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THE CHARACTERIZATION OF DIFFERENTIAL OPERATORS WITH RESPECT TO THE CHARACTERISTIC CAUCHY PROBLEM

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1. Introduction. Let $L(\lambda, \eta)=\sum_{j=0}^{M}\sum_{k=0}^{N} a_{j,k}\lambda^j\eta^k$ be a polynomial of λ and η with degrees M and N respectively. Then we can define a constant $\alpha(L)$ as follows. When $L(\lambda, 0)\not\equiv 0$, we set

$$\alpha(L) = \max_{a_{j,k}\neq 0, k>0} \frac{m-j}{k},$$

where m is the degree of $L(\lambda, 0)$. In this case we have $j+k\alpha(L)\leqq m$ if $a_{j,k}\neq 0$ and $j_0=k_0\alpha(L)=m$ for some (j_0, k_0) such that $k_0>0$ and $a_{j_0,k_0}\neq 0$. When $L(\lambda, 0)\equiv 0$, we define $\alpha(L)=-\infty$. It is easily shown by the definition of $\alpha(L)$ that the line $t=0$ is characteristic with respect to the differential operator $L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ if and only if $\alpha(L)<1$. L. Hörmander [3] proved that there exist null solutions[1] of the differential equation $Lu=0$ with respect to the half plane $\Pi=\{(t, x)\,;\, t\leqq 0\}$ if and only if the line $t=0$ is characteristic.

In this note we shall characterize the differential operator L by the smallest (largest) function class $G_x(\cdot)$[2] of Gevrey's to which null solutions are (not) able to belong. In theorem 1, using the same method as L. Hörmander's in [2], we construct a null solution which belongs to $G_x(\alpha+\varepsilon)$ for any $\varepsilon>0$ if $0<\alpha$[3] <1, and to $G_x(\alpha)$ if $-\infty\leqq\alpha\leqq 0$. In theorem 2, we prove the uniqueness of the solution of the Cauchy

1) A solution $u(t, x)$ of the equation $Lu=0$ is called a null solution with respect to the half plane Π, if $u\in C^\infty(R^2)$ and $u\not\equiv 0$ in R^2 but $u=0$ in Π.

2) A C^∞-function $f(t, x)$ is called to be in $G_x(\alpha)$ in $(T_1, T_2)\times(x_1, x_2)$, $-\infty\leqq x_1<x_2\leqq +\infty$, if it satisfies

$$\left|\frac{\partial^k}{\partial x^k} f(t, x)\right| \leqq K^{k+1}(k!)^\alpha \qquad (k=0, 1, 2, \cdots)$$

in any finite interval $[a, b]$ in (x_1, x_2) for some constant K.

3) In what follows we write $\alpha=\alpha(L)$.

problem in the function class $G_x(\alpha)$ if $0<\alpha<1$ and in $G_x(\alpha-\varepsilon)$ for any $\varepsilon>0$ if $\alpha\leqq 0$.

When $\alpha\leqq 0$, it is impossible to reduce the differential equation $Lu=f$ to a system of the form $\frac{\partial}{\partial t}U=P\left(\frac{\partial}{\partial x}\right)U+F$ with a matrix $P(\eta)$ of differential polynomials. Accordingly it becomes impossible to use the method of A. Friedman [1] which reduces the problem to the property of the fundamental solution of a system of first order ordinary differential equations.

We remark for example that $\alpha\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)=1/2$, $\alpha\left(\frac{\partial^2}{\partial t\,\partial x}+\frac{\partial}{\partial t}\right)=0$, $\alpha\left(\frac{\partial^2}{\partial t\,\partial x}+1\right)=-1$, and $\alpha\left(\frac{\partial^2}{\partial t\,\partial x}\right)=-\infty$.

2. Preliminary lemmas.

Lemma 1. *Let* $-\infty<\alpha<1$. *Then there exists a function* $\eta(\lambda)$ *which satisfies the following conditions*:

i) $L(\lambda, \eta(\lambda))=0$.

ii) *There exist constants* C_0 *and* K_0 *such that if* $\Im m\,\lambda$[4] $\geqq K_0$, $\eta(\lambda)$ *is analytic and satisfies the inequality*

$$|\eta(\lambda)| \leqq C_0|\lambda|^{\alpha}. \tag{1}$$

Proof. Set

$$L(\lambda, \eta) = Q_N(\lambda)\eta^N+\cdots+Q_0(\lambda),$$

then we have $Q_N(\lambda)\not\equiv 0$ and

$$\begin{cases} \deg^{5)} Q_0(\lambda) = \deg L(\lambda, 0) = m\geqq 0, \\ \deg Q_k(\lambda)+\alpha k\leqq m, \qquad (k = 1, 2, \cdots, N), \\ \deg Q_{k_0}(\lambda)+\alpha k_0 = m. \end{cases} \tag{2}$$

Let $\eta_j(\lambda)$ $(j=1, 2, \cdots, N)$ be the roots of the equation $L(\lambda, \eta)=0$. Then every $\eta_j(\lambda)$ has the Puiseux series expantion at infinity:

$$\eta_j(\lambda) = \sum_{n=-\infty}^{l_j} \alpha_{j,n}\lambda^{n/p_j}, \qquad (\alpha_{j,l_j}\neq 0). \tag{3}$$

Hence, for a sufficiently large constant K_0, $\eta_j(\lambda)$ is analytic in $\Im m\,\lambda\geqq K_0$. By (1) and (2) we have

$$|Q_N(\lambda)\eta_1(\lambda)\cdots\eta_N(\lambda)| = |Q_0(\lambda)|\leqq K_1|\lambda|^m$$

and

4) $\Im m\,\lambda$ means the imaginary part of a complex number λ.

5) $\deg Q_0(\lambda)$ means the degree of $Q_0(\lambda)$.

$$|Q_N(\lambda)\sum_{i_1\cdots i_{N-k_0}}\eta_{i_1}(\lambda)\cdots\eta_{i_{N-k_0}}(\lambda)| = |Q_{k_0}(\lambda)| \geqq K_2|\lambda|^{m+k_0}.$$

Without loss of generality we may assume

$$|Q_N(\lambda)\eta_{k_0+1}(\lambda)\cdots\eta_N(\lambda)| \geqq K_3|\lambda|^{m-\alpha k_0},$$

hence we have

$$\begin{aligned}|\eta_1(\lambda)\cdots\eta_{k_0}(\lambda)| &\leqq K_1|\lambda|^m|Q_N(\lambda)\eta_{k_0+1}(\lambda)\cdots\eta_N(\lambda)|^{-1}\\ &\leqq (K_1/K_2)|\lambda|^{\alpha k_0}.\end{aligned}$$

Using (3) and this we have

$$\sum_{j=1}^{k_0}(l_j/p_j) \leqq \alpha k_0.$$

This shows that $l_j/p_j \leqq \alpha$ for some j, and by this, if we choose K_0 large enough, we have

$$|\eta_j(\lambda)| \leqq C_0|\lambda|^\alpha \quad \text{if} \quad \Im m\,\lambda \geqq K_0.$$

Q. E. D.

DEFINITION. We call a function $f(t, x)$ to be in a class $G(\nu, \mu)$ in a domain $\Omega \subset R^2$, where ν and μ are real numbers, if $f \in C^\infty(\Omega)$ and satisfies

$$\left|\frac{\partial^{j+k}}{\partial t^j \partial x^k}f(t, x)\right| \leqq KC^{j+k}(j!)^\nu(k!)^\mu, \qquad (j, k = 0, 1, 2, \cdots) \tag{4}$$

for some constants K and C.

Let H be a integro-differential operator of the form

$$\begin{aligned}(Hf)(t, x) &= \sum_{j+\alpha k\leqq m,\ j\leqq m-1} a_{j,k}\int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!}\frac{\partial^k}{\partial x^k}f(\tau, x)\,d\tau\\ &+\sum_{j+\alpha k\leqq m,\, j\geqq m,\, k>0} a_{j,k}\frac{\partial^{j-m+k}}{\partial t^{j-m}\partial x^k}f(t, x), \qquad (-\infty<\alpha\leqq 0),\end{aligned} \tag{5}$$

where m is a non-negative integer and $0 \leqq j \leqq M$, $0 \leqq k \leqq N$. Then we have the following

Lemma 2. *Let Ω be a rectangular domain $(0, T)\times(x_1, x_2)$; $0<T<+\infty$, $-\infty \leqq x_1 < x_2 \leqq +\infty$, and let a function $f(t, x)$ belong to $G(\rho, \rho\alpha-\varepsilon_0)$ in Ω for some constants $\rho>1$ and $0<\varepsilon_0\leqq 1$. Then the equation $v-Hv=f$ has a unique solution in the same class.*

Proof. It suffices to prove that the series $\sum_{m=0}^{+\infty} H^n f$ converges to a function in $G(\rho, \rho\alpha-\varepsilon_0)$. If we write

$$(H_{j,k}f)(t,\, x) = \int_0^t \frac{(t-\tau)^{j-1}}{(j-1)!}\frac{\partial^k}{\partial x^k} f(\tau,\, x)\, d\tau\,,$$

and

$$(H'_{j,k}f)(t,\, x) = \frac{\partial^{j+k}}{\partial t^j \partial x^k} f(t,\, x)\,,$$

then we have

$$\begin{aligned} H_{j,k}H_{j',k'} &= H_{j+j',k+k'}\,, \\ H'_{j,k}H'_{j',k'} &= H_{j+j',k+k'}\,, \\ H'_{j',k'}H_{j,k} &= \begin{cases} H_{j-j',k+k'}, & \text{when } j>j' \\ H'_{j'-j,k+k'}, & \text{when } j'\leqq j\,. \end{cases} \end{aligned}$$

If we write

$$\frac{\partial^{j+k}}{\partial t^j \partial x^k} H^n f = \sum a_{j_1,k_1} a_{j_2,k_2} \cdots a_{j_n,k_n} H_{j_1,k_1,\cdots,j_n,k_n} f\,, \tag{6}$$

then, each term of the summation in the right hand side takes one of the following two forms:

a) $$H_{j_1k_1\cdots j_nk_n} f = H_{J-j,K+k} H'_{J',K'} f$$

where $J=\sum_{i=1}^q (m-j_i)$, $K=\sum_{i=1}^q k_i$, $J'=\sum_{i=q+1}^n (j_i-m)$, $K'=\sum_{i=q+1}^n k_i$,

b) $$H_{j_1k_1\cdots j_nk_n} f = H'_{J'+j,K'+k} f\,,$$

where $J'=\sum_{i=1}^n (j_i-m)$, $K'=\sum_{i=1}^n k_i$.

For the case a), let $\{j_{i_1},\cdots,j_{i_r}\}$ be the set of all elements which are contained in $\{j_1,\cdots,j_q\}$ and smaller than m, and let

$$\begin{aligned} J_1 &= \textstyle\sum_{l=1}^r (m-j_{i_l})\,, \\ K_1 &= \textstyle\sum_{l=1}^r k_{i_l}\,, \\ J_2 &= J_1 - J\,, \\ K_2 &= K - K_1\,. \end{aligned}$$

Then in view of (5) we have

$$\begin{cases} J = J_1 + J_2,\ K = K_1 + K_2;\ J_1 \geqq r,\ K_2 \geqq q-r\,, \\ J_i + \alpha K_i \leqq 0\,, \qquad (i = 1,2)\,. \end{cases} \tag{7}$$

We also have

$$\begin{cases} J' = J'_1 - J'_2,\ K' = K'_1 + K'_2;\ J'_2 \geqq s\,,\ K'_1 \geqq (n-q)-s\,, \\ J'_i + \alpha K'_i \leqq 0\,, \qquad (i = 1,2)\,. \end{cases} \tag{8}$$

For the case b), we have as above

$$(9)\qquad \begin{cases} J' = J_1'-J_2',\ K' = K_1'+K_2';\ J_2'\geqq s,\ K_1'\geqq n-s, \\ J_i'+\alpha K_i' \leqq 0 \qquad (i=1,2). \end{cases}$$

Since f satisfies (4), we have for the case a)

$$(10)\qquad \begin{aligned} |H_{J-j,K+k}H'_{J',K'}f| &\leqq K_0 C^{J-j+J'+K+k+K'}\times \\ &\times[(J-j)!]^{-1}[(K+k)!]^{\alpha\rho-\varepsilon_0}(J'!)^{\rho}(K'!)^{\alpha\rho-\varepsilon_0}, \end{aligned}$$

and for the case b)

$$(11)\qquad |H'_{J'+j,K'+k}f| \leqq K_0 C^{J'+j+K'+k}[(J'+j)!]^{\rho}[(K'+k)!]^{\alpha\rho-\varepsilon_0}.$$

When $J+\alpha K\leqq 0$, using Stirling's formula we have

$$(12)\qquad (J!)(K!)^{\alpha}\leqq C^K$$

for some constant C. Using (7)-(12) with the inequality

$$(n-q)!q! \leqq n! \leqq 2^n(n-q)!q!,$$

we have for both cases

$$|H_{j_1k_1\cdots j_nk_n}f| \leqq K_0 C^{n+j+k}(j!)^{\rho}(k!)^{\alpha\rho-\varepsilon_0}(n!)^{-\varepsilon_0},$$

and this completes the proof.

3. Main theorems.

Theorem 1. *For every positive constant ε there exists a null solution U_ε of the equation $Lu=0$ with respect to the half plane Π, which satisfies one of the following inequalities for some constants K and C depending on ε,*

$$(13)\qquad \left|\frac{\partial^{j+k}}{\partial t^j\partial x^k}U_\varepsilon(t,x)\right| \leqq \exp\{K(t+|x|^{-1/(1-\alpha-\varepsilon)})\}\,C^{j+k}(j!)^{1+\varepsilon}(k!)^{\alpha+\varepsilon},$$
$$\text{if}\quad 0<\alpha<1,$$

$$(14)\qquad \left|\frac{\partial^{j+k}}{\partial t^j\partial x^k}U_\varepsilon(t,x)\right| \leqq \exp\{K(1+t)(1+x^{1/(1-\alpha)})\}\,C^{j+k}e^{kt}(j!)^{1+\varepsilon}(k!)^{\alpha}$$
$$\text{if}\quad -\infty<\alpha\leqq 0.$$

REMARK. When $\alpha=-\infty$, we can write $L=L_0\eta$ with a polynomial L_0. Then, if we set

$$\psi(t) = \begin{cases} 0, & \text{when}\quad t\leqq 0, \\ \exp\{-t^{-1/\varepsilon}\}, & \text{when}\quad t>0, \end{cases}$$

$\psi(t)$ is a null solution and satisfies

$$\left|\frac{\partial^j}{\partial t^j}\psi(t)\right| \leqq K_\varepsilon^{j+1}(j!)^{1+\varepsilon} \qquad \text{(See [3] p. 257)}.$$

This means that the statement of the theorem is also valid for the limiting case of (14).

Proof of the theorem. Take a positive constant ρ, $\alpha<1/\rho<1$, and set

$$u(t, x) = \int_{-\infty+iK_0}^{+\infty+iK_0} \exp\{-it\lambda - ix\eta(\lambda) - (\lambda/i)^{1/\rho}\}\, d\lambda\,, \tag{15}$$

where $\eta(\lambda)$ is the function defined in lemma 1 and $(\lambda/i)^{1/\rho}$ is defined real and positive on the positive imaginary axis. Then by L. Hörmander [4], p. 121, $u(t, x)$ is a null solution of the equation $Lu=0$ with respect to the half plane Π. If we set $C_1=\cos(\pi/(2\rho))$, we have

$$\Re e(\lambda/i) \geqq C_1|\lambda|^{1/\rho} \quad \text{when} \quad \Im m\,\lambda>0\,. \tag{16}$$

When $0<\alpha<1$, we have using (1) and Young's inequality,

$$|x\eta(\lambda)| \leqq C_0|x|\cdot|\lambda|^\alpha \leqq \frac{1}{2}C_1|\lambda|^{1/\rho} + K|x|^{1/(1-\rho\alpha)}$$

Hence we have

$$\left|\frac{\partial^k}{\partial x^k}u(t, x)\right| \leqq \exp\{tK_0 + K|x|^{1/(1-\rho\alpha)}\}C_0\int_{-\infty}^{+\infty}\exp\{-C_1|x|^{1/\rho}/2\}\,dy\,,$$
$$\text{where} \quad \lambda = y+iK_0\,.$$

Using the inequality

$$r^n e^{-\gamma r^{1/\rho}} \leqq C_{\gamma,\rho}^n(n!)^\rho \qquad (r>0)\,, \tag{17}$$

we have

$$\left|\frac{\partial^k}{\partial x^k}u(t, x)\right| \leqq \exp\{C'(t+|x|^{1/(1-\rho\alpha)})\}\,C'^{k+1}(k!)^{\rho\alpha}\,. \tag{18}$$

When $-\infty<\alpha\leqq 0$, if we take $|\lambda|=|x|^{1/(1-\alpha)}$, we have

$$|x\eta(\lambda)| \leqq C_0|x|\,|\lambda|^\alpha = C_0|x|\cdot|\lambda|^{\alpha/(1-\alpha)} = C_0|x|^{1/(1-\alpha)}\,.$$

If we replace the path of the integration in (15) by the path from $-\infty+i\tau$ to $\infty+i\tau$ where $\tau=K_0+|x|^{1/(1-\alpha)}+k$, we get

$$\left|\frac{\partial^k}{\partial x^k}u(t, x)\right| \leqq \exp\{t(K_0+|x|^{1/(1-\alpha)}+k) + C_0|x|^{1/(1-\alpha)}\}\times$$
$$\times C_0^k\int_{-\infty}^{+\infty}|\lambda|^{\alpha k}\exp\{-C_1|\lambda|^{1/\rho}\}\,dy\,, \qquad \lambda = y+i\tau\,.$$

Since $|\lambda|^{\alpha k} \leqq k^{k\alpha}$, using Stirling's formula we have

$$\left|\frac{\partial^k}{\partial x^k} u(t, x)\right| \leqq C'' \exp\{C''(1+t)(1+|x|^{1/(1-\alpha)})\} \cdot (C'' e^t)^k (k!)^\alpha . \tag{19}$$

Let $\varphi(t, x) \not\equiv 0$ be a function of the class $G(1+\varepsilon, 1+\varepsilon)$ in R^2 such that supp[6] $\varphi(t, x) \subseteq \{(t, x) ; t^2+x^2<1, t>0\}$ and $\varphi(t, x) \geqq 0$. Such a function is easily constructed using the function given in the above remark. We write $\varphi_\delta(t, x) = \varphi(t/\delta, x/\delta)$, then φ_δ is also in $G(1+\varepsilon, 1+\varepsilon)$. Set

$$U_\delta(t, x) = \varphi_\delta * u = \int \varphi_\delta(t-\tau, x-y) u(\tau, y) \, d\tau \, dy .$$

Then, $LU_\delta = 0$, $U_\delta = 0$ for $t \leqq 0$ and $U_\delta \not\equiv 0$ in R^2 for sufficiently small $\delta > 0$. Hence U_δ be a null solution of the equation $Lu=0$. When $0<\alpha<1$, we take $1<\rho<1+\varepsilon/\alpha$, and write

$$\frac{\partial^{j+k}}{\partial t^j \partial x^k} U_\delta(t, x) = \int \left\{\frac{\partial^j}{\partial t^j} \varphi_\delta(t-\tau, y)\right\} \left\{\frac{\partial^k}{\partial x^k} U(\tau, x-y)\right\} d\tau \, dy .$$

Then by (18) and (19) we get the desired estimates (13) and (14).

Q. E. D.

Next, we shall prove in the sharper form that we can not construct any null solution in the class $G_x(\alpha)$ if $0<\alpha<1$, and $G_x(\alpha-\varepsilon)$ for every $\varepsilon>0$ if $-\infty<\alpha\leqq 0$.

Theorem 2. *For any $T>0$ we have the following results:*
i) *When $0<\alpha<1$, let u be a distribution solution of the equation $Lu=0$ in $(-\infty, T) \times (-\infty, \infty)$ such that* supp $u \subset \{(t, x) ; t \geqq 0\}$. *Furthermore assume that u is a function satisfying*

$$|u| \leqq K \exp\{K x^{1/(1-\alpha)}\}$$

for some constant K. Then $u \equiv 0$ in $(-\infty, T) \times (-\infty, \infty)$.
ii) *When $-\infty<\alpha\leqq 0$, let u be a distribution solution of the equation $Lu=0$ in $(-\infty, T) \times (x_1, x_2)$, $-\infty \leqq x_1 < x_2 \leqq +\infty$, such that* supp $u \subset \{(t, x) ;$ $t \geqq 0\}$. *Furthermore assume that $\frac{\partial^k}{\partial x^k} u$ $(k=0, 1, 2, \cdots)$ are functions satisfying*

$$\left|\frac{\partial^k}{\partial x^k} u\right| \leqq K^{k+1} (k!)^{\alpha-\varepsilon}$$

for some constants $\varepsilon>0$, K. Then $u \equiv 0$ in $(-\infty, T) \times (x_1, x_2)$.

6) supp $\varphi(t, x)$ is the closure of $\{(t, x) ; \varphi(t, x) \neq 0\}$.

Proof. The proof of i) is given in [1] and [5]. So we shall prove ii) using lemma 2. Set $\varepsilon_0=\varepsilon/2$ and determine $\rho>1$ by $\alpha-\varepsilon=\rho\alpha-\varepsilon_0$, i.e. $\rho=1+(\varepsilon-\varepsilon_0)/(-\alpha)$. Let $\varphi(t, x)\not\equiv 0$ be a function of the class $G(\rho, \rho)$ in R^2 such that supp $\varphi\subseteq\{(t, x)\,;\, t\geqq 0\}$, and set $\varphi_\delta(t, x)=\varphi(t/\delta, x/\delta)$ for $\delta>0$. Then $u_\delta=\varphi_\delta*u$, where u is a function of ii), is defined in $(-\infty, T-\delta)\times(x_1+\delta, x_2-\delta)$ and satisfies the following condition:

$$
\left\{\begin{array}{l} Lu_\delta = 0\,, \qquad \operatorname{supp} u_\delta\subset\{t, x)\,;\, t>0\} \\ \left|\dfrac{\partial^{j+k}}{\partial t^j\partial x^k}u_\delta(t, x)\right|\leqq K'^{j+k+1}(j!)^\rho(k!)^{\alpha\rho-\varepsilon_0}. \end{array}\right. \tag{20}
$$

Now setting $v_\delta=\dfrac{\partial^m}{\partial t^m}u_\delta$, we have $v_\delta\in G(\rho, \rho\alpha-\varepsilon_0)$ in $(0, T-\delta)\times(x_1+\delta, x_2-\delta)$ and

$$
\frac{\partial^j}{\partial t^j}u_\delta = \left\{\begin{array}{ll} \dfrac{\partial^{j-m}}{\partial t^{j-m}}v_\delta & \text{for } j\geqq m \\ \displaystyle\int_0^t\frac{(t-\tau)^{m-j-1}}{(m-j-1)!}v_\delta(\tau, x)\,d\tau & \text{for } j<m\,. \end{array}\right. \tag{21}
$$

Hence we have

$$
0 = a_{m,0}^{-1}Lu_\delta = v_\delta-Hv_\delta
$$

where H is the operator given by (5). As $v_\delta\in G(\rho, \rho\alpha-\varepsilon_0)$, we can apply lemma 2 and get $v_\delta\equiv 0$ in $(0, T-\delta)\times(x_1+\delta, x_2-\delta)$. Hence $u_\delta\equiv 0$ in the same domain.

Letting $\delta\to 0$, we have $u\equiv 0$ in $(-\infty, T)\times(x_1, x_2)$. Q.E.D.

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