, Y)\mid(X, Y)\in F\times F\}=2p>0$, there exists an increasing net $\{k_m(X, Y)\mid m\in D$, a directed set$\}$ of finite continuous functions such that $\lim_m k_m(X, Y)=k(X, Y)$ and $k_m(X, Y)>p$ for any points X and Y of F. Taking a positive number a which is less than

$$\frac{p\cdot\min\{\Im f(X)\mid X\in F\}}{\max\{\Im f(X)\mid X\in F\}\cdot\max\{|n(X, Y)|\mid(X, Y)\in F\times F\}},$$

by Lemma 3, there exist measures $\alpha_m=\mu_m+i\nu_m\in\mathfrak{M}(a, F, 1)$ and real constants γ_m and γ'_m such that

(1) $k_m\mu_m(X)-n\nu_m(X)-\gamma'_m\cdot\Re f(X)\geqq\gamma_m$ on F,

(2) $k_m\mu_m(X)-n\nu_m(X)-\gamma'_m\cdot\Re f(X)=\gamma_m$ on the support of μ_m,

(3) $k'_m\nu_m(X)+n'\mu_m(X)\geqq\gamma'_m$ on F, and
(4) $k'_m\nu_m(X)+n'\mu_m(X)=\gamma'_m$ on the support of ν_m.

In the first place, we are going to see the boundedness of the net $\{\gamma'_m\,|\,m\in D\}$. Obviously $\gamma'_m>0$ for any m. Supposing that $\overline{\lim_m}\,\gamma'_m=+\infty$, we can take a subnet $\{\gamma'_{m_i}\,|\,m_i\in D'$, a directed set$\}$ such that $\nu_{m_i}\to\nu$, $\mu_{m_i}\to\mu$, $\gamma'_{m_i}\to+\infty$, and $k_{m_i}(X, Y)\uparrow k(X, Y)$ along D' for any points X and Y of F. $\check{k}'(X, Y)$ satisfying the continuity principle, we have, by the above inequality (3),

$$k'\nu(X)+n'\mu(X)\geqq\underline{\lim_{m_i}}\,k'_{m_i}\nu_{m_i}(X)+\lim_{m_i}\,n'\mu_{m_i}(X)\geqq\lim_{m_i}\,\gamma'_{m_i}=+\infty$$

on F with a possible exception of a set of k-transfinite diameter zero. Then we have that $k\nu(X)=+\infty$ on F with a possible exception of a set of k-transfinite diameter zero, which is a contradiction by Lemma 1. Using the boundedness of the net $\{\gamma'_m\,|\,m\in D\}$, we can see the boundedness of the net $\{\gamma_m\,|\,m\in D\}$ by the same way as above. Consequently, considering an adequate directed set E, we have that $\gamma'_{l_i}\to\gamma_2$, $\gamma_{l_i}\to\gamma_1$, $\mu_{l_i}\to\mu_0$, $\nu_{l_i}\to\nu_0$, and $k_{l_i}(X, Y)\uparrow k(X, Y)$ along E. Hence we have, by the same way as M. Kishi ([2] and [3])

(1) $k\mu_0(X)-n\,\nu_0(X)-\gamma_2\cdot\Re f(X)\geqq\gamma_1$ on F with a possible exception of a set of k-transfinite diameter zero,
(2) $k\mu_0(X)-n\nu_0(X)-\gamma_2\cdot\Re f(X)\leqq\gamma_1$ on the support of μ_0,
(3) $k'\nu_0(X)+n'\mu_0(X)\geqq\gamma_2$ on F with a possible exception of a set of k-transfinite diameter zero, and
(4) $k'\nu_0(X)+n'\mu_0(X)\leqq\gamma_2$ on the support of ν_0.

By the property of the number a, γ_2 is strictly positive. Putting $\mu=\frac{\mu_0}{\gamma_2}$, $\nu=\frac{\nu_0}{\gamma_2}$, and $\gamma=\frac{\gamma_1}{\gamma_2}$, we have

(1) $k\mu(X)-n\nu(X)\geqq\Re f(X)+\gamma$ on F with a possible exception of a set of k-transfinite diameter zero,
(2) $k\mu(X)-n\nu(X)\leqq\Re f(X)+\gamma$ on the support of μ,
(3) $k\nu(X)+n\mu(X)\geqq\Im f(X)$ on F with a possible exception of a set of k-transfinite diameter zero, and
(4) $k\nu(X)+n\mu(X)\leqq\Im f(X)$ on the support of ν.

Thus, the measure $\alpha=\mu+i\nu$, and the real constant γ are what Lemma 4 needs. Finally, we prove the theorems.

Proof of Theorem 1. As $k(X, Y)$ is a lower semi-continuous function such that $-\infty<k(X, Y)\leqq+\infty$, there exists an increasing net $\{k_m(X, Y)\,|\,m\in D$, a directed set$\}$ of finite continuous functions such that $\lim_m k_m(X, Y)=k(X, Y)$ for any points X and Y of F. Then, by Lemma 2, there exist measures $\alpha_m=\mu_m+i\nu_m$ of $\mathfrak{M}(a, F, b)$ and complex constants $\gamma_m=\gamma'_m+i\gamma''_m$ such that

(1) $k_m\mu_m(X)-n\nu_m(X)\geqq\Re f(X)+\gamma'_m$ on F,
(2) $k_m\mu_m(X)-n\nu_m(X)=\Re f(X)+\gamma'_m$ on the support of μ_m,
(3) $k_m\nu_m(X)+n\mu_m(X)\geqq\Im f(X)+\gamma''_m$ on F, and
(4) $k_m\nu_m(X)+n\mu_m(X)=\Im f(X)+\gamma''_m$ on the support of ν_m.

By the same way as Lemma 4, there exist a measure $\alpha=\mu+i\nu$ of $\mathfrak{M}(a, F, b)$ and a complex constant $\gamma=\gamma_1+i\gamma_2$ such that

(1) $\Re K\alpha(X)\geqq\Re\{f(X)+\gamma\}$ on F with a possible exception of a set of k-transfi nite diameter zero,
(2) $\Re K\alpha(X)\leqq\Re\{f(X)+\gamma\}$ on the support of $\Re\alpha$,
(3) $\Im K\alpha(X)\geqq\Im\{f(X)+\gamma\}$ on F with a possible exception of a set of k-transfinite diameter zero, and
(4) $\Im K\alpha(X)\leqq\Im\{f(X)+\gamma\}$ on the support of $\Im\alpha$.

Proof of Theorem 2. Let $\{f_m(X)\,|\,m\in D\}$ and $\{g_m(X)\,|\,m\in D\}$ be decreasing nets of positive finite continuous functions on F such that $f_m(X)\downarrow\Re f(X)$ and $g_m(X)\downarrow\Im f(X)$. Taking an adequate positive number a, by Lemma 4, there exist measures $\alpha_m=\mu_m+i\nu_m$ of $\mathfrak{M}(a, F, 1)$ and real constants γ'_m and γ''_m such that

(1) $k\mu_m(X)-n\nu_m(X)-\gamma''_m\cdot f_m(X)\geqq\gamma'_m$ on F with a possible exception of a set of k-transfinite diameter zero,
(2) $k\mu_m(X)-n\nu_m(X)-\gamma''_m\cdot f_m(X)\leqq\gamma'_m$ on the support of μ_m,
(3) $k\nu_m(X)+n\mu_m(X)\geqq\gamma''_m\cdot g_m(X)$ on F with a possible exception of a set of k-transfinite diameter zero, and
(4) $k\nu_m(X)+n\mu_m(X)\leqq\gamma''_m\cdot g_m(X)$ on the support of ν_m.

By the same way as Lemma 4, there exist a measure $\alpha=\mu+i\nu$ of $\mathfrak{M}(F)$ and a real constant γ such that

(1) $k\mu(X)-n\,\nu(X)\geqq\Re f(X)+\gamma$ on F with a possible exception of a set of k-transfinite diameter zero,
(2) $k\mu(X)-n\,\nu(X)\leqq\Re f(X)+\gamma$ on the support of μ.
(3) $k\nu(X)+n\mu(X)\geqq\Im f(X)$ on F with a possible exception of a set of k-transfinite diameter zero, and
(4) $k\nu(X)+n\mu(X)\leqq\Im f(X)$ on the support of ν.

Thus, the measure $\alpha=\mu+i\nu$, and the real constant γ are what Theorem 2 needs. The analogous arguments will give us the latter part of Theorem 2.

Corollary. *Let F be a compact subset of positive k-transfinite diameter, and $f(X)$ be a real-valued upper semi-continuous function with lower bound on F, and a be a positive number. If the adjoint kernel $\check{k}(X, Y)$ satisfies the continuity principle, then there exist a measure μ of $m(a, F)$ and a real constant γ such that*

(1) *$k\mu(X)\geqq f(X)+\gamma$ on F with a possible exception of a set of k-transfinite diameter zero, and*
(2) *$k\mu(X)\leqq f(X)+\gamma$ on the support of μ.*

REMARK. In above Theorem 2, we can not always reduce the constant γ to zero. We may consider the following example : let Ω be a finite space consisting of two points X_1 and X_2, and $\Re K(X, Y)$ and $\Im K(X, Y)$ be given by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively, and $\Re f(X)$ and $\Im f(X)$ be equal to 1 everywhere. Then, for the compact set $F=\Omega$, we have no measure α such that

(1) $\Re K\alpha(X) \geqq \Re f(X)$ on F,
(2) $\Re K\alpha(X) = \Re f(X)$ on the support of $\Re\alpha$,
(3) $\Im K\alpha(X) \geqq \Im f(X)$ on F, and
(4) $\Im K\alpha(X) = \Im f(X)$ on the support of $\Im\alpha$.

REMARK. Putting $n(X, Y) = \Im K(X, Y) \equiv 0$, we can assert that our Theorem 2 contains the existence theorem obtained by M. Kishi and M. Nakai ([2], [3] and [4]).

SHIZUOKA UNIVERSITY

References

[1] K. Fan: *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.

[2] M. Kishi: *Maximum principles in the potential theory*, Nagoya Math. J. **23** (1963), 165–187.

[3] M. Kishi: *An existence theorem in potential theory*, Nagoya Math. J. **27** (1966), 133–137.

[4] M. Nakai: *On the fundamental existence theorem of Kishi*, Nagoya Math. J. **23** (1963), 189–198.

[5] N. Ninomiya: *On the potential taken with respect to complex-valued and symmetric kernels*, Osaka J. Math. **9** (1972), 1–9.