

REPRESENTATION FORMULAS OF THE SOLUTIONS TO THE CAUCHY PROBLEMS FOR FIRST ORDER SYSTEMS

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Abstract

Representation formulas of the solutions to the Cauchy problems for first order systems of the forms $\partial u/\partial t - \sum_{j=1}^d A_j(t)\partial u/\partial x_j - A_0(t)u = f$ are established. The coefficients A_j 's are assumed to be matrix-valued functions of the forms $A_j(t) = \alpha_j(t)I + \beta_j(t)M_j$, where $\alpha_j(t), \beta_j(t)$, $j = 1, \dots, d$, are real-valued continuous functions, the eigenvalues of the matrices M_j , $j = 1, \dots, d$, are real, and the commutators $[M_j, M_l] = 0$ for all $j, l = 0, 1, \dots, d$. No restrictions on the multiplicities of the characteristic roots are imposed.

1. Introduction

In this note we establish a representation formula for the Cauchy problem for a first order system of the form

$$(1.1) \quad \frac{\partial u}{\partial t} - \sum_{j=1}^d A_j(t) \frac{\partial u}{\partial x_j} - A_0(t)u = f(t, x) \quad \text{in } (0, T) \times \mathbb{R}_x^d$$

$$(1.2) \quad u(0, x) = \eta(x)$$

where $A_j(t)$, $j = 0, 1, \dots, d$, is a matrix-valued function of the form

$$(1.3) \quad A_j(t) = \alpha_j(t)I + \beta_j(t)M_j$$

with scalar valued functions $\alpha_j(t)$, $\beta_j(t)$ on the interval $[0, T]$, and a k by k complex matrix M_j . The k by k identity matrix is denoted by I . Both $f(t, x) = {}^t[f_1(t, x), \dots, f_k(t, x)]$ and $\eta(x) = {}^t[\eta_1(x), \dots, \eta_k(x)]$ are given functions.

We introduce notation in order to state the main theorem. For k by k matrices A and B , $[A, B]$ denotes the commutator:

$$[A, B] = AB - BA.$$

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The Fourier transform with respect the variable x is denoted by

$$\hat{\varphi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \varphi(x) dx.$$

- ASSUMPTION (A). (i) $[M_j, M_l] = 0$ for all $j, l = 0, 1, \dots, d$.
(ii) For each $j = 1, 2, \dots, d$, the eigenvalues of M_j are real.
(iii) For each $j = 1, 2, \dots, d$, the functions $\alpha_j(t)$ and $\beta_j(t)$ are real-valued continuous functions, and $\alpha_0(t)$ and $\beta_0(t)$ are possibly complex-valued continuous functions.

Theorem 1.1. *Let Assumption (A) be verified and let f be a function such that $\hat{f}(t, \xi)$ is continuous with respect to t in the interval $[0, T]$ for each $\xi \in \mathbb{R}^d$. Suppose that there exist a constant C and a function $\psi \in L^1(\mathbb{R}^d)$ such that*

$$(1.4) \quad \langle \xi \rangle^{(m-1)d+1} (|\hat{f}(t, \xi)| + |\hat{\eta}(\xi)|) \leq C\psi(\xi)$$

for all $(t, \xi) \in [0, T] \times \mathbb{R}^d$, where $m = 1$ if all M_j , $j = 1, \dots, d$ are semisimple, and otherwise

$$(1.5) \quad m := \max\{n \mid n \text{ equals the algebraic multiplicity of an eigenvalue of some } M_j, 1 \leq j \leq d\}.$$

Then the solution of the Cauchy problem (1.1)–(1.2) is given by

$$(1.6) \quad u(t, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{B(t, \xi)} \left(\hat{\eta}(\xi) + \int_0^t e^{-B(s, \xi)} \hat{f}(s, \xi) ds \right) d\xi,$$

which is a C^1 -function in $[0, T] \times \mathbb{R}^d$. Here

$$(1.7) \quad B(t, \xi) = i \sum_{j=1}^n \xi_j \int_0^t A_j(s) ds + \int_0^t A_0(s) ds.$$

It might be worthwhile to note that we do not need any restriction on the multiplicities of the characteristic roots for the equation (1.1), i.e. the roots of the characteristic polynomial

$$\det \left(\lambda I - \sum_{j=1}^d \xi_j A_j(t) \right).$$

If all the α_j 's and β_j 's are C^∞ , and $f(t, x)$ and η satisfy some suitable conditions, then the solution $u(t, x)$ becomes C^∞ . However, we shall not go into the discussions about this.

2. Proof of the main theorem

We start with ordinary differential equations for $M_k(\mathbb{C})$ -valued functions, where $M_k(\mathbb{C})$ denotes the set of all k by k complex matrices.

Let an $M_k(\mathbb{C})$ -valued continuous function $L(t)$ on the interval $[0, T]$ be given, and consider the ordinary differential equation

$$(2.1) \quad \frac{dU}{dt} = L(t)U$$

with the initial condition

$$(2.2) \quad U(0) = I.$$

Here the unknown function $U(t)$ is an $M_k(\mathbb{C})$ -valued function.

The solution $U(t)$ to the equation (2.1) subject to (2.2), which is called the fundamental solution, can be expressed in the form

$$(2.3) \quad U(t) = e^{\Omega(t)},$$

where $\Omega(t)$ is generally given as an infinite series (cf. [2, Theorem III]).

As a preliminary to the proof of Theorem 1.1, we need the following

Lemma 2.1. *Suppose that $[L(t), L(t')] = 0$ for all $t, t' \in [0, T]$. Then the solution to the initial value problem (2.1) and (2.2) is written as*

$$(2.4) \quad U(t) = e^{\int_0^t L(s) ds} \quad (0 \leq t \leq T).$$

Proof. It is easy to see that

$$(2.5) \quad \left[\int_0^t L(s) ds, L(t') \right] = 0$$

for all $t, t' \in [0, T]$. If we appeal to the definition

$$e^{\int_0^t L(s) ds} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\int_0^t L(s) ds \right)^j,$$

then (2.5) enables us to show that

$$(2.6) \quad \frac{d}{dt} e^{\int_0^t L(s) ds} = L(t) e^{\int_0^t L(s) ds} = e^{\int_0^t L(s) ds} L(t),$$

which implies (2.4). □

We should like to remark that the assertion of Lemma 2.1 is implicitly mentioned in [2]. See also [1].

Proof of Theorem 1.1. We shall give the proof only in the case $m \geq 2$. The proof in the case $m = 1$ is easier.

Taking the Fourier transform of (1.1) and (1.2), we obtain the ordinary differential equation

$$(2.7) \quad \frac{d}{dt} \hat{u}(t, \xi) = \left(i \sum_{j=1}^d \xi_j A_j(t) + A_0(t) \right) \hat{u}(t, \xi) + \hat{f}(t, \xi)$$

subject to the initial condition

$$(2.8) \quad \hat{u}(0, \xi) = \hat{\eta}(\xi),$$

where ξ should be regarded as a parameter. In view of (1.3), it follows from the assumptions of Theorem 1.1 that

$$(2.9) \quad i \sum_{j=1}^d \xi_j A_j(t) + A_0(t)$$

satisfies the assumption of Lemma 2.1. From Lemma 2.1, we see that the solution to (2.7), (2.8) is given by

$$(2.10) \quad e^{B(t, \xi)} \left(\hat{\eta}(\xi) + \int_0^t e^{-B(s, \xi)} \hat{f}(s, \xi) ds \right).$$

We now compute $e^{B(s, \xi)}$. For this purpose, we put

$$(2.11) \quad \tilde{\alpha}_j(t) = \int_0^t \alpha_j(s) ds, \quad \tilde{\beta}_j(t) = \int_0^t \beta_j(s) ds$$

for $j = 0, 1, \dots, d$. Then

$$(2.12) \quad B(t, \xi) = i \sum_{j=1}^d \xi_j (\tilde{\alpha}_j(t) I + \tilde{\beta}_j(t) M_j) + \tilde{\alpha}_0(t) I + \tilde{\beta}_0(t) M_0.$$

By Assumption (A) (i), we have

$$(2.13) \quad e^{B(t, \xi)} = \left(\prod_{j=1}^d e^{i \xi_j \tilde{\alpha}_j(t)} e^{i \xi_j \tilde{\beta}_j(t) M_j} \right) e^{\tilde{\alpha}_0(t)} e^{\tilde{\beta}_0(t) M_0}.$$

For each j , the matrix M_j can be expressed as the sum of a semisimple matrix S_j and a nilpotent matrix N_j that commutes with S_j :

$$(2.14) \quad M_j = S_j + N_j,$$

which implies that

$$(2.15) \quad e^{i\xi_j \tilde{\beta}_j(t) M_j} = e^{i\xi_j \tilde{\beta}_j(t) S_j} \sum_{q=0}^{m-1} \frac{1}{q!} (i\xi_j \tilde{\beta}_j(t) N_j)^q$$

for $j = 1, \dots, d$, and that

$$(2.16) \quad e^{\tilde{\beta}_0(t) M_0} = e^{\tilde{\beta}_0(t) S_0} \sum_{q=0}^{m-1} \frac{1}{q!} (\tilde{\beta}_0(t) N_0)^q,$$

where m is the integer defined in (1.5). It is now straightforward to show that the function defined by (1.6) is C^1 and gives the solution to the Cauchy problem (1.1)–(1.2). \square

3. Examples

The following proposition is useful in constructing examples to which Theorem 1.1 are applicable.

Proposition 3.1. *Let M be a k by k matrix of which eigenvalues are real, and let $M_j := p_j(M)$, $j = 0, 1, \dots, d$, be real polynomials of M . Then Assumption (A) (i) is verified.*

EXAMPLE 3.1. We deal with the Cauchy problem for the partial differential equation of the form

$$(3.1) \quad \frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ 0 & a(t) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix} u,$$

$$u(0, x) = \eta(x) = {}^t[\eta_1(x), \eta_2(x)]$$

where the coefficients are $M_2(\mathbb{C})$ -valued continuous functions, and $a(t)$ and $b(t)$ are real-valued. (Note that (3.1) is the case where $k = 2$ and $d = 1$ in (1.1).) The aim here is to obtain the representation of the solution.

We would like to mention that the equation (3.1) is a special case of the equation that Matsumoto [3] investigated in order to study the hyperbolicity of systems with

double characteristic roots. Indeed, the equation he dealt with is reduced to the equation of the form

$$(3.2) \quad \frac{\partial u}{\partial t} = \begin{bmatrix} a(t, x) & b(t, x) \\ 0 & a(t, x) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t, x) & \beta(t, x) \\ \gamma(t, x) & \delta(t, x) \end{bmatrix} u,$$

where all the coefficients are C^∞ -functions of (t, x) and $b(t, x)\gamma(t, x) \equiv 0$.

Taking into account this, we shall establish a representation formula for the equation (3.1) in the case where

$$b(t) \equiv 0, \quad \alpha(t) \equiv \delta(t), \quad \beta(t) \equiv \gamma(t).$$

In this case, the equation (3.1) becomes

$$(3.3) \quad \frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & 0 \\ 0 & a(t) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \alpha(t) \end{bmatrix} u.$$

It is straightforward to check that the coefficients of (3.3) satisfy Assumption (A). Indeed, with

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have $A_1(t) = a(t)I + M_1$, $A_0(t) = \alpha(t)I + \beta(t)M_0$. Furthermore, we have

$$e^{B(t, \xi)} = e^{i\xi\tilde{\alpha}(t)+\tilde{\alpha}(t)} \left\{ \frac{e^{\tilde{\beta}(t)} + e^{-\tilde{\beta}(t)}}{2} I + \frac{e^{\tilde{\beta}(t)} - e^{-\tilde{\beta}(t)}}{2} M_0 \right\},$$

where we have used the fact that $M_0^2 = I$. Since M_0 is semisimple, the assumption (1.4) on η becomes that $\langle \xi \rangle \hat{\eta} \in L^1(\mathbb{R})$. Then (1.6) defines a C^1 -function and leads to the representation of the solution of the Cauchy problem (3.1):

$$\begin{aligned} u(t, x) &= \frac{1}{2} \left(e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} + e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} \right) \begin{bmatrix} \eta_1(x + \tilde{\alpha}(t)) \\ \eta_2(x + \tilde{\alpha}(t)) \end{bmatrix} \\ &\quad + \frac{1}{2} \left(e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} - e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} \right) \begin{bmatrix} \eta_2(x + \tilde{\alpha}(t)) \\ \eta_1(x + \tilde{\alpha}(t)) \end{bmatrix} \end{aligned}$$

Following the proof of Theorem 1.1, we can deduce a direct generalization of the theorem in the manner described in the next theorem.

Theorem 3.1. *Let*

$$(3.4) \quad A_j(t) = \alpha_j(t)I + \sum_{p: \text{finite}} \beta_{jp}(t)M_{jp}$$

for $j = 0, 1, \dots, d$, with $[M_{jp}, M_{lq}] = 0$ for all pairs (j, p) and (l, q) . Suppose that for each $j = 1, 2, \dots, d$, the eigenvalues of M_{jp} 's are real. Suppose, in addition, that for each $j = 1, 2, \dots, d$, $\alpha_j(t)$ and $\beta_{jp}(t)$'s are real-valued continuous functions, and $\alpha_0(t)$ and $\beta_{0p}(t)$'s are possibly complex-valued continuous functions. Furthermore, suppose that $f(t, x)$ and $\eta(x)$ satisfy the same assumptions as in Theorem 1.1, where $m = 1$ if all the M_{jp} 's are semisimple, and otherwise with m in (1.5) replaced by

$$(3.5) \quad m := \max\{n \mid n \text{ equals the algebraic multiplicity of an eigenvalue of some } M_{jp}, 1 \leq j \leq d, p, \}.$$

Then the solution of the Cauchy problem (1.1)–(1.2) is given by (1.6), which is a C^1 -function in $[0, T] \times \mathbb{R}^d$.

EXAMPLE 3.2 ($d = 1$). We consider the Cauchy problem

$$(3.6) \quad \frac{\partial u}{\partial t} = A(t) \frac{\partial u}{\partial x}$$

$$(3.7) \quad u(0, x) = \eta(x)$$

where

$$(3.8) \quad A(t) = \alpha(t)I + \beta_1(t)M_1 + \beta_2(t)M_2$$

$$(3.9) \quad M_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$(3.10) \quad J_l = \begin{bmatrix} \lambda_l & 1 & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_l & 1 \\ 0 & & & & & \lambda_l \end{bmatrix}, \quad (l = 1, 2)$$

I being the k by k identity matrix, J_1 and J_2 being k_1 by k_1 and k_2 by k_2 matrices respectively, and $k_1 + k_2 = k$. We suppose that $\alpha(t)$, $\beta_1(t)$ and $\beta_2(t)$ are real-valued continuous functions. It is easy to see that all the assumptions of Theorem 3.1 are verified. Thus, if the initial data is assumed to satisfy that $\langle \xi \rangle^m \hat{\eta} \in L^1(\mathbb{R})$, where $m = \max\{k_1, k_2\}$, then Theorem 3.1 gives the C^1 -solution $u(t, x)$ to the Cauchy problem (3.6), (3.7).

We shall compute $u(t, x)$. To this end, we write

$$J_1 = \lambda_1 I_1 + N_1, \quad J_2 = \lambda_2 I_2 + N_2$$

where I_l is the k_l by k_l identity matrix ($l = 1, 2$). Noting that $N_l^{k_l} = 0$, we have

$$(3.11) \quad e^{i\xi \tilde{\beta}_l(t) J_l} = e^{i\xi \tilde{\beta}_l(t) \lambda_l} \sum_{q=0}^{k_l-1} \frac{1}{q!} (i\xi \tilde{\beta}_l(t) N_l)^q, \quad l = 1, 2.$$

Here $\tilde{\beta}_l(t)$ is defined similarly to (2.11). Since

$$e^{B(t,\xi)} = e^{i\xi\tilde{\alpha}(t)} \begin{bmatrix} e^{i\xi\tilde{\beta}_1(t)J_1} & 0 \\ 0 & e^{i\xi\tilde{\beta}_2(t)J_2} \end{bmatrix},$$

the formula (1.6), together with (3.11), leads to the representation

$$u(t, x) = \begin{bmatrix} \sum_{q=0}^{k_1-1} \frac{1}{q!} \tilde{\beta}_1(t)^q N_1^q (\partial_x^q \eta^*) (x + \tilde{\alpha}(t) + \lambda_1 \tilde{\beta}_1(t)) \\ \sum_{q=0}^{k_2-1} \frac{1}{q!} \tilde{\beta}_2(t)^q N_2^q (\partial_x^q \eta^{**}) (x + \tilde{\alpha}(t) + \lambda_2 \tilde{\beta}_2(t)) \end{bmatrix},$$

where $\eta^*(x) = {}^t[\eta_1(x), \dots, \eta_{k_1}(x)]$, $\eta^{**}(x) = {}^t[\eta_{k_1+1}(x), \dots, \eta_k(x)]$ and

$$\eta(x) = \begin{bmatrix} \eta^*(x) \\ \eta^{**}(x) \end{bmatrix}.$$

REMARK. Neither of Theorems 1.1 and 3.1 is applicable to the system of the form

$$\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{bmatrix} \frac{\partial u}{\partial x}$$

which is a special case of the system that was studied in Nishitani [4].

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References

- [1] A. Iserles: *Expansions that grow on trees*, Notices Amer. Math. Soc. **49** (2002), 430–440.
- [2] W. Magnus: *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. **7** (1954), 649–673.
- [3] W. Matsumoto: *On the conditions for the hyperbolicity of systems with double characteristic roots I*, J. Math. Kyoto Univ. **21** (1981), 47–84.
- [4] T. Nishitani: *Hyperbolicity of two by two systems with two independent variables*, Comm. Partial Differential Equations **23** (1998), 1061–1110.

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