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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, III

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In the previous paper we defined a function $N(z, p)$ and ideal boundary points and studied some properties of superharmonic functions in $\bar{R}$, but the mass distributions are only slightly discussed. In the present article, we rewrite pages from 174 to 176 of II[1] in more precise form and continue the previous work. We use the same notations and definitions as in II.

Theorem 1. *Let p be a minimal point and $v(p)$ be a neighbourhood of p. Let $V^M(z)$ be a harmonic function in $v(p)$ such that $V^M(z)=$ min $(M, N(z, p))$ on $\partial v(p)$ and $V^M(z)$ has M.D.I. over $v(p)$. Put $V(z)= \lim_{M=M'} V^M(z) : M'=\sup N(z,p)$. Then $N(z,p)-V(z)=N'(z,p)>0$ and $N'(z,p)$ has the same properties as $N(z,p)$.*

Suppose $\sup N(z,p)=\infty$, i.e. p is of capacity zero. Assume $V(z)\equiv N(z, p)$. Then $N(z, p)=\int\limits_{\bar{R}-v(p)} N(z, q)\, d\mu(q)$. Since $N(z, p)$ is harmonic in R, $V(z)=\int\limits_{B-v(p)} N(z, q)\, d\mu(q)$. If μ is a point mass, $N(z,p)=N(z,q) : q\notin v(p)$, which implies $p=q\notin v(p)$. This is a contradiction. Hence μ is not a point mass. Therefore there exist two positive mass distributions μ_1 and μ_2 such that $\mu=\mu_1+\mu_2$ and both $V_1(z)=\int N(z, q)d\mu_1(q)$ and $V_2(z)=\int N(z, q)d\mu_2(q)$ are not multiples of $N(z, p)$. Because, if every μ_i presents a multiple of $N(z,p)$ and whose kernel k_i tends to a point $q\notin v(p)$. Then $\lim_{i=\infty} \frac{\mu_i}{\text{total mass of } \mu_i}$ represents $N(z, p)=N(z, q) : q\notin v(p)$. This is also a contradiction. Therefore $N(z,p)-V_1(z)$ (>0) and $V_1(z)$ (>0) are superharmonic in $\bar{R}$, whence $N(z,p)$ is not minimal. Hence $V(z)<N(z,p)$. Next we show that $V(z)$ *has no mass at p* in any canonical mass distribution[2]. To the contrary, suppose $V(z)$ has a positive mass at p. Then

1) Z. Kuramochi: Mass distributions on the ideal boundaries, II. Osaka Math. Jour., 8, 1956.
2) At present we cannot prove the uniqueness of canonical mass distributions.

$V(z)=KN(z,p)+U(z)$, where $0<K<1$ and $U(z)$ is superharmonic in $\bar{R}$. Then $U(z)=(1-K)N(z,p)$ on $\partial v(p)$ and superharmonic, whence $V_{R-v(p)}(z)\leqq V(z)$. Now $V^M(z)$ has M.D.I. over $v(p)$ with value $V^M(z)=\min(M, N(z,p))$ on $\partial v(p)$, hence

$$V(z)\geqq [V^M_{R-v(p)}(z)]=V^M(z)$$

and

$$V(z)\geqq V_{R-v(p)}(z)=\lim_{M=\infty} V^M_{R-v(p)}(z)\geqq \lim_{M=\infty} V^M(z)=V(z)\,,$$

whence

$$V_{R-v(p)}(z)\equiv V(z)=KN_{R-v(p)}(z,p)+U_{R-v(p)}(z) \qquad \text{in} \quad v(p)$$

i.e.

$$V(z)=KV(z) \qquad\qquad +U_{R-v(p)}(z) \qquad \text{in} \quad v(p)\,.$$

But $V(z)<N(z,p)$ and $U_{R-v(p)}(z)\leqq U(z)$. Hence

$$V(z)=KN(z,p)+U(z)>KV(z)+U_{R-v(p)}(z)=V(z)\,.$$

This is a contradiction. Hence $V(z)$ *has no mass at* p.

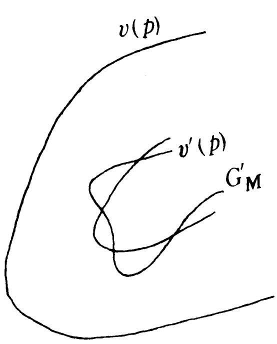

Fig. 1

Put $N'(z,p)=N(z,p)-V(z)$. Then $N'(z,p)=0$ on $\partial v(p)$. Let $G'_M=E[z\in R: N'(z,p)\geqq M]$ and let $v'(p)$ be a neighbourhood of p such that $v'(p)\subset v(p)$. Then

$$\begin{aligned} N(z,p) &\leqq N_{v'(p)\cap G'_M}(z,p)+N_{v'(p)\cap CG'_M}(z,q) \\ &= N_{v'(p)\cap G'_M}(z,p)+V_{v'(p)\cap CG'_M}(z) \\ &\quad +N_{v'(p)\cap CG'_M}(z,p)\,. \end{aligned}$$

In page 158 (II), we proved that if p is of capacity zero, $V(z)-V_p(z)$ is superharmonic. If $V_p(z)>0$, then $V(z)$ has a positive mass at p. This is a contradiction. *Hence* $V_p(z)=0$.

Since $N'(z,p)=M$ on $\partial(v(p)\cap CG'_M)$ and p is of capacity zero,

$$\lim_{v'(p)\to p} N_{v'(p)\cap CG'_M}(z,p)\leqq \lim_{v'(p)\to p}(M\omega(v'(p),z)+V_{v'(p)}(z))=0\,,$$

where $\omega(v'(p),z)$ is C.P. (equilibrium potential) of $v'(p)$.

Hence $\quad N(z,p)\leqq \lim_{v'(p)\to p} N_{v'(p)\cap G'_M}(z,p)\leqq N(z,p)\,,\quad$ whence

$$N(z,p)=N_{v'(p)\cap G'_M}(z,p)\leqq N_{v'(p)}(z,p)\leqq N(z,p)\,, \tag{1}$$

$$N(z,p)=N_{v'(p)\cap G'_M}(z,p)\leqq N_{G'_M}(z,p)\leqq N(z,p)\,. \tag{2}$$

Suppose $S\subset v(p)$ and let ${}^*V^M_S(z)$ be a harmonic function in $v(p)-S$ such that ${}^*V_S^M(z)=\min(M,\ V(z))$ on $\partial S+\partial v(p)$ and has M.D.I. over $v(p)-S$. Then by the definition of $V(z)$, we have $V(z)=\lim_{M=\infty} {}^*V_S^M(z)$ and, for $N(z, p)$ we have by (1) and (2) the following

$$N_{v'(p)\cap G'_M}(z, p) = N(z, p) \quad \text{and} \quad \lim_{M=\infty} N'^M_{v'(p)\cap G'_M}(z, p) = N'(z, p)\,,$$

where $N'^M_{v'(p)\cap G'_M}(z, p)$ is a harmonic function in $v(p)-(v'(p)\cap G'_M)$ such that $N'^M_{v'(p)\cap G'_M}(z, p) = \min(M,\ N'(z, p))$ on $\partial v(p)+\partial(v'(p)\cap G'_M)$ and $N'^M_{v'(p)\cap G'_M}(z, p)$ has M.D.I. over $v(p)-(v'(p)\cap G'_M)$. Hence we have the following

Property 1. $N'_{v'(p)\cap G'_M}(z, p)=N'_{v'(p)}(z, p)=N'_{G'_M}(z, p)=N'(z, p)$.

As in page 153 (II) $N'_{G'_M}(z, p)=\lim_n N'_{G'_M\cap R_n}(z, p)$ and $N'_{G'_M\cap R_n}(z, p)$ $=\lim_m N'_{n,m}(z, p)$, where $N'_{n,m}(z, p)$ is a harmonic function in $R_m-R_0-(G'_M\cap R_n)$ $(m>n)$ such that $N'_{n,m}(z, p)=N'(z, p)$ on $G'_M\cap R_n$, $\frac{\partial}{\partial n}N'_{n,m}(z, p)=0$ on ∂R_m. Let $V_n(z)$ be a harmonic function in $v(p)$ such that $V_n(z)$ $=N_{G'_M\cap R_n}(z, p)$ on $\partial v(p)$ and $V_n(z)$ has M.D.I. over $v(p)$. Then $V_n(z)=\lim_m {}_nV_m(z)$, where ${}_nV_m(z)$ is a harmonic function in $v(p)\cap R_n$ such that ${}_nV_m(z)=V(z)$ on $\partial v(p)\cap R_n$ and $\frac{\partial}{\partial n}{}_nV_m(z)=0$ on $\partial R_m\cap v(p)$.

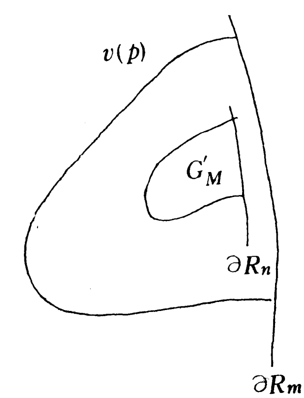

Fig. 2

Since $M=N(z, p)-V(z)=N'(z, p)$ on $\partial G'_M$, ${}_mG'_{M-\varepsilon}=E[z\in R:\ N_{n,m}(z,\ p)-{}_nV_m(z)>M-\varepsilon)\supset$ $(R_n\cap G'_M=E[z\in R:\ N(z,\ p)-V(z)>M])$ for sufficiently large number $m(\varepsilon, n)$ for any given positive number ε and n.

Since $2\pi\geqq \int\limits_{\partial_m G'_{M-\varepsilon}} \frac{\partial}{\partial n}N_{n,m}(z, p)ds=\int\limits_{\partial R_0}\frac{\partial}{\partial n}N_{n,m}(z, p)ds\geqq 2\pi-\varepsilon$ and

$$\int\limits_{\partial_m G'_{M-\varepsilon}}\frac{\partial}{\partial n}{}_nV_m(z)ds=\int\limits_{v(p)\frown\partial R_m}\frac{\partial}{\partial n}{}_nV_m(z)ds=0,$$

$$2\pi\geqq\int\limits_{\partial_m G'_{M-\varepsilon}}\frac{\partial}{\partial n}(N_{n,m}(z, p)-{}_nV_m(z))\geqq 2\pi-\varepsilon\,.$$

Thus $D_{v(p)-{}_mG'_{M-\varepsilon}}(N_{n,m}(z, p)-{}_nV_m(z))\leqq 2\pi(M-\varepsilon)$.
Let $m\to\infty$ and $n\to\infty$. Then $\{N_{n,m}(z, p)-{}_nV_m(z)\}\to\{N(z, p)-V(z)\}$ and let $\varepsilon\to 0$. Then

$$D_{v(p)}(\min(M,\ N'(z, p))\leqq 2\pi M\,.$$

On the other hand, since $\frac{\partial}{\partial n}N_{n,m}(z,p)=\frac{\partial}{\partial n}{}_nV_m(z)=0$ on $\partial R_n\cap v(p)$,

$$D_{v(p)}(\min(M,N_n(z,p)-{}_nV(z))\geqq D_{v(p)-mG'_{M-\varepsilon}}(N_{n,m}(z,p)-{}_nV_m(z))$$
$$\geqq(2\pi-\varepsilon)(M-\varepsilon).$$

Hence $D_{v(p)}(\min(M,N'(z,p))\geqq(2\pi-\varepsilon)(M-\varepsilon)$. Thus we have

Property 2. $D_{v(p)}(\min(M,N'(z,p))=2\pi M,\ M<\infty.$

Now $N(z,p)$ has the same properties 1 and 2 as $N(z,p)$. Therefore we can use $N'(z,p)$ in stead of $N(z,p)$ in $\bar{R}$. As in case of $\sup N(z,p)=\infty$ we have next.

If $\sup N(z,p)<\infty$ and minimal, we have more easily the properties 1 and 2.

Another definition of the value of a superharmonic function at a minimal point.

In the previous paper, we defined the value of a superharmonic function $U(z)$ at a minimal point p by

$$U(p)=\lim_{M\uparrow M'}\frac{1}{2\pi}\int_{C_M}U(z)\frac{\partial}{\partial n}N(z,p)ds,^{3)}$$

where $M'=\sup N(z,p)$ and $C_M=E[z\in R:N(z,p)=M]$ is regular i.e. $\int_{C_M}\frac{\partial}{\partial n}N(z,p)ds=2\pi$.

Above definition is inconverient in the sense as follows: *every regular curve C_M encloses a neighbourhood $v(p)$ but $v(p)$ des not necessarily contain the set $E[z\in R:N(z,p)>M]$ for any large number M.* In the above definition $U(p)$ depends on a larger set than $v(p)$. It is better to define $U(p)$ on the behaviour of $U(z)$ in $v(p)$. Therefore we shall give more useful definition of $U(p)$. $N'(z,p)$ in $v(p)$ in Theorem 1 has the properties. We can prove as in case of $N(z,p)$ that there exists a set E in the interval $(0,M)$ $(M=\sup N'(z,p))$ such that $\operatorname{mes}E=0$ and $E\not\ni\delta$ implies that $C_\delta=E[z\in R:N'(z,p)=\delta]$ is regular i.e. $\int_{C_\delta}\frac{\partial}{\partial n}N'(z,p)ds=2\pi$.

Theorem 2. *Let $U(z)$ be a superharmonic function in $\bar{R}$ and let $v(p)$ be a neighbourhood of p. Let $N'(z,p)$ be the function in Theorem 1.*

3) In II we defined for $N(z,p)$, but the same facts hold for $U(z)$. It is easily seen that $U(p)=\int_{B_1}N(p,q)d\mu(q)$, where $U(z)=\int_{B_1}N(z,p)d\mu(q)$, i.e., μ is a canonical distribution of $U(z)$.

Then $U^*(z)=\lim_{\beta\uparrow M'}\frac{1}{2\pi}\int_{C_\beta}U(z)\frac{\partial}{\partial n}N'(z,p)ds$ *exists and*

$$U(p)=U^*(p)\,,$$

where $M'=\sup N'(z,p)$ *and* C_β *is a regular curve of* $N'(z,p)$.

Lemma 1. *Let* $U(z)$ *be a harmonic function in* $\bar{R}\cap E[z\in R: N(z,p)>\alpha]$ $(=V(p))$ *with continuous boundary value on* $\partial V(p)$. *Then*

$$U(p)=U^*(p)\,.$$

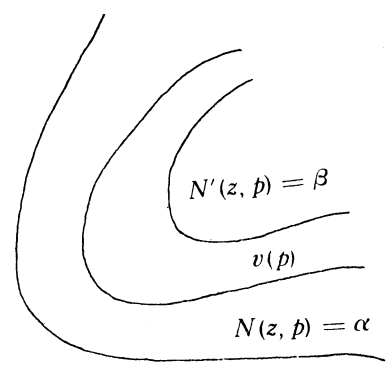

Fig. 3

Suppose $C_\alpha=E[z\in R: N(z,p)=\alpha]$ and $C_\beta=E[z\in R: N'(z,p)=\beta]$ are regular. Let $U_n^S(z)$ be a harmonic function in $R_n\cap V(p)$ such that $U_n^S(z)=\min(U(z),S)$ on $\partial V(p)\cap R_n$ and $\frac{\partial}{\partial n}U_n^S(z)=0$ on $\partial R_n\cap V(p)$. Then $U_n^S(z)\to U^S(z)$ in mean and $U^S(z)\uparrow U(z)$ as $S\to\infty$.

Let $V_n^L(z)$ be a harmonic function in $v(p)\cap R_n$ such that $V_n^L(z)=\min(L,N(z,p))$ on $R_n\cap\partial v(p)$ and $\frac{\partial}{\partial n}V_n^L(z)=0$ on $\partial R_n\cap v(p)$. Then $V_n^L(z)\to V^L(z)$ in mean and $V^L(z)\uparrow V(z)$. Let $N_n^L(z,p)$ be a harmonic function in $(E[z\in R: N(z,p)>\alpha]-E[z\in R: N'(z,p)>\beta]\cap R_n$ such that $N_n^L(z,p)=\alpha$ on $C_\alpha\cap R_n$, $N_n^L(z,p)=\beta+V_n^L(z)$ on $C_\beta\cap R_n$ and $\frac{\partial}{\partial n}N_n^L(z,p)=0$ on $\partial R_n\cap(E[z\in R: N(z,p)>\alpha]-E[z\in R: N'(z,p)>\beta])$.

Then $N_n^L(z,p)-V_n^L(z)=N_n'^L(z,p)$ and $N_n^L(z,p)\to N^L(z,p)$, $N_n'^L(z,p)\to N'_n(z,p)$ in mean and $N^L(z,p)\uparrow N(z,p)$, $N'^L(z,p)\uparrow N'(z,p)$.
Now it is proved (similarly as page 151 (II)) that C_α and C_β are also regular for $N^L(z,p)$ and $N'^L(z,p)$ respectively, i.e. $\int_{C_\alpha}U(z)\frac{\partial}{\partial n}N_n^L(z,p)ds\to\int_{C_\alpha}U(z)\frac{\partial}{\partial n}N^L(z,p)ds$ and $\int_{C_\beta}U(z)\frac{\partial}{\partial n}N_n'^L(z,p)ds\to\int_{C_\beta}U(z)\frac{\partial}{\partial n}N'^L(z,p)ds$.
Apply the green's formula to $U_n^S(z)$ and $N_n^L(z,p)$ in $E[z\in R: N(z,p)>\alpha]-v(p)$. Then

$$\int_{C_\alpha+\partial v(p)\cap R_n}U_n^S(z)\frac{\partial}{\partial n}N_n^L(z,p)ds=\int_{C_\alpha+\partial v(p)\cap R_n}N_n^L(z,p)\frac{\partial}{\partial n}U_n^S(z)ds\,.$$

By $\int_{C_\alpha\cap R_n}\frac{\partial}{\partial n}U_n^S(z)ds=\int_{\partial R_n\cap E[z\in R:\,N(z,p)>\alpha]\cap v(p)}\frac{\partial}{\partial n}U_n^S(z)ds=0,$

$$\int_{C_\alpha \cap R_n} U_n^S(z) \frac{\partial}{\partial n} N_n^L(z, p) ds = \int_{\partial v(p) \cap R_n} (N_n^L(z, p) \frac{\partial}{\partial n} U_n^S(z) - U_n^S(z) \frac{\partial}{\partial n} N_n^L(z, p)) ds , \tag{3}$$

Next

$$\int_{(C_\beta + \partial v(p)) \cap R_n} U_n^S(z) \left(\frac{\partial}{\partial n} N_n^L(z, p) - \frac{\partial}{\partial n} V_n^L(z) \right) ds = \int_{(C_\beta + \partial v(p)) \cap R_n} (N_n^L(z, p) - V_n^L(z)) \frac{\partial}{\partial n} U_n^S(z) ds .$$

By $\int_{C_\beta \cap R_n} \frac{\partial}{\partial n} U_n^S(z) = 0$,

$$\int_{C_\beta \cap R_n} U_n^S(z) \frac{\partial}{\partial n} (N_n^L(z, p) - V_n^L(z)) ds = \int_{\partial v(p) \cap R_n} \{N_n^L(z, p) \frac{\partial}{\partial n} U_n^S(z) - V_n^L(z) \frac{\partial}{\partial n} U_n^S(z) - U_n^S(z) \frac{\partial}{\partial n} N_n^L(z, p) + U_n^S(z) \frac{\partial}{\partial n} V_n^L(z)\} ds . \tag{4}$$

But $\int_{C_\beta \cap R_n} V_n^L(z) \frac{\partial}{\partial n} U_n^S(z) ds = \int_{C_\beta \cap R_n} U_n^S(z) \frac{\partial}{\partial n} V_n^L(z) ds$, hence the term on the left hand side of (4) = the term on the right hand side of (3). Since $0 < U_n^S(z) \leqq S$ and by the regularity of C_α and C_β, we have by letting $n \to \infty$ and then $L \to \infty$ and then $S \to \infty$, we have

$$U(z) = \int_{C_\alpha} U(z) \frac{\partial}{\partial n} N(z, p) ds = \int_{C_\beta} U(z) \frac{\partial}{\partial n} N'(z, p) ds = U^*(p) .$$

Proof of the theorem. Let $U(z)$ be superharmonic in $\bar{R}$. In every $V(p) = E[z \in R: N(z, p) > \alpha]$ there exists a $v(p) \subset V(p)$. Let $U^V(z)$ be a $\overset{*}{\text{h}}$armonic function in $\bar{R} \cap V(p)$[4] with value $U^V(z) = U(z)$ on $\partial V(p)$ and let $U^v(z)$ be $\overset{*}{\text{h}}$armonic in $\bar{R} \cap v(p)$ with value $U^v(z) = U(z)$ on $\partial v(p)$. Then $U^v(z) \leqq U^V(z)$. Hence by Lemma 1

$$U^V(p) = U^{*V}(p) \leqq U^{*v}(p) . \tag{5}$$

Clearly $U^{G_\beta}(p) = \int_{C_\beta} U(z) \frac{\partial}{\partial n} N'(z, p) ds \leqq \int_{C_{\beta'}} U(z) \frac{\partial}{\partial n} N'(z, p) ds \leqq U^{G_{\beta'}}(p)$ for regular C_β and $C_{\beta'}$ $(\beta < \beta')$ by the superharmonicity of $U(z)$, where $G_\beta = E[z \in R: N'(z, p) > \beta]$. Hence $\lim_{\beta \uparrow M'} \int_{C_\beta} U(z) \frac{\partial}{\partial n} N'(z, p) ds$ exists. We

4) If $U(z) = \lim_{M=\infty} U^M(z)$, we say $U(z)$ is *harmonic in G, where $U^M(z) = (\min(M, U(z))$ on ∂G and $U^M(z)$ has M.D.I. over G.

define $U^*(z)$ by this limit. Thus

$$U(p) \leqq U^*(p).$$

Next we show $U(p) \geqq U^*(p)$. We suppose $U^*(p) < \infty$. Then by definition of $U^*(p)$, there exist $v(p)$, $N'(z, p)$ and a regular curve C of $N'(z, p)$ for any given positive number ε such that

$$U^*(p) - \varepsilon \leqq \frac{1}{2\pi} \int_C U(z) \frac{\partial}{\partial n} N'(z, p)\, ds.$$

Let $U_A(z)$ be the lower envelope of superharmonic functions in $\bar{R}$ larger than $U(z)$ on A. Then by the superharmonicity of $U(z)$, $U_{Cv(v) \cap R_n}(z) \uparrow U_{Cv(p)}(z) = U^{v(p)}(z)$ as $n \to \infty$. $V_M(p) = E[z \in R: N(z, p) > M]$ clusters at the ideal boundary as $M \uparrow \sup N(z, p)$. Therefore we can find a number n_0 and M such that

$$U^*(p) - 2\varepsilon \leqq \frac{1}{2\pi} \int_{C \cap R_n} U(z) \frac{\partial}{\partial n} N'(z, p)\, ds \quad (n \geqq n_0)$$

and $(R - R_{n_0}) \supset V_M(p)$ for the same $\varepsilon > 0$. Since $(Cv(p) \cap R_n) \subset Cv(p) + CV_M(p)$ $U_{Cv(p) \cap R_n}(z) \leqq U_{Cv(p)+CV_M(p)}(z)$ and $U^*(p) - 2\varepsilon \leqq \frac{1}{2\pi} \int_{C \cap R_n} U_{Cv(p)+CV_M(p)}(z) \frac{\partial}{\partial n} N'(z, p)\, ds$ for every $V_M(p)$ such that $V_{M(p)} \subset (R - R_n)$. Now $N'(z, p) = N(z, p) - V(z)$, where $V(z)$ is harmonic in $\bar{R} \cap v(p)$ such that $V(z) = N(z, p)$ on $\partial v(p)$, i.e. $V(z) = N_{Cv(p)}(z, p)$ in $v(p)$. Hence $N_{CV_M(p)+Cv(p)}(z, p) \uparrow N_{Cv(p)}(z, p) = V(z)$ as $V_M(p) \to 0$. Hence the niveau curve $C' = E[z \in R: N(z, p) - N_{CV_M(p)+Cv(p)}(z, p) = k]$ tend to $E[z \in R: N(z, p) - V(z) = k]$ and further,

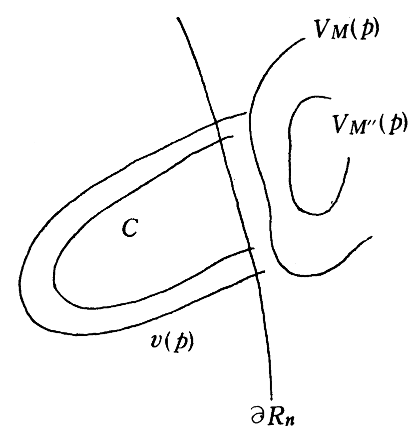

Fig. 4

$\frac{\partial}{\partial n}(N(z, p) - N_{CV_M(p)+Cv(p)}(z, p))$ on C' tends to $\frac{\partial}{\partial n} N'(z, p)$ on C as $M \uparrow \sup N(z, p)$. Hence there exists $M' > M$ such that

$$U^*(p) - 2\varepsilon \leqq \frac{1}{2\pi} \int_{C \cap R_n} U_{Cv(p)+CV_{M'}(p)}(z) \frac{\partial}{\partial n} N'(z, p)\, ds$$

$$\leqq \frac{1}{2\pi} \int_{C^* \cap R_n} U_{Cv(p)+CV_{M'}(p)}(z) \frac{\partial}{\partial n} (N(z, p) - N_{Cv(p)+CV_{M'}(p)}(z, p))\, ds + \varepsilon,$$

$$U^*(p) - 3\varepsilon \leqq \frac{1}{2\pi} \int_{C^*} U_{Cv(p)+CV_{M'}(p)}(z) \frac{\partial}{\partial n} (N(z, p) - N_{Cv(p)+CV_{M'}(p)}(z, p))\, ds, \qquad (6)$$

where $C^*=E[z\in R;\ N(z,p)-N_{Cv(p)+CV_{M}'(p)}(z,p)=k]$.
Next suppose $v(p)+V_{M'}\supset V_{M'}\supset V_{M''}(M''>M')$ and $V_{M''}=E[z\in R: N(z,p)>M'']$ and $\partial V_{M''}$ is regular. Then similarly it is proved that

$$\frac{1}{2\pi}\int_{C^*} U_{Cv(p)+CV_{M}'}(z)\frac{\partial}{\partial n}(N(z,p)-N_{Cv(p)+CV_{M}'}(z,p))ds$$

$$\leq\frac{1}{2\pi}\int_{C''} U_{CV_{M}''}(z)\frac{\partial}{\partial n}(N(z,p)-N_{CV_{M}''}(z,p))ds\,, \tag{7}$$

where C^* and C'' are regular niceau curves of $N(z,p)-N_{Cv(p)+CV_{M}''}(z,p)$ and $N(z,p)-N_{CV_{M}''}(z,p)$ respectively. Since $N(z,p)-N_{CV_{M}''}(z,p)=N(z,p)-M''$. Hence by letting $\varepsilon\to 0$, the last term of $(7)=\frac{1}{2\pi}\int_{C''} U(z)\frac{\partial}{\partial n}N(z,p)\leq U(p)$,

whence $$U(p)=U^*(p)\,.$$

In case $U^*(p)=\infty$, we can prove similarly.

Properties of functiontheoretic equilibrium potential.

Let G be a non compact domain in $R-R_0$ and let $\omega_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G\cap(R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on ∂R_0, $\omega_{n,n+i}(z)=1$ on $\partial(G\cap(R_{n+i}-R_n))$ and $\frac{\partial\omega_{n,n+i}}{\partial n}(z)=0$ on $\partial R_{n+i}-G$. Then it is proved (pp, 145 and 154) that $\omega_{n,n+i}(z)\to\omega_n(z)$ in mean as $i\to\infty$ and $\omega_n(z)\to\omega(z)$ in mean as $n\to\infty$ and that $\omega(z)$ is superharmonic function in $\bar{R}$. We call $\omega(z)$ the (functiontheoretic) equilibrium potential of the ideal boundary $(B\cap G)$ determined by G. Let F be a closed set. Put $F_m=E[z\in\bar{R}: \delta(z,F)\leq\frac{1}{m}]$ and $\omega_m(z)$ C.P. (equilibrium potential) of F_m. Then $\omega_m(z)\to\omega(z)$ in mean.

Lemma 2. *If $\omega(z)$, C.P. of $(G\cap B)$ determined by G is not zero, $\sup_{z\in G}\omega(z)=1$. Put $G_\delta=E[z\in R: \omega(z)<1-\delta]$, $\delta>0$. Then $(B\cap G\cap G_\delta)$ is of capacity zero.*

Since $\omega_n(z)\to\omega(z)$ in mean, $\omega(z)=\omega'_n(z)$, where $\omega'_n(z)$ is a harmonic function in $R-R_0-((R-R_n)\cap G)$ such that $\omega'_n(z)=\omega(z)$ on $\partial(G\cap(R-R_n))$ and $\omega'_n(z)$ has M.D.I.[5] Suppose $\sup_{z\in G}\omega(z)\leq K<1$. Let $\omega'_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(G\cap(R-R_n))$ such that $\omega'_{n,n+i}(z)=\omega(z)$ on $\partial R_0+\partial(G\cap(R-R_n))$ and $\frac{\partial}{\partial n}\omega'_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$. Then

$$K\omega_{n,n+i}(z)\geq\omega'_{n,n+i}(z)\,.$$

5) We abbreviate minimal Dirichlet integral by M.D.I.

Let $i\to\infty$ and $n\to\infty$. Then

$$K\omega(z) > \omega'_n(z) = \omega(z)\,,$$

whence $\omega(z)=0$. This is a contradiction.
Next let $\omega^\delta(z)$ be C.P. of $(B\cap G\cap G_\delta)$. Then $\omega^\delta(z)\leqq\omega(z)$ and $\sup\limits_{z\in G\delta}\omega^\delta(z)\leqq 1-\delta$. This implies $\omega^\delta(z)\equiv 0$. Hence we have Lemma 2.

Let $\omega(z)$ be C.P. of closed set F. Then $\omega(z)$ is superharmonic in $\bar{R}$ and the value of $\omega(z)$ is defined in $\bar{R}(=(R+B))$ (see Theorem 1) and it is proved that $\omega(z)$ is lower semicontinuous in $\bar{R}$. (see II).

Theorem 3. *Let F be a closed set of positive capacity and let $\omega(z)$ be C.P. of F. Then $\omega(z)=1$ except at most an F_σ of capacity zero.*

Lemma 3. *Let $\omega(z)$ be C.P. of F of positive capacity. Then $\sup\limits_{z\in F}\omega(z)=1$.*
Since $F=\bigcap\limits_{m>0}F_m$ and F_m can be considered as a non compact domain, it is clear $\sup\limits_{z\in F_m}\omega(z)=1$ for every n, but our assertion is not clear. If F has a closed subset F' of F of positive capacity in R, our assertion is trivial. *Hence we suppose $F\subset B$.* Put $G_K=E[z\in R:\ \omega(z)<K<1]$. Then $G_K\cap R$ is an open set. Let G'_K be a component of G_K. Assume that G_K has a positive distance from ∂R_0, then $\omega(z)<K$ in $G'_K\cap R$ and $\omega(z)=K$ on $\partial G'_K\cap R$. But by the superharmonicity of $\omega(z)$, $\omega(z)\geqq H(z)\equiv K$, where $H(z)$ is harmonic in $R\cap G'_K$ such that $H(z)=K$ on $\partial G'_K\cap R$ and $H(z)$ has M.D.I. On the other hand, $\omega(z)<K$ in G'_K, whence $\omega(z)\equiv K$ in G'_K. But $\omega(z)$ is a non constant is R. This is a contradiction. Hence G'_K has a subset of ∂R_0 as its boundary. Now $0<\omega(z)<\varepsilon$ in a neighbourhood of ∂R_0 for any positive number $\varepsilon>0$. Therefore G'_K has ∂R_0 in its boundary which implies that G_K consists *of only one component.*

Assume $\omega(p)=K(<1)$ and that $p(\in R+B_1)$ has a positive distance from G_K. Then there exists a neighbourhood $v(p)\subset CG_K\cap\bar{R}$. Then

$$K=\omega(p)\geqq\frac{1}{2\pi}\int\limits_C\omega(z)\frac{\partial}{\partial n}N'(z,p)ds>K\,,$$

by the non-constancy of $\omega(z)$ in R. Hence every point $p\in(R+B_1)$ such that $\omega(p)=K$ is a limit point of a sequence $\{z_i\}$ $(z_i\in G_K)$.

Let $p\in(R+B_1)$ such that $\omega(p)=K$. Then $p\in G_K\subset\bar{G}_{K+\delta}$ (closure of $G_{K+\delta}$), where $G_{K+\delta}=E[z\in R:\ \omega(z)<K+\delta]$ for any given positive number $\delta>0$. Since $\omega(z)$ is lower semicontinuous, there exists a neighbourhood $v(p)$ $(\subset F_m)$ such that $\omega(z)>K-\varepsilon:\ z\in v(p)$ for any given positive number $\varepsilon>0$.

$$\omega(p) \geqq \frac{1}{2\pi}\left(\int_{C\cap G_K+\delta} \omega(z)\frac{\partial}{\partial n}N'(z,p)\,ds + \int_{C\cap CG_{K+\delta}} \omega(z)\frac{\partial}{\partial n}N'(z,p)\,ds\right) \tag{8}$$

Assume $\int_{C\cap CG_{n+\delta}} \frac{\partial}{\partial n}N'(z,p)ds > \pi$. Then

$$\omega(p) \geqq \frac{1}{2\pi}(K-\varepsilon)\pi + (K-\delta)\pi \geqq \frac{\delta-\varepsilon}{2}+K.$$

This is acontradittion for $\varepsilon<\frac{\delta}{3}$. Hence we have the following assertion.

Let $p\in B_1+R$ with $\omega(p)=K<1$. Then for any $\delta>0$, we can find a $v(p)$ in F_m such that whose $N'(z,p)$ satisfies the following condition

$$\int_{C\cap G_{K+\delta}} \frac{\partial}{\partial n}N'(z,p)ds \geqq \pi \tag{9}$$

for every regular curve C of $N'(z,p)$.

Put $H_K=E[z\in \bar{R}: \omega(z)\leqq K]$. Then H_K is closed by the lower semicontinuity of $\omega(z)$. Then $F\cap H_K$ is also closed. We show that $F\cap H_K$ is *a set of capacity zero.* Let $\omega_m(z)$ be a superharmonic function in $R-R_0$ such that $\omega_m(z)=0$ on ∂R_0, $\omega_m(z)=1$ on $F_m\cap G_{K+\delta}\cap(R-R_m)$ and $\omega_m(z)$ has M.D.I. Then $\omega_m(z)\to\omega'(z)$, where $\omega'(z)$ is C.P. of the boundry determined by $\bigcap_{m>1} F_m \bigcap G_{K+\delta}$ $(m=1,2,\cdots)$. Hence by Lemma 2, $\omega'(z)=0$. Choose a sequence $m_1, m_2, \cdots$ such that $\int_{\partial R_0}\frac{\partial\omega_{mi}}{\partial n}(z)ds \leqq \frac{1}{2^m}$. Then

$$\omega^*(z) = \sum_{m_i}^{\infty} \omega_{mi}(z) < \infty$$

and

$$\lim \omega^*(z) = \infty \quad \text{as } z \text{ tends to } F \text{ inside of } G_{K+\delta}.$$

Let $p\in(F\cap H_K\cap B_1)$. Then $\omega^*(p)\geqq\frac{1}{2\pi}\int_C \omega^*(z)\frac{\partial}{\partial n}N'(z,p)$ whence by (9) $\omega^*(p)=\infty$ and the lower semicontinuity of $\omega^*(z)$,

$$\lim_{z\to q\in(F\cap H_K)} \omega^*(z) = \infty$$

B_0 (set of non minimal points) is a sum of closed sets of capacity zero. We can construct as above a superharmonic function $\omega^{**}(z)$ such that $\lim_{z\to G\in B_0} \omega^{**}(z)=\infty$.

Proof of Lemma 3. Suppose $\underset{z\in F}{\omega}(z)\leqq K<1$. Then $\lim_{z\to q\in F}\varepsilon(\omega^*(z)+\omega^{**}(z))=\infty$ for any $\varepsilon>0$. Put $\Delta_\varepsilon=E[z\in\bar{R}: \varepsilon(\omega^*(z)+\omega^{**}(z))\leqq 2]$. Then Δ_ε is also closed and $\Delta_\varepsilon\cap F=0$, which implies dist $(\Delta_\varepsilon, F)>d_\varepsilon>0$.

Put $F_{d\varepsilon}=E[z\in R: \delta(z, F)\leqq d_\varepsilon]$. Let $\omega_\varepsilon(z)$ be C.P. of $F_{d\varepsilon}$. Then

$$\varepsilon(\omega^*(z)+\omega^{**}(z))\geqq \omega_\varepsilon(z)\geqq \omega(z)\ .$$

By letting $\varepsilon\to 0$. We have $\omega(z)\equiv 0$.
This a contradiction. Hence $\sup_{z\in F}\omega(z)=1$.

Proof of Theorem 3. Let $\omega_k(z)$ be C.P. of $E_k=E[z\in(\bar{R}\cap F): \omega(z)\leqq 1-\frac{1}{k}]$ $(k=1, 2, \cdots)$. Then $\omega_k(z)\leqq\omega(z)$, whence $\sup_{z\in E_k}\omega_k(z)\leqq 1-\frac{1}{k}$. Hence by lemma 3 E_k is a set of capacity zero. Then $E=\bigcup_{k>0}E_k$ is an F_σ of capacity zero.

Theorem 4. *Let $\omega(z)$ be C.P. of a closed set F of positive capacity. $R-F$ consists at most enumerably infinite number of domains. Let G be one containing ∂R_0 in its boundary. Then $\omega(z)<1$ in G except at most capacity zero.*

Since $\omega(z)$ is harmonic in $R-F$, $\omega(z)<1$ in $G\cap R$. Suppose p is a point in $(B\cap G\cap B_1)$.[6] Then there exists a neighbourhood $v(p)$ such that $v(p)\subset G$. Then

$$\omega(p)=\frac{1}{2\pi}\int_{C\cap R}\omega(z)\frac{\partial}{\partial n}N'(z, p)ds<1$$

because $\omega(z)$ is non constant harmonic in $G-F$, i.e. $\omega(z)$ has M D.I. over $v(p)$. On the other hand, B_0 is a set of capacity zero. Hence we have the theorem.

Mass distribution on $\bar{R}$. We have seen that $N(z, p)$ and $N'(z, p)$ have the essential properties of the logarithmic potential: lower semi-continuity in $\bar{R}$, symmetry and superharmonicity in the sense as follow: $N(q, p)\geqq\frac{1}{2\pi}\int_C N(z, q)\frac{\partial}{\partial n}N'(z, p)ds$ for every $v(p)$ of $p\in R+B_1$, where $N'(z, p)$ is the function in $v(p)$ in Theorem 1. But there exists a fatal difference between our space and the euclidean space, that is, in our space there may exist points of B_0 where we cannot distribute any *true* mass. A distribution μ on B_0 may be called a *pseudo distribution* in the sense that μ can be replaced, by Theorem 8 of II, by a canonical distribution on B_1+R without any change of $U(z)=\int N(z, p)d\mu(p)$. Hence it is sufficient to consider only *canonical distributions.*

6) G is open with respect to Martin's topology, whence G may contain points of the ideal boundary.

Energy integral $I(\mu)$ of a canonical mass distribution on $R+B_1$ is defined as

$$I(\mu)=\iint N(q,\,p)\,d\mu(p)\,d\mu(q)\,.$$

$\overset{*}{\text{C}}$**apacity (potentialtheoretic)** of a closed set F in $\bar{R}$ is defined by $\dfrac{1}{\inf I(\mu)}$, where μ is a canonical distribution of $F\frown(R+B_1)$ of total mass unity.

Lemma 4. *Cap* $(F)>0$ *implies* $\overset{*}{\text{C}}\text{ap}\,(F)>0$ *for every closed subset* F *of* $\bar{R}$.

In fact, if Cap $(F)>0$, there exists C.P. of F such that $\omega(z)=\omega_F(z)$ and $\omega_F(z)$ is represented by a mass distribution μ on F. $\omega_{F\frown(R+B_1)}(z)\leqq\omega_F(z)\leqq\omega_{F\frown(R+B_1)}(z)+\omega_{B_0}(z)$. But $\omega_{B_0}(z)=0$ by Theorem 8 in (II), hence $\omega(z)=\omega_{F\frown(R+B_1)}(z)$ and $\omega(z)$ is represented by a canonical distribution on $F\frown(R+B_1)$ and the total mass is given by $\displaystyle\int_{\partial R_0}\frac{\partial\omega}{\partial n}(z)ds$. Since $\sup\omega(z)=1$, $I(\mu)<\infty$. This implies $\overset{*}{\text{C}}\text{ap}\,(F)>0$.

Theorem 5. *Let* μ *be a canonical distribution on a closed set* F *of capacity zero such that its potential* $U(z)=U_F(z)>0$. *Then* $\sup\limits_{z\in F}U(z)=\infty$.

It is clear $\sup\limits_{z\in R}U(z)=\infty$, but our assertion is not so clear. Suppose $\sup\limits_{z\in F}U(z)\leqq M$. Let p be a point in $(R+B_1)\frown F$. Then

$$\frac{1}{2\pi}\int_C U(z)\frac{\partial}{\partial n}N'(z,\,p)ds\leqq M$$

for every regular curve of $N'(z,\,p)$.

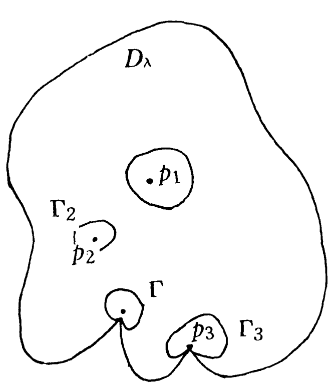

Fig. 5

Let $p_1,p_2,\cdots,p_i$ be points in $F\frown(R+B_1)$ and put $D_\lambda=E[z\in R:\sum c_iN(z,\,p_i)>\lambda]$, where $c_i>0$ and $\sum c_i=$total mass of μ.

Let $U^{D_\lambda}(z)$ be a harmonic function in D_λ such that $U^{D_\lambda}(z)=U(z)$ on ∂D_λ and $U^{D_\lambda}(z)$ has M.D.I. over D_λ. Then $U^{D_\lambda}(z)\leqq U(z)$. Similarly as in Theorem 2 and by $\displaystyle\int_{\partial D_\lambda}\frac{\partial}{\partial n}\sum c_iN(z,\,p_i)ds\leqq 2\pi\sum c_i$,

$$\frac{1}{2\pi}\int_{\partial D_\lambda}U(z)\frac{\partial}{\partial n}(\sum c_iN(z,\,p_i))ds\leqq\frac{1}{2\pi}\sum_i\int_{\Gamma_i}c_iU(z)\frac{\partial}{\partial n}N(z,\,p_i)ds\leqq M\sum c_i\,,$$

where Γ_i is a regular curve of $N(z,\,p_i)$ and contained in D_λ. By the continuity of $N(z, p)$ there exists a linear form $\sum c'_k N(z, p_k): p_k \in F \cap (R+B_1)$ such that $|U(z)-\sum c'_k N(z, p_k)|<\varepsilon: z \in R_m$ for any given R_m and $\varepsilon>0$. Hence there exists a sequence $\{U_j(z)\}$ of the above linear form such that $U_j(z) \rightarrow U(z)$ in $R-R_0$. Now $U_j(z) \rightarrow U(z)$ implies $\frac{\partial}{\partial n} U_j(z) \rightarrow \frac{\partial}{\partial n} U(z)$ in R and $C_\lambda^j=E[z \in R: U_j(z)=\lambda]$ tends to $C_\lambda=E[z \in R: U(z)=\lambda]$. Then by Fatou's lemma

$$\sum c_k M \geqq \varlimsup_j \int_{C_\lambda^j} U(z) \frac{\partial}{\partial n} U_j(z) ds \geqq \int_{C_\lambda} U(z) \frac{\partial}{\partial n} U(z) ds, \quad \text{for every } \lambda \tag{10}$$

On the other hand, $U(z)=U_F(z)=U_{D_\lambda}(z)$ implies $U(z)=\lambda \omega(z)$, where $\omega(z)$ is C.P. of D_λ. Hence for almost all λ

$$\int_{C_\lambda} \frac{\partial}{\partial n} U(z) ds=\text{total mass of } \mu,$$

whence
$$\lim_{\lambda=\infty} \int_{C_\lambda} U(z) \frac{\partial}{\partial n} U(z) ds=\infty \tag{11}$$

(10) contradicts to (11). Hence we have the theorem.

At present, we cannot prove the uniqueness of canaonical mass distribution but we shall prove

Theorem 6. *Let $U(z)$ be a superharmonic function in $\bar{R}$ such that $U(z)=U_F(z)$. Then $U(z)=\int_{F \cap (R+B_1)} N(z, p) d\mu(p)$. The mass distribution μ cannot be replaced by any other canonical distribution on F' such that dist $(F, F')>0$ without any change of $U(z)$.*

As for the part of μ on R, the uniqueness of mass distribution is clear. We suppose both F and F' are contained in B. We cover F by a finite number of closed discs $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_{i_0}$ with diameter $<\frac{1}{n}$. Put $\mu=\mu_1+\mu_2+\cdots+\mu_{i_0}$, where μ_i is the restriction of μ on $\mathfrak{F}_i$. Hence there exist μ_i and $\mathfrak{F}_i$ such that $\int N(z, p) d\mu_i(p)={}_{F'}\left(\int N(z, p) d\mu_i(p)\right)>0$. We denote $\mathfrak{F}_i$ and μ_i by $\mathfrak{F}_1$ and μ_1 respectively. As above we chosse $\mathfrak{F}_2$ and μ_2 such that $\mu_2>0$, dia $\mathfrak{F}_2(\subset \mathfrak{F}_1)<\frac{1}{2^2}$ and ${}_{F'}\left(\int N(z, p) d\mu_2(p)\right)=\int_{\mathfrak{F}_2} N(z, p) d\mu_2(p)>0$. Hence we can find sequences $\left\{\frac{\mu_i}{m_i}\right\}$ and $\mathfrak{F}_1 \supset \mathfrak{F}_2 \cdots$ such that $\bigcap_i \mathfrak{F}_i=p \in (R+B_1) \cap F$, where m_i is the total mass of μ_i. Since ${}_{F'}\left(\int N(z, p) d\mu_i(p)\right)$ $=\int N(z, p) d\mu_i(p)$, $\frac{1}{m_i} \int N(z, p) d\mu_i$ is represented by a mass distribution

μ_i^* on F'. There exists a subsequence $\{\mu_{i'}^*\}$ such that $\{\mu_{i'}^*\}$ converges to μ^* on F'. On the other hand, $\left\{\frac{\mu_i}{m_i}\right\}$ tends to a point mass $N(z, p)$. Hence

$$N(z, p) = \int_{F'} N(z, q) d\mu^*(q) \ : \ \text{dist } (F', p) > 0 .$$

Now we can prove as in Theorem 1 that $N(z, p)$ is not minimal. This is a contradiction. Hence we have the theorem.

Lemma 5. *Let μ be a mass distribution and let μ_C be its canonical distribution (on $R+B_1$), i.e, $\int_{\bar{R}} N(z, p) d\mu(p) = \int_{R+B_1} N(z, p) d\mu_C(p)$. Then $I(\mu) = I(\mu_C)$. Hence $I(\mu)$ does not depend on a choise of particular distribution.*

Suppose p and and q are not minimal. Then $N(z, p) = \int_\alpha N(z, \alpha) d\mu_p(\alpha)$ and $N(z, q) = \int_\beta N(z, \beta) d\mu_q(\beta)$, where α and $\beta \in R+B_1$. (12)

Then $I(\mu) = \iint N(p, q)\, d\mu(p)\, d\mu(q) = \int_q \int_p \int_\alpha N(\alpha, q)\, d\mu_p(\alpha)\, d\mu(p)\, d\mu(q)$

$= \int_q \int_p \int_\alpha \int_\beta N(\alpha, \beta)\, d\mu_p(\alpha)\, d\mu_q(\beta)\, d\mu(p)\, d\mu(q) = \int_\alpha \int_\beta N(\beta, \alpha) \int_p d\mu_p(\alpha)\, d\mu(p)$

$\times \int_q d\mu_q(\beta)\, d\mu(q) = \int_\alpha \int_\beta N(\alpha, \beta)\, d\mu(\alpha)\, d\mu(\beta) = I(\mu_C)$,

because (12) means that a unit mass on p is replaced by $\mu_p(\alpha)$ on α, whence $\int_p d\mu_p(\alpha)\, d\mu(p) = d\mu(\alpha)$ and $\int d\mu_q(\beta)\, d\mu(q) = d\mu(\beta)$.

Lemma 6. If $\mu_n \to \mu$, *then* $I(\mu) \leq \varliminf_{n=\infty} I(\mu_n)$.

$I(\mu) = \lim_{M=\infty} \iint N^M(p, q)\, d\mu(p)\, d\mu(q) \leq \lim_{M=\infty} \lim_{n=\infty} \iint N^M(p, q)\, d\mu(p)\, d\mu(q)$

$\leq \varliminf_{n=\infty} \iint N(p, q)\, d\mu(p)\, d\mu(q) = \varliminf_{n=\infty} I(\mu_n)$.

Theorem 7. (Fundamental theorem 1). *Let F be a closed set in $\bar{R}$ of positive *capacity. Then there exists a unit mass canonical distribution μ on F (on $F \cap (R+B_1)$) whose energy integral is minimal and its potential $U(z)$ satisfies the following conditions:*

1) $U(z) \geq V$ in F except at most a set of *capacity zero.
2) $U(z) \leq V$ in F^* (kernel of μ).
3) $U(z) \geq V\omega(z)$
4) $U(z) = V$ on $F^* \cap R$

where $V=I(\mu)$ and $\omega(z)$ is C.P. of F.

In our space, the potential $N(z, p)$ is continuous in $R-p$ and lower semicontinuous in $\bar{R}$ but $N(z, p)$ is not necessarily continuous in $\bar{R}-p$ and the *continuity principle* cannot be proved. Therefore we cannot prove the above theorem in usual manner.

Let $\{\mu_n\}$ be a sequence of canonical distributions on F such that $I(\mu_n) \downarrow V$, where V is the infinimum of energy integrals of all canonical distributions on F of total mass unity. Put $\mu=\lim\limits_n \mu_n$. Then by Lemma 6, $I(\mu)=V$. If μ is not canonical, we replace by a canonical distribution μ_C. Then also by Lemma 5, $I(\mu_C)=V$. Assume that there exists a closed set $\mathfrak{F}$ in CF such that $\int\limits_{\mathfrak{F}} N(z, p)d\mu''(p)>0$, where μ'' is the restriction of μ_C on $\mathfrak{F}$. Then μ'' cannot be replaced by any canonical distribution on F by Theorem 6. Hence every canonical distribution which is equivalent to μ has a positive mass on $\mathfrak{F}$. This contradicts to $\mu=\lim\limits_n \mu_n$. Hence such μ_C has no mass except on F. Thus there exists a canonical distribution μ of total mass unity on F such that $I(\mu)=V$.

Let F^* be the kernel of the distribution μ. Then clearly $F^* \subset F$ and closed. By symmetry of $N(p, q)$, $I(\mu)=\int U(p)d\mu(p)$ and $U(z) \not\leqq V-\varepsilon$ on F^*, because $I(\mu)=V$. Hence there exists a point $p_0 \in F^*$ such that $U(p_0)>V-\varepsilon$ and there exists a neighbourhood $v(p_0)$ such that $U(z)> V-\varepsilon$ $(z \in v(p_0))$ by the lower semicontinuity of $U(z)$ and that the restriction of μ in $v(p_0)$ has a positive mass m in $v(p_0)$. Assume $U(z) \leqq V-2\varepsilon$ on a set F' of positive $\overset{*}{\text{capacity}}$ in F. We define a new canonical mass distribution $\hat{\mu}$ on F' whose energy integral is finite and whose total mass is m. Define another distribution σ as follows:

$\sigma=-\mu$ on $v(p_0)$, $\sigma=\hat{\mu}$ on F' and $\sigma=0$ outside of $v(p_0)$ and F'.

Then $\mu+h\sigma>0$ for $h<1$ and the total mass is unity. Then the variation

$$\begin{aligned}
\delta I &= I(\mu+h\sigma)-I(\mu) \geqq 0 \quad \text{and} \\
\delta I &= 2h\int U(p)d\mu(p)+h^2 I(\sigma) < 2h\,[m(V-2\varepsilon)-m(V-\varepsilon)]+h^2 I(\sigma) \\
&= -h\,[2m-hI(\sigma)]\,.
\end{aligned}$$

This is a contradiction for sufficiently small h. Hence by letting $\varepsilon \to 0$, we have (1).

Put $F'=E[z \in F\colon U(z) \leqq V-\varepsilon]$. Then F' is closed and $\overset{*}{\mathrm{Cap}}(F')=0$ and the restriction of μ' on F' has no mass, because $I(\mu') \leqq I(\mu) \leqq V$. Hence μ has no mass on a set $E[z \in F\colon U(z) \leqq V+\varepsilon]$ for any $\varepsilon>0$. Next assume μ has a positive mass m on a set $E[z \in F\colon U(z)>V+\varepsilon]$.

Then $I(\mu) > V$. This also a contradiction. Hence by letting $\varepsilon \rightarrow 0$, $U(z) = V$ where the mass is distributed. Thus $U(z) = V$ on F^* by the lower semicontinuity of $U(z)$, whence we have (2).

$R-F$ consists of at most enumerably infinite number of domains $G_1, G_2, \cdots$, where G_1 is the domain containing ∂R_0 in its boundary. At first, consider $U(z)$ in G_1. $U(z) = V$ on F^* except at most a set F' of capacity zero (by Lemma 4, capacity zero). Hence the exists a superharmonic function $\omega^*(z)$ in $\bar{R}$ (as in Theorem 3) such that $\varlimsup_{z \to p \in F'} \omega^*(z) = \infty$. Hence as in case of Lemma 3, $U(z) \geqq V\omega(z)$ in G_1. Let G_2 be one of other domains. Then $\partial G_2 \subset F$. $\partial G_2 \cap R$ consists of continum boundary Γ_i $(i = 1, 2, \cdots)$ and others Γ^*. Put $G_2 + \Gamma^* = G_2^*$. Then G_2^* is also a domain. Since for every point $p \in \Gamma_i \cap R$, there exists a neighbourhood $v(p)$ which is conformally equivalent to a disc in the z-plane. Hence the continuity principle is valid in $v(p)$, whence $U(z) \geqq V$ on $\Gamma_i \cap R$. Then by the superharmonicity of $U(z)$, $U(z) \geqq U_{CG_2^*}(z) = V$ in G_2^*, whence $U(z) \geqq V\omega(z)$ in G_2, because $\omega(z)$ is clearly $= V$ in G_2. Hence in every G_i $(i = 3, 4, \cdots)$ $U(z) \geqq V\omega(z)$. $U(z) \geqq V$ on F except a set capacity zero. Similarly as in Lemma 3, $U(z) \geqq V\omega(z)$. By considering sequences $\{z_i\}$: $z_i \in R$ which clusters at B, we have $U(z) \geqq V\omega(z)$ in B. Thus we have $U(z) \geqq V\omega(z)$ in $\bar{R}$ and we have (3). Clearly by the continuity principle, $U(z) = V$ on $F \cap R$. Hence we have (4).

We know the property of $U(z)$ very little, i.e. at present we don't know whether $U(z)$ is bounded in $\bar{R}$ or not. We shall prove the next

Theorem 8. (Fundamental theorem 2)

$$U(z) \equiv V\omega(z) \text{ in } \bar{R} \text{ and}$$
$$I(\mu) = D(U(z)) = D(V\omega(z)) = V.$$

Lemma 7. *Let $U(z)$ be a function in Theorem 7. Put $G_a = E[z \in \bar{R} : U(z) > V + a]$ $(a > 0)$ and $G_n^* = E[z \in \bar{R} : \delta(z, F^*) < \frac{1}{n}]$. Then $g_n = G_a \cap G_n^*$ is open. Let $\omega^{g_n}(z)$ be C.P. of g_n. Then $\lim_{n=\infty} \omega^{g_n}(z) = 0$.*

Let $\{\mathfrak{F}_i\}$ be a sequence of closed subsets of g_n such that $\mathfrak{F}_i \uparrow g_n$. Let $\omega^{\mathfrak{F}_i}(z)$ be C.P. of $\mathfrak{F}_i$. Then $\omega^{\mathfrak{F}_i}(z) \uparrow \omega^{g_n}(z)$ in mean (see page 154, II). Hence $\omega^{g_n}(z)$ is superharmonic in $\bar{R}$. Put $H_n^i = E[z \in \bar{R} : \omega^{\mathfrak{F}_i}(z) = 1]$ and $H_n = E[z \in \bar{R} : \omega^{g_n}(z) = 1]$. Then H_n^i and H_n are closed. Clealy by Theorem 3 and 4 $\omega^{\mathfrak{F}_i}(z) = \omega^{H_n^i}(z)$, where $\omega^{H_n^i}(z)$ is C.P. of H_n^i. By the superharmonicity of $\omega^{g_n}(z)$, $\omega^{g_n}(z) \geqq \omega^{H_n^i}(z)$, where $\omega^{H_n}(z)$ is C.P. of H_n. On the other hand, $\omega^{\mathfrak{F}_i}(z) \leqq \omega^{H_n}(z)$ for every i. Hence $\omega^{g_n}(z) \geqq \omega^{H_n}(z)$. Thus

$$\omega^{g_n}(z) = \omega^{H_n}(z) .$$

Clearly $\omega^{\bar{g}_n}(z)\geqq\omega^{g_n}(z)$ ($\bar{g}_n$ is the closure of g_n). By (4) of Theorem 7, $F^*\cap g_n\cap R=0$. Hence the complementary set Cg_n of g_n consists of only one component containing ∂R_0 in its boundary. Hence by Theorem 4 $\omega^{g_n}(z)\leqq\omega^{\bar{g}_n}(z)<1$ in Cg_n except at most a subset of Cg_n of capacity zero, whence $H_n\subset\bar{g}_n$ except a subset of H_n of capacity zero.

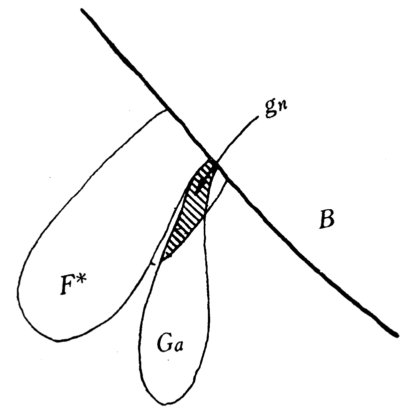

Fig. 6

Next $\omega^{g_n}(z)\downarrow\omega^g(z)(g=\bigcap_{n>1}g_n)$ in mean as $n\to\infty$ and $H_n\downarrow H$ and $H\subset\bar{g}_n\subset F^*$. Assume $\omega^g(z)>0$. Then $\omega^g(z)=1$ on H except a subset of H of capacity zero, whence by Theorem 4 there exists at least a point $z_0\in F^*$ such that $\omega^g(z_0)=1$. Since $F\supset F^*$,

$$U(z)=U_{F_n^*}(z)\geqq(V+a)\omega^{g_n}(z)\quad\text{and}\quad U(z)\geqq(V+a)\omega^g(z)\,,$$

where $F_n^*=E[z\in\bar{R}:\delta(z,F^*)\leqq\frac{1}{n}]$. Hence $U(z_0)\geqq(V+a):z_0\in F^*$. This contradicts to (2) of Theorem 7. Hence $\omega^g(z)=0$.

Lemma 8. *Put* $G^N=E[z\in\bar{R}:U(z)>N]$. *Then* $\lim_{N=\infty}U_{G^N}(z)=0$.

By the superharmonicity of $U(z)$, $U(z)\geqq N\omega^{G^N}(z)$, where $\omega^{G^N}(z)$ is C.P. of G^N. Hence $\lim_{N=\infty}\omega^{G_N}(z)=0$. i.e. $\cap G^N=G_\delta$ is a set of capacity zero. Assume $\lim_{N=\infty}U_{G^N}(z)=U^*(z)>0$. Then $U^*(z)$ is represented by a canonical distribution μ^* on G_δ and the kernel k^* of μ^* is closed and $\subset G_\delta$. Hence k^* is a set of capacity zero and $U^*_{k^*}(z)=U^*(z)$. Suppose dist$(F^*,k^*)>0$. Then by Theorem 6, μ^* cannot be replaced by a distribution of F^*. On the other hand, $U(z)-U_{k^*}(z)$ is superharmonic (see p. 158, II), whence $U(z)$ has μ^* on k^*, which implies $k^*\subset F^*$. Now by Theorem 5, $\sup_{z\in F^*}U^*_{F^*}(z)=\infty$. Hence there exists a point z_0 in $k^*\subset F^*$ for any large number N such that

$$U(z_0)>U_{k^*}(z_0)>N\,.$$

This contradicts to (2) of Theorem 7. Hence $U^*(z)=0$.

Proof of theorem 8. Let $U_{G^N}(z)$ be function in Lemma 8. Then there exists a number N such that $U_{G^N}(z_0)<\varepsilon$ for given number $\varepsilon>0$ and a point z_0. Put $G_\delta=E[z\in\bar{R}:U(z)>V+\delta]$ and $F_n^*=E[z\in\bar{R}:\delta(z,F^*)\leqq\frac{1}{n}]$ and $g_n=F_n^*\cap G_\delta$. Let $\omega^{g_n}(z)$ and $\omega^{F_n^*}(z)$ be C.P. of g_n and

F_n^* respectively. Then since $F^* \subset F_n^*$,

$$U(z) = U_{F_n^*}(z) \leqq \delta + V\omega^{F_n^*}(z) + N\omega^{g_n}(z) + U_{G^N}(z) .$$

Let $n \to \infty$. Then $N\omega^{g_n}(z) \to 0$, by Lemma 8, Hence

$$U(z_0) < \delta + V\omega^{F^*}(z) + \varepsilon < \delta + \delta V\omega(z) + \varepsilon .$$

Then by letting $\varepsilon \to 0$ and then $\delta \to 0$, we have

$$U(z_0) \leqq V\omega(z_0) .$$

On the other hand, $U(z) \geqq V\omega(z)$, hence we have $U(z) = V\omega(z)$ and

$$D(U(z)) = D(V\omega(z)) = V^2 \frac{1}{2\pi} \int_{\partial R_0} \frac{\partial\omega}{\partial n}(z)ds = V^2 D(\omega(z)) = V = I(\mu) .$$

By Theorem 8 we have the following

Corollary. $\overset{*}{\mathrm{Cap}}\,(F) = \mathrm{Cap}\,(F)$, and

$$\overset{*}{\mathrm{Cap}}\,(F) > 0 \quad \text{implies} \quad \mathrm{Cap}\,(F) > 0 .$$

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