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ON GALOIS ALGEBRA OVER A COMMUTATIVE RING

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In [1], M. Auslander and O. Goldman introduced the notion of a Galois extension of a commutative ring. Galois theory for separable extension of a commutative ring has been developed by S. U. Chase, D. K. Harrison and A. Rosenberg in [3]. The author, in [6] and [7], generalized the notion of a Galois extension of a commutative ring to the case of non commutative ring, and developed the Galois theory for separable algebra over a commutative ring. We call here an algebra Λ over a commutative ring R a *Galois algebra* if Λ is a Galois extension of R. The study of Galois algebra over a commutative ring has been done by F. R. DeMeyer in [4] and [5], and Y. Takeuchi in [11]. In this note we investigate the structure of such Galois algebra over a commutative ring.

In §2 we prove that if Λ is a Galois algebra over a commutative ring R with group G and if C is the center of Λ then Λ is a direct sum of C-submodule J_σ of Λ with $\sigma \in G$ where $J_\sigma = \{a \in \Lambda \mid \sigma(x)a = ax$ for every x in $\Lambda\}$. Using this fact, we give shorter proofs of the results of F. DeMeyer in [4] and [5] and Y. Takeuchi in [11]. In §3 we prove that if Λ is a Galois algebra over R with group G then, for each σ in G, $c_\sigma = J_\sigma J_{\sigma^{-1}}$ is an idempotent ideal of the center of Λ which is generated by an idempotent element. As corollary to this theorem, we reduce the following Harrison-DeMeyer's theorems. If Λ is a Galois algebra over R with group G and if the center C of Λ is indecomposable then Λ is a Galois algebra over C and C is a Galois algebra over R. If Λ is a Galois algebra over R with cyclic group G, then Λ is commutative.

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Throughout this note we assume that every ring has an identity element.

1. Definitions and Preliminary results.

Let Λ be a ring, G a finite group of ring automorphisms of Λ, and let $\Delta = \Delta(\Lambda, G) = \sum_{\sigma \in G} \oplus \Lambda U_\sigma$ be the crossed product of Λ and G with trivial factor set, i.e. $\{U_\sigma\}$ is a Λ-free basis of Δ and $U_\sigma U_\tau = U_{\sigma\tau}$, $U_\sigma\lambda = \sigma(\lambda)U_\sigma$ for $\lambda\in\Lambda$. We let Λ^G denote the totality of elements of Λ which are left invariant by G. For λ in Λ, we let λ_r (or λ_l) denote the right (or left) multiplication by λ on Λ and Γ_r (or Γ_l) denote the totality of λ_r (or λ_l) with $\lambda\in\Gamma$. In [6] we generalized the notion of Galois extension defined first by M. Auslander and O. Goldman [1] to the non commutative case. Our definition of Galois extension is as follows. A ring Λ is called a Galois extension of a ring Γ relative to G, if the following conditions are satisfied;

I. $\Gamma = \Lambda^G$,

II. Δ is finitely generated projective Γ_r-module.

III. $\delta: \Delta(\Lambda, G)\to \mathrm{Hom}_{\Gamma_r}(\Lambda, \Lambda)$ is and isomorphism where δ is defined by $\delta(\lambda U_\sigma)=\lambda_r\sigma$ for $\lambda\in\Lambda$.

If Λ is an algebra over a commutative ring R, and if Λ is a Galois extension of R relative to G, then we call Λ a *Galois algebra* over R with group G. If Λ is a Galois algebra over R with Group G, and if R is the center of Λ, then we call Λ *central Galois algebra* over R with group G. In [3], Chase, Harrison and Rosenberg gave another definition of Galois extension for the case of commutative ring which is equivalent to the definition by Auslander and Goldman [1]. We consider the following Deffinition the case of non commutative ring; Λ is called a Galois extension of Γ with group G, if the following conditions are satisfied;

I′. $\Gamma = \mathrm{Tr}(\Lambda)$, where $\mathrm{Tr}(x)=\sum_{\sigma\in G}\sigma(x)$ for $x\in\Lambda$,

II′. there exist $x_1, x_2, \cdots, x_s$ and $y_1, y_2, \cdots, y_s$ in Λ such that for $\sigma\in G$

$$\sum_{i=1}^{s} x_i\sigma(y_i)=\begin{cases}1, & \text{if } \sigma=1\\ 0, & \text{if } \sigma\neq 1.\end{cases}$$

In [7], we have seen that if Λ is an algebra over R then "Galois extension Λ of R" in our sense and that in their sense are equivalent.

Now, let Λ be an arbitrary ring, and C the center of Λ. We generalize the argument for J_σ in [10], § 3. For any ring automorphism σ of Λ, let $J_\sigma=\{a\in\Lambda \mid \sigma(x)a=ax \text{ for every } x\in\Lambda\}$. Then J_σ is a C-submodule of Λ and we may show easily the following properties. If σ and τ are ring automorphisms of Λ, then

1) $J_\sigma J_\tau\subset J_{\sigma\tau}$,

2) $\tau(J_\sigma)=J_{\tau\sigma\tau^{-1}}$,

3) $J_\sigma\Lambda=\Lambda J_\sigma$ is a two sided ideal of Λ,

4) for the identity mapping 1 of Λ, $J_1=C$.

For a central separable algebra Λ over C, using the result in Rosenberg and Zelinsky [10], we have

Lemma 1. *Let Λ be a central separable algebra over C and σ a ring automorphism of Λ which leaves C element wise fixed. Then we have*

1) $J_\sigma \Lambda = \Lambda J_\sigma = \Lambda$,
2) $J_\sigma J_{\sigma^{-1}} = J_{\sigma^{-1}} J_\sigma = C$,
3) *σ is an inner automorphism of Λ if and only if there is an element x in J_σ such that $xC = J_\sigma$ (cf. Lemma in 5 in [10]),*
4) *if C is a semi-local ring then σ is an inner automorphism of Λ.*

Proof. 1). By Theorem 3. 1 in [1], the homomorphism $g: \Lambda \underset{C}{\otimes} J_\sigma \to \Lambda$, defined by $g(\lambda \otimes a) = \lambda a$ for $\lambda \in \Lambda$ and $a \in J_\sigma$, is an isomorphism as C-module. Therefore, we have $\Lambda J_\sigma = \Lambda$. 2). $\mathfrak{c}_\sigma = J_\sigma J_{\sigma^{-1}}$ is an ideal of C, and $\mathfrak{c}_\sigma \Lambda = J_\sigma J_{\sigma^{-1}} \Lambda = J_\sigma \Lambda = \Lambda$. Since Λ is central separable, $\mathfrak{c}_\sigma = \mathfrak{c}_\sigma \Lambda \cap C = C$. 3) is calear by 1). 4). We suppose that C is semi-local. Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ be the maximal ideals of C. We first show that there is an element x in J_σ such that $x \notin \mathfrak{p}_i J_\sigma$ for every maximal ideal $\mathfrak{p}_i$ of C. Since $J_\sigma J_{\sigma^{-1}} = C$, we have $\mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_{i-1} \mathfrak{p}_{i+1} \cdots \mathfrak{p}_r J_\sigma \not\subset \mathfrak{p}_i J_\sigma$ for $i = 1, 2, \cdots, r$. For each i, there is an element x_i in J_σ such that

$$x_i \in \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_{i-1} \mathfrak{p}_{i+1} \cdots \mathfrak{p}_r J_\sigma \text{ and } x_i \notin \mathfrak{p}_i J_\sigma .$$

Put $x = \sum_{i=1}^{r} x_i$. Then x is contained in J_σ, but is not contained in $\mathfrak{p}_i J_\sigma$ for every $\mathfrak{p}_1$. Now, we shall show $xC = J_\sigma$. Since, by Proposition 4 in [10], J_σ is a finitely generated projective and rank one C-module, we have $[J_\sigma \underset{C}{\otimes} C/\mathfrak{p}_i : C/\mathfrak{p}_i] = 1$ for every $\mathfrak{p}_i$. Since $xC \not\subset \mathfrak{p}_i J_\sigma$, $J_\sigma = xC + \mathfrak{p}_i J_\sigma$ for $i = 1, 2, \cdots, r$. By Nakayama's Lemma, we have $J_\sigma = xC$. By 3), this completes the proof.

2. Structure theorem.

Proposition 1. *If Λ is a Galois extension of Γ relative to G, then*

$$V_\Lambda(\Gamma) = \sum_{\sigma \in G} \oplus J_\sigma$$

where $V_\Lambda(\Gamma)$ is the commutor ring of Γ in Λ.

Proof. From our definition of Galois extension, we may identify $\Delta = \Delta(\Lambda, G) = \sum_{\sigma \in G} \oplus \Lambda U_\sigma$ and $\mathrm{Hom}_{\Gamma_r}(\Lambda, \Lambda)$ by the isomorphism δ. Then we may denote $\Delta(\Lambda, G) = \sum_{\sigma \in G} \oplus \Lambda_l \sigma$. It follows that $V_\Delta(\Lambda) = V_{\mathrm{Hom}_{\Gamma_r}(\Lambda, \Lambda)}(\Lambda_l) = \mathrm{Hom}_{\Lambda_l \Gamma_r}(\Lambda, \Lambda) = (V_\Lambda(\Gamma))_r$. On the other hand, an easy computation shows $V_\Delta(\Lambda) = \sum_{\sigma \in G} \oplus J_{\sigma^{-1}} U_\sigma = (\sum_{\sigma \in G} \oplus J_{\sigma^{-1}})_r$. Therefore, we have $V_\Lambda(\Gamma) = \sum_{\sigma \in G} \oplus J_\sigma$. From this proposition we have immediately

Theorem 1. *Let Λ be a Galois algebra over a commutative ring R with group G. Then we have $\Lambda=\sum_{\sigma\in G}\oplus J_\sigma$.*

Proposition 2. *Let Λ be a Galois algebra over R with group G, G the center of Λ, and let $\mathfrak{c}_\sigma=J_\sigma\Lambda\cap C$ for each σ in G. Then $\mathfrak{c}_\sigma$ is an ideal of C and $\mathfrak{c}_\sigma\Lambda=J_\sigma\Lambda$. For σ, τ in G, we have the following properties;*

1) $\mathfrak{c}_\sigma=0$ *if and only if* $J_\sigma=0$,
2) $J_\sigma J_\tau=\mathfrak{c}_\sigma J_{\sigma\tau}=\mathfrak{c}_\tau J_{\sigma\tau}$,
3) $J_\sigma J_{\sigma^{-1}}=J_{\sigma^{-1}}J_\sigma=\mathfrak{c}_\sigma$, *therefore* $\mathfrak{c}_\sigma=\mathfrak{c}_{\sigma^{-1}}$,
4) $\mathfrak{c}_\sigma J_\sigma=J_\sigma$,
5) $\mathfrak{c}_\sigma^2=\mathfrak{c}_\sigma$,
6) $\mathfrak{c}_\sigma=C$ *if and only if σ leaves each element of the center C invariant, i.e.* $\sigma|C=1$,
7) *if* $\sigma|C=1$ *or* $\tau|C=1$ *then* $J_\sigma J_\tau=J_{\sigma\tau}$.

Proof. Let Λ be a Galois algebra over R with group G. Then Λ is separable over R (cf. Proposition 4 in [6]), therefore Λ is central separable over C and C is separable over R. From the central separability of Λ, we obtain $\mathfrak{c}_\sigma\Lambda=J_\sigma\Lambda$ for $\mathfrak{c}_\sigma=C\cap J_\sigma\Lambda$. Since $\Lambda=\sum_{\sigma\in G}\oplus J_\sigma$, we have $\mathfrak{c}_\sigma\Lambda=\sum_{\tau\in G}\oplus\mathfrak{c}_\sigma J_\tau$ and $J_\sigma\Lambda=\sum_{\tau\in G}J_\sigma J_\tau$. Since $J_\sigma J_\tau\subset J_{\sigma\tau}$ and $\mathfrak{c}_\sigma\Lambda=J_\sigma\Lambda$, we have $J_\sigma\Lambda=\sum_{\tau\in G}\oplus J_\sigma J_\tau$ and $\mathfrak{c}_\sigma J_{\sigma\tau}=J_\sigma J_\tau$. Similarly, $\mathfrak{c}_\tau J_{\sigma\tau}=J_\sigma J_\tau$. In paticular, taking $\tau=\sigma^{-1}$ or $\tau=1$, we have $J_\sigma J_{\sigma^{-1}}=\mathfrak{c}_\sigma=\mathfrak{c}_{\sigma^{-1}}$ or $\mathfrak{c}_\sigma J_\sigma=J_\sigma$, and $\mathfrak{c}_\sigma^2=\mathfrak{c}_\sigma J_\sigma J_{\sigma^{-1}}=J_\sigma J_{\sigma^{-1}}=\mathfrak{c}_\sigma$. If σ is an automorphism of the central separable algebra Λ over C which leaves C element wise fixed, then by Lemma 1 we have $\Lambda J_\sigma=\Lambda$, therefore $\mathfrak{c}_\sigma=C\cap\Lambda J_\sigma=C$. Conversely, if $\mathfrak{c}_\sigma=C$, then by definition of J_σ we have $(\sigma(x)-x)a=0$ for every x in C and a in J_σ. Since $\mathfrak{c}_\sigma=J_\sigma J_{\sigma^{-1}}$, $(\sigma(x)-x)C=(\sigma(x)-x)\mathfrak{c}_\sigma=0$ for every x in C. Therefore $\sigma(x)=x$ for every x C. If $\sigma|C=1$ then by 6) and 2) $J_\sigma J_\tau=\mathfrak{c}_\sigma J_{\sigma\tau}=J_{\sigma\tau}$.

From Theorem 1 and Proposition 2, we obtain easily the following

Corollary 1. *If Λ is a central Galois algebra over C with group G, then $\Lambda=\sum_{\tau\in G}\oplus J_\sigma$, $J_\sigma J_\tau=J_{\sigma\tau}$ and $\mathfrak{c}_\sigma=J_\sigma J_{\sigma^{-1}}=C$ for every σ in G.*

Corollary 2. (De Meyer and Takeuchi) *If Λ is a central Galois algebra over C with group G, and if every element σ of G is an inner automorphism of Λ associated with a unit u_σ in Λ, then $J_\sigma=Cu_\sigma$ and $\Lambda=\sum_{\sigma\in G}\oplus Cu_\sigma$.*

Proposition 3. *Let Λ be a Galois algebra over R with group G, C the center of Λ, and $H=\{\sigma\in G\,|\,\sigma(x)=x$ for every x in $C\}$. Then Λ is a central Galois algebra over C with group H if and only if $J_\tau=0$ for every τ in G such that $\tau\notin H$, and then C is a Galois algebra over R with group G/H.*

Proof. Let Λ be Galois algebra over R with G. Then $\Lambda=\sum_{\sigma\in G}\oplus J_\sigma$. If Λ is acentral Galois algebra over C with group H, then $\Lambda^H=C$ and C is a Galois extension of R with group G/H (cf. proof of Theorem 3. 1 in [3], or Theorem 1 in [11]). If Λ is a central Galois algebra over C with group H, then, by Theorem 1, $\Lambda=\sum_{\sigma\in H}\oplus J_\sigma$, therefore $J_\tau=0$ for $\tau\notin H$. Conversely, if $J_\tau=0$ for every $\tau\notin H$, then $\Lambda=\sum_{\sigma\in H}\oplus J_\sigma$. Since by Theorem 3 in [6] Λ is a Galois extension of Λ^H relative to H, by Proposition 1 we have $V_\Lambda(\Lambda^H)=\sum_{\sigma\in H}\oplus J_\sigma$. therefore $\Lambda=V_\Lambda(\Lambda^H)$, and $\Lambda^H\subset C$. Since $C\subset\Lambda^H$, we have $\Lambda^H=C$, thus Λ is a central Galois algebra over. C. This completes the proof.

Proposition 4. *Let Λ be a Galois algebra over R with group G, and let $N(\sigma)=\{\tau\in G\,|\,\tau\sigma=\sigma\tau\}$ for each $\sigma\in G$, then we have the following statements*;

1) *for $J_\sigma\neq 0$ and J_τ, $J_\sigma=J_\tau$ if and only if $\sigma=\tau$,*
2) *each element of $N(\sigma)$ induces an automorpxism of C-module J_σ, and if $J_\sigma\neq 0$ then τ is contained in $N(\sigma)$ if and only if $\tau(J_\sigma)=J_\sigma$,*
3) *for $\sigma\neq 1$ in G and x in J_σ, if $\tau(x)=x$ for every τ in $N(\sigma)$, then $x=0$.*
4) *for $\sigma\neq 1$ and for every x in J_σ, $\sum_{\tau\in N(\sigma)}\tau(x)=0$,*
5) *for $\sigma\neq 1$ in G and for every x in J_σ, $\mathrm{Tr}(x)=0$.*

Proof. 1) and 2) are clear. To prove 3), let $\tau_1, \tau_2, \cdots, \tau_r$ be the right coset representatives of G modulo $N(\sigma)$. If x in J_σ satisfies $\tau(x)=x$ for every τ in $N(\sigma)$, then we put $y=\sum_{i=1}^r\tau_i(x)$. Since $\nu(y)=\sum_i\nu\tau_i(x)=\sum_i\tau_i(x)=y$ for every ν in G, y is contained in $\Lambda^G=R$, and therefore $y\in J_1=C$. On the other hand, $\tau_i(x)\in\tau_i(J_\sigma)=J_{\tau_i\sigma\tau_i^{-1}}\neq J_1$, and by 2) $\tau_i(J_\sigma)\neq\tau_i(J_\sigma)$ if $i\neq j$. Since Λ is a direct sum of J_σ for $\sigma\in G$, we have $\tau_i(x)=0$ $i=1, 2, \cdots, r$, and therefore $x=0$. 4) is easily proved by 3). Now, for every element x in J_σ, $\mathrm{Tr}(x)=\sum_{\sigma\in G}\sigma(x)=\sum_{i=1}^r\tau_i(\sum_{\nu\in N(\sigma)}\nu(x))=0$, therefore we have 5).

Using this proposition we have

Proposition 5. *Let Λ be a Galois algebra over R with group G, C the center of Λ, and let $H=\{\sigma\in G\,|\,\sigma|C=1\}$. Then the order $|H|$ of H is a unit in R.*

Proof. By 5) in Proposition 4, $\mathrm{Tr}(J_\sigma)=0$ for $\sigma\neq 1$ in G. Therefore $\mathrm{Tr}(\Lambda)=\mathrm{Tr}(\sum_{\sigma\in G}J_\sigma)=\sum_{\sigma\in G}\mathrm{Tr}(J_\sigma)=\mathrm{Tr}(J_1)=\mathrm{Tr}(C)$, and $R=\mathrm{Tr}(C)$. Then there is an element a in C such that $\mathrm{Tr}(a)=1$. Let $G=\sigma_1 H+\cdots+\sigma_r H$ be the right decomposition of G modulo H. We have $\mathrm{Tr}(a)=\sum_{\sigma\in G}\sigma(a)=|H|(\sum_{i=1}^{r}\sigma_i(a))=1$. However, $\sum_{i=1}^{r}\sigma_i(a)$ is contained in $\Lambda^G=R$. Therefore $|H|$ is a unit in R.

Corollary 3. (De Meyer and Takeuchi) *Let Λ be a central Galois algebra over C with group G. Then the order $|G|$ of G is a unit in C.*

Corollary 4. *Let Λ be a central Galois algebra over C with group R. Then Λ is a strongly separable algebra over C in the sense of* [9].

Proof. By Theorem 1 in [9], Λ is a strongly separable algebra over C if and only if $\Lambda/\mathfrak{p}\Lambda$ is a strongly separable algebra over $C/\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of C[1]. For a maximal ideal $\mathfrak{p}$ of C, $\Lambda/\mathfrak{p}\Lambda$ is a central simple algebra with minimum condition over $C/\mathfrak{p}$, and $[\Lambda/\mathfrak{p}\Lambda : C/\mathfrak{p}]=[\Lambda\otimes_C C_\mathfrak{p} : C_\mathfrak{p}]=|G|$. Therefore the degree of the central simple algebra $\Lambda/\mathfrak{p}\Lambda$ is a unit in $C/\mathfrak{p}$. Thus the degree of $\Lambda/\mathfrak{p}\Lambda$ is prime to the characteristic of $C/\mathfrak{p}$. By definition of strongly separbility in [8], $\Lambda/\mathfrak{p}\Lambda$ is a strongly separable algebra over $C/\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of C, which completes the proof.

3. Main theorem.

Proposition 6. *Let Λ be a Galois algebra over R with group G, C the center of Λ, and $\mathfrak{c}_\sigma$ the ideal defined in Proposition 2 for each $\sigma\in G$. Then we have the following statemements;*

1) $\mathfrak{c}_\sigma\mathfrak{c}_\tau=\mathfrak{c}_\sigma\mathfrak{c}_{\sigma\tau}=\mathfrak{c}_\tau\mathfrak{c}_{\sigma\tau}$,
2) $\mathfrak{c}_\sigma\subset\mathfrak{c}_{\sigma^i}$ *for any integer i, therefore* $\mathfrak{c}_\sigma\neq 0$ *implies* $\mathfrak{c}_{\sigma^i}\neq 0$,
3) *for* $\tau\in G$, $\tau(\mathfrak{c}_\sigma)=\mathfrak{c}_{\tau\sigma\tau^{-1}}$,
4) *for* $H=\{\sigma\in G\,|\,\sigma|C=1\}$, *if* $\sigma\equiv\tau \pmod{H}$ *then* $\mathfrak{c}_\sigma=\mathfrak{c}_\tau$.

Proof. 1) and 3) are clear by Proposition 2, and 2) and 4) are easily proved by 1).

Lemma 2. *Let C be a commutative algebra over R, and $\mathfrak{c}$ an ideal of C such that $\mathfrak{c}$ is idempotent and finitely generated over R. Then $\mathfrak{c}$ is generated by an idempotent element iu C.*[2]

Proof. Let $\mathfrak{c}=\sum_{i=1}^{r}Rx_i$. Since $\mathfrak{c}$ idempotent, $\mathfrak{c}^2=\mathfrak{c}=\sum_{i=1}^{r}\mathfrak{c}x_i$. Then, we

1) Let A be a central separable algebra over C. Then A is strongly separable over C if and only if $A/\mathfrak{p}A$ is strongly separable over $C/\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of C. (Cf. proof of Theorem 1 in [9].)

2) This lemma suggested to me by M. Harada. I express here my thanks to him,

have $x_i=\sum_j a_{ij}x_j$ with some a_{ij} in $\mathfrak{c}$. Let be the deteriminant of the matrix $E-(a_{ij})$, where E is the unit matrix. Then, we can easily see that $d=1-e$ with some e in $\mathfrak{c}$ and $xd=0$ for every x in $\mathfrak{c}$. Therefore, we have $e^2=e$ and $ex=x$ for every $\mathfrak{c}$, thus $\mathfrak{c}=eC$.

From this lemma, we have the following main theorem;

Theorem 2. *Let Λ be a Galois algebra over R with group G, C, the center of Λ. Then $\mathfrak{c}_\sigma=J_\sigma J_{\sigma^{-1}}$ is generated by an idempotent element e_σ in C for each σ in G.*

As a corollary of Theorem 2, we have

Theorem 3. (Harrison, De Meyer) *Let Λ be a Galois algebra over R with group G, and let C be the center of Λ. If C is indecomposable, then Λ is a central Galois algebra over C with group H, and C is a Galois algebra over R with group G/H, where $H=\{\sigma\in G \mid \sigma \mid C=1\}$.*

Proof. Since the idempotent elements in C are only 0 and 1, for each $\sigma\in G$, by Theorem 2, $\mathfrak{c}_\sigma$ is either 0 or C. Therefore, if $\tau\notin H$ then $J_\tau=0$. By Proposition 3, the proof is completed.

Proposition 7. *Let Λ be a Galois algebra over R with group G, and let $\mathfrak{a}_\sigma=\{x\in C \mid x\mathfrak{c}_\sigma=0\}$. Then we have the following statements;*

1) $\mathfrak{a}_\sigma=\mathfrak{a}_{\sigma^{-1}}\supset\mathfrak{a}_{\sigma^i}$ *for any integer i,*
2) $\mathfrak{a}_\sigma\Lambda=\{x\in\Lambda \mid xJ_\sigma=0\}$,
3) $\mathfrak{a}_\sigma\Lambda\cap J_\tau=\mathfrak{a}_\sigma J_\tau$,
4) *for $x\in J_\sigma$, $x=0$ if and only if $xJ_\sigma=0$ (or $x\mathfrak{c}_\sigma=0$).*
5) *if $x\in J_\sigma$ and $xJ_{\sigma^i}=0$ for some integer i, then $x=0$.*

Proof. 1) and 2) are clear by 4) in Proposition 6. Since $\Lambda=\sum_{\sigma\in G}\oplus J_\sigma$, we have $\mathfrak{a}_\sigma\Lambda=\sum_{\tau\in G}\oplus\mathfrak{a}_\sigma J_\tau$, therefore $\mathfrak{a}_\sigma\Lambda\cap J_\tau=\mathfrak{a}_\sigma J_\tau$. In particular, taking $\sigma=\tau$, we have $\mathfrak{a}_\sigma\Lambda\cap J_\sigma=\mathfrak{a}_\sigma J_\sigma=\mathfrak{a}_\sigma\mathfrak{c}_\sigma J_\sigma=0$, which proves 4). 5) is clear by 1).

For a Galois algera with abelian group, we have the following proposition with a weaker assumption than Theorem 3.

Proposition 8. *Let Λ be a Galois algebra over R with abelian group G. Then Λ is a strongly separable algebra over R. If R is indecomposable, then Λ is a central Galois algebra over the center C and the center C is a Galois algebra over R.*

Proof. We prove first the second part. Since G is abelian, for every τ in G, $\tau(\mathfrak{c}_\sigma)=\mathfrak{c}_{\tau\sigma\tau^{-1}}=\mathfrak{c}_\sigma$. If $\mathfrak{c}_\sigma\neq0$, then there is a non zero idempotent element e_σ in C such that $\mathfrak{c}_\sigma=e_\sigma C$, and for every τ in G, $\tau(e_\sigma)=e_\sigma$, Therefore, e_σ is contained in $\Lambda^G=R$. It must be $e_\sigma=1$. Thereforec $\mathfrak{c}_\sigma=C$. By Proposition 3, this completes the proof of the second part. By Theorem 1 in [9] and the second part of this proposition, we can prove the first part; for every maximal ideal $\mathfrak{p}$ of R, $\Lambda\underset{R}{\otimes}R\mathfrak{p}$, is strongly separable over $R\mathfrak{p}$, therefore Λ is strongly seprarable over R[3].

Proposition 9. *Let Λ be a Galois algebra over R with group G. If Λ is a strongly separable algebra over R, then we have the following statements;*

1) *for each $\sigma\in G$, $\sigma|J_\sigma=1$, i.e. $\sigma(x)=x$ for all x in J_σ,*
2) *for each integer i, if $a\in J_\sigma$ and $b\in J_{\sigma^i}$ then $ab=ba$.*

Proof. If Λ is a strongly separable algebra over R, then by Proposition 1 in [9], $\Lambda=C\oplus[\Lambda, \Lambda]$ where C is the center of Λ and $[\Lambda, \Lambda]$ is a C-submodule of Λ generated by $xy-yx$ for every x, y in Λ. For any x, y in J_σ and z in $J_{\sigma^{-1}}$, it follows that $\sigma(x)yz=yxz=xzy$. Since zy and yz are in $J_\sigma J_{\sigma^{-1}}=J_{\sigma^{-1}}J_\sigma=\mathfrak{c}_\sigma\subset C$, we have $zy-yz\in[\Lambda, \Lambda]\cap C=0$, and therefore $zy=yz$. Thus $\sigma(x)yz=xyz$, and $(\sigma(x)-x)yz=0$. Therefore, $(\sigma(x)-x)J_\sigma J_{\sigma^{-1}}=(\sigma(x)-x)\mathfrak{c}_\sigma=0$, and hence $\sigma(x)=x$. Thus we have 1). By 1), we obtain the statement 2); for every $a\in J_\sigma$ and $b\in J_{\sigma^i}$, $ab=\sigma^i(a)b=ba$.

We now obtain the following Harrison- De Meyers, Theorem.

Theorem 4. (Harrison-De Meyer) *Let Λ be a Galois algebra over R with cyclic group G. Then Λ is commutative.*

Proof. Since G is abelian, by Proposition 8, Λ is strongly separable over R. Now, suppose Λ is non commutative. Let $G=(\sigma)$. Since $\Lambda=\sum_i\oplus J_{\sigma^i}$, there is $J_{\sigma^i}\neq 0$. Let $k=\min(i>0|J_{\sigma^i}\neq 0)$. If $k\not| i$ then, by 1) in Proposition 6, $\mathfrak{c}_{\sigma^k}\mathfrak{c}_{\sigma^i}=\mathfrak{c}_{\sigma^k}\mathfrak{c}_{\sigma^{i-nk}}=0$ where n is an integer such that $0<i-nk<k$. Therefore, if $k\not| i$ then $J_{\sigma^i}J_{\sigma^k}=J_{\sigma^k}J_{\sigma^i}=0$. If $k|i$, i.e. $i=kr$, then by 2) in Proposition 9, $ab=ba$ for every $a\in J_{\sigma^k}$ and $b\in J_{\sigma^{kr}}=J_{\sigma^i}$. Thus $J_{\sigma^k}\neq 0$ is contained in the center $C=J_1$, this is a contradiction. Therefore Λ is commutative.

Now, let Λ be a Galois algebra over R with group G, and C the center of Λ. Then for each $\sigma\in G$, there is an idempotent element e_σ such that $e_\sigma C=\mathfrak{c}_\sigma$. Let $e_\sigma=\sum_{i-1}^r a_ib_i$, $a_1\in J_\sigma$, $b_i\in J_{\sigma^{-1}}$. Then we have

Proposition 10. *Under the above assumption, $e'_\sigma=\sum_{i=1}^r b_ia_i$ is an element in $\mathfrak{c}_\sigma$, and satisfies the following conditions;*

3) By Theorem 1 in [9], if Λ is a separable algebra over R, then Λ is strongly separable over R if and only if $\Lambda\otimes_R R\mathfrak{p}$ is strongly separable over $R\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of R.

1) *$\sigma(x)=e'_\sigma x$ for every $x\in J_\sigma$,*

2) *$e'^2_\sigma=e_\sigma$ and $e'_\sigma C=\mathfrak{c}_\sigma$, therefore $\sigma^2|J_\sigma=1$.*

Proof. Since $\sigma(x)\in J_\sigma$ for every $x\in J_\sigma$, we have $\sigma(x)=e_\sigma\sigma(x)=\sum_{i=1}^{r}a_ib_i\sigma(x)=\sum_{i=1}^{r}b_i\sigma(x)a_i=\sum_{i=1}^{r}b_ia_ix=e'_\sigma x$ for $x\in J_\sigma$. Now, $e'^2_\sigma=\sum_{ij}b_i(a_ib_j)a_j=\sum_{ij}(a_ib_j)b_ia_j=\sum_{ij}a_ib_j(b_ia_j)=\sum_{ij}a_i(b_ia_j)b_j=e^2_\sigma=e_\sigma$. It follows that $e'_\sigma C=\mathfrak{c}_\sigma$ and $\sigma^2(x)=\sigma(\sigma(x))=e'_\sigma\sigma(x)=e'^2_\sigma x=e_\sigma x=x$ for all x in J_σ.

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References

[1] M. Auslander and O. Goldman: *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[2] H. Cartan and S. Eilenberg: Homological algebra, Princeton, 1956.

[3] S. U. Chase, D. K. Harrison and A. Rosenberg: Galois theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc. no. 52 (1965).

[4] F. R. De Meyer: *Galois theory in algebra over a commutative ring*, to appear in Ill. J. Math.

[5] ――――――: *Some note on the general Galois theory of ring*, Osaka J. Math. **2** (1965) 117–127.

[6] T. Kanzaki: *On commutor ring and Galois theory of separable algebras*, Osaka J. Math. **1** (1964) 103–115.

[7] ――――――: *On Galois extension of rings*, Nagoya Math. J. (1965)

[8] ――――――: *A type of separable algebra*, J. Math. Osaka City Univ. **13** (1962) 39–43.

[9] ――――――: *Special type of separable algebra over a commutative ring*, Proc. Japan Acad. **40** (1964) 781–786.

[10] A. Rosenberg and D. Zelinsky: *Automorphisms of separable algebras*, Pacific. J. Math. **11** (1957) 1109–1118.

[11] Y. Takeuchi: *On Galois extension over a commutative ring*, Osaka J. Math. **2** (1965) 137–145.