ON THE DIRICHLET PROBLEM IN THE AXIOMATIC THEORY OF HARMONIC FUNCTIONS

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In the frame of the recent axiomatic theories of harmonic functions [2], [3], [1], it has been shown that the continuous bounded functions on the boundaries of relatively compact open sets are resolutive [5], [1]. The aim of the present paper is to substitute in these results the continuous functions by Borel-measurable functions and to leave out the restriction that the open sets are relatively compact. H. Bauer has replaced the axiom 3 of Brelot's axiomatic by two weaker axioms: the axiom of separation (Trennungsaxiom) and the axiom K_1. Since the axiom of separation is not fulfilled in some important cases (e.g. the compact Riemann surfaces) we shall weaken this axiom too, substituting it by one of its consequences: the minimum principle for hyperharmonic functions.*)

0. Notations and terminology. We shall use the following notations and terms. A real (resp. numerical) function is a map in the real axis (resp. real axis completed with the points $+\infty$, $-\infty$). For a topological space T we denote by $\mathscr{C}(T)$ the set of real continuous functions on T. If U is an open set in a topological space T, $\partial_T U = \partial U$ (resp. $\overline{U}^T = \overline{U}$) will stand for the boundary (resp. for the closure) of U in T. For a locally compact but non-compact space T we shall denote by $\mathfrak{A}_T$ the filter of sets with relatively compact complements.

1. The axioms. *Let X be a locally compact Hausdorff space and $\mathscr{H}$ a sheaf on X of real vector spaces of real continuous functions called harmonic functions.* We shall suppose that $\mathscr{H}$ will satisfy the axioms $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$ stated in this paragraph.

Let U be an open set of X. An open relatively compact set of X is called **regular in** U *if:*

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*) A similar axiom has been proposed by M. Brelot [4].

a) *$\overline{V} \subset U$ and ∂V is not empty;*

b) *for any $f \in \mathscr{C}(\partial V)$ there exists a unique continuous extension on $\overline{V}$ whose restriction to V, denoted by $H_f^V = H_f$, is harmonic;*

c) *from $f \in \mathscr{C}(\partial V)$ and $f \geq 0$ it follows $H_f^V \geq 0$.*

We shall say simply **regular** *instead of regular in X.*

Let V be regular and $x \in V$. The map $f \to H_f^V(x)$ is a positive linear functional on $\mathscr{C}(\partial V)$ and so, *there exists a Radon measure ω_x^V on ∂V, called* the **harmonic measure of V at the point x,** *such that for any $f \in \mathscr{C}(\partial V)$*

$$H_f^V(x) = \int f d\omega_x^V.$$

Axiom H_1. *The regular sets form a basis of X.*

A point $x \in X$ will be called a **zero-point** *if any harmonic function on a neighbourhood of x vanishes at x; we shall denote by X_0 the set of zero-points.*

THEOREM 1. *X_0 is closed, nowhere dense and totally disconnected.*

Let $x \in \overline{X}_0$ and let u be a harmonic function defined on a neighbourhood U of x. Since u vanishes at the points of $X_0 \cap U$ it vanishes also at x. Hence $x \in X_0$ and X_0 is closed.

Let $x \in X_0$ and V be a regular neighbourhood of x. Then

$$\left\{ y \in V \mid H_1^V(y) < \frac{1}{2} \right\}$$

is an open neighbourhood of x contained in V whose boundary does not intersect X_0. X_0 is therefore nowhere dense and totally disconnected.

A numerical function on an open non-empty set U is called **hyperharmonic on** *U if:*

a) *it does not take the value $-\infty$;*

b) *it is lower semi-continuous;*

c) *for any point $x \in U$ there exists a neighbourhood $U_s(x) \subset U$ of x, such that for any regular set V in $U_s(x)$ and any $y \in V$*

$$s(y) \geq \int^* s d\omega_y^V.$$

A numerical function s on U is called **hypoharmonic on** *U if $-s$ is hyperharmonic on U.*

The harmonic functions are hyperharmonic and a function which is simultaneously hyperharmonic and hypoharmonic is harmonic. The sum and the minimum of two hyperharmonic functions and the product of a positive real number with a hyperharmonic function is also hyperharmonic. The hyperharmonic functions form a sheaf on X.

We shall say that the **minimum principle** *is valid on an open non-empty set U if one of the following conditions is fulfilled:*

a) *U is compact and all hyperharmonic functions on U are non-negative;*

b) *U is non-compact and any hyperharmonic function s on U for which*

$$\lim\ \inf_{\mathfrak{A}_U} s \geq 0$$

is non-negative.

The open non-empty sets on which the minimum principle is valid will be called **M. P.-sets.**

THEOREM 2. *Let U, U' be open non-empty sets, $U' \subset U$, and s (resp. s') be a hyperharmonic function on U (resp. U'). We suppose that the function s^*, defined on U equal to s on $U - U'$ and equal to min (s, s') on U', is lower semi-continuous. If any point $x \in U \cap \partial U'$ possesses a neighbourhood U_x such that either $s \leq s'$ on $U_x \cap U'$ or $U_x \cap U'$ is an M.P.-set, then s^* is a hyperharmonic function.*

The conditions a) and b) of the definition of the hyperharmonic functions are satisfied trivially by s^* and the same is true for the condition c) at the points of $U - \partial U'$. Let x be a point of $U \cap \partial U'$ and U_x be a neighbourhood which satisfies the condition from the statement. There exists a neighbourhood W of x, $W \subset U_x$, which satisfies the condition c) for s. We shall prove that W fulfils the condition c) also for s^*. This is trivial if $s \leq s'$ on $U_x \cap U'$ since then $s^* = s$ on W. On the contrary case let V be a regular set in W and $f \in \mathscr{C}(\partial V)$, $f < s^*$. The function $s^* - H_f^V$ is lower semi-continuous on V, non-negative on $V - U'$, hyperharmonic on $V \cap U'$, and

$$\liminf_{V \ni y \to z} (s^*(y) - H_f^V(y)) > 0$$

for any $z \in \partial V$. The function s_0 defined on $U_x \cap U'$ equal to 0 on $(U_x \cap U') - V$ and equal to min $(s^* - H_f^V, 0)$ on $V \cap U'$ is hyperharmonic by the above proof, where $U_x \cap U'$ (resp. $V \cap U'$) replaces U (resp. U') and 0 (resp. $s^* - H_f^V$) replaces s (resp. s'). If $U_x \cap U'$ is non-compact then it can be easily verified that

$$\lim \inf_{\mathfrak{A}_{U_x \cap U'}} s_0 \geq 0.$$

Since $U_x \cap U'$ is an M.P.-set we get $s_0 \geq 0$, $s^* \geq H_f^V$ on V. f being arbitrary we get for any $y \in V$

$$s^*(y) \geq \int^* s^* d\omega_y^V.$$

COROLLARY 1. *Let U be an M.P.-set and U' be an open non-empty subset of U. If any point $x \in U \cap \partial U'$ possesses a neighbourhood U_x such that either there exists a finite hyperharmonic function s_x on $U_x \cap U'$ with* $\inf s^* > 0$ *or $U_x \cap U'$ is an M.P.-set, then U' is an M.P.-set. The intersection of any regular set with U is an M.P.-set.*

Let s' be a hyperharmonic function on U' such that if U' is non-compact

$$\lim \inf_{\mathfrak{A}_{U'}} s' \geq 0.$$

We suppose firstly that there exists a finite hyperharmonic function s_0 on U' with $\inf s_0 > 0$. Let ε be a positive number and s denote the function on U equal to 0 on $U - U'$ and equal to $\min (s' + \varepsilon s_0, 0)$ on U'. From the theorem it follows that s is hyperharmonic. If U is non-compact it can be verified easily that

$$\lim \inf_{\mathfrak{A}_U} s \geq 0.$$

Since U is an M.P.-set we get $s \geq 0$, $s' \geq -\varepsilon s_0$. ε being arbitrary and s_0 finite it follows $s' \geq 0$; U' is therefore an M.P.-set.

From this proof we see that for any point $x \in U \cap \partial U'$ the set $U_x \cap U'$ is an M.P.-set. Let s^* be the function on U equal to 0 on $U - U'$ and equal to $\min (s', 0)$ on U'. From the theorem it follows that s^* is hyperharmonic. If U is non-compact it can be verified easily that

$$\lim \inf_{\mathfrak{A}_U} s^* \geq 0.$$

Since U is an M.P.-set we get $s^* \geq 0$, $s' \geq 0$; U' is therefore an M.P.-set.

If V is a regular set, the set $U' = V \cap U$ fulfils the required conditions taking $U_x = U - \left\{ y \in V | H_1^V \leq \frac{1}{2} \right\}$ and $s_x = H_1^V$ for any $x \in U \cap \partial U' = U \cap \partial V$.

COROLLARY 2. *Let U be an open non-empty set, s be a hyperharmonic function on U and V be a regular set in U. If V is an M.P.-set then the function s^V defined on U equal to s on $U-V$ and equal to*

$$x \to \int^* s d\omega_x^V$$

on V, is hyperharmonic and not greater than s.

Let $f \in \mathscr{C}(\partial V)$, $f \leq s$. Since V is an M.P.-set $s \geq H_f^V$ on V

$$s \geq \sup_f H_f^V = s^V.$$

Being on $\overline{V}$ the least upper bound of a family of continuous functions, s^V is lower semi-continuous on $\overline{V}$. It follows immediately that s^V is lower semi-continuous on U. In order to show that s^V is hyperharmonic on V let us take a regular set V' in V. We have for any $x \in V'$ and $f \in \mathscr{C}(\partial V)$, $f \leq s$,

$$s^V(x) \geq H_f^V(x) = \int H_f^V d\omega_x^{V'}, \qquad s^V(x) \geq \sup_f \int H_f^V d\omega_x^{V'} = \int^* s^V d\omega_x^{V'}.$$

Taking $U_x = U$ for any $x \in \partial V$ it follows from the theorem that s^V is hyperharmonic.

Axiom $\mathbf{H}_2$. *The M.P.-sets form a covering of X.*

This axiom is a theorem in Brelot's [2], [3] and Bauer's [1] axiomatic since in these axiomatics all regular sets are M.P.-sets (Theorem 3 (ii) and 4 ([3] part IV) and Korollar of Lemma 2 [1]).

We shall denote by $\mathfrak{B}$ the set of regular M.P.-sets. From $\mathbf{H}_1$, $\mathbf{H}_2$ and the corollary 1 it follows that $\mathfrak{B}$ is a basis of X. For any open non-empty set U and any hyperharmonic function s on U and $V \in \mathfrak{B}$, $\overline{V} \subset U$, we see by corollary 2 that s^V is hyperharmonic and for any $x \in V$

$$s(x) \geq \int^* s d\omega_x^V.$$

From this fact it follows that any M.P.-set can be taken instead of $U_s(x)$ in the definition of hyperharmonic functions, *this means independently of s.*

THEOREM 3. *The least upper bound of an upper directed set of hyperharmonic functions is also hyperharmonic.*

Let $\mathcal{S}$ be such a set of hyperharmonic functions and s_0 its least upper bound. s_0 is lower semi-continuous, does not take the value $-\infty$, and for any $V \in \mathfrak{B}$ and $x \in V$ we have

$$s_0(x) = \sup_{s \in \mathcal{S}} s(x) \geq \sup_{s \in \mathcal{S}} \int^* s d\omega_x^V = \int^* s_0 d\omega_x^V.$$

Axiom $\mathbf{H}_3$*). *For any open non-empty set U the least upper bound of any upper directed set of equally bounded harmonic functions on U is harmonic.*

An equivalent statement of this axiom is: *on any regular set V the function*

$$x \to \int f d\omega_x^V$$

is harmonic on V for any bounded lower semi-continuous function f.

Let U be an open non-empty set. *A set $\mathcal{S}$ of hyperharmonic functions on U is called a* **Perron** set *if it is lower directed and for any $V \in \mathfrak{B}$, $\overline{V} \subset U$, and $s \in \mathcal{S}$ it follows $s^V \in \mathcal{S}$, where s^V denotes the function defined in Corollary 2.*

THEOREM 4. ([1] Satz 11) *The greatest lower bound of a locally equally bounded Perron set is harmonic.*

Let u denote the greatest lower bound of a locally equally bounded Perron set $\mathcal{S}$ on an open non-empty set U and let $V \in \mathfrak{B}$, $\overline{V} \subset U$, such that $\mathcal{S}$ is equally bounded on $\overline{V}$. Then u is equal on U to the greatest lower bound of the set $\{s^V \mid s \in \mathcal{S}\}$ and therefore harmonic by $\mathbf{H}_3$.

A potential *is a non-negative hyperharmonic function for which any hypoharmonic minorant is non-positive.* If $\underline{s}$ is a hypoharmonic minorant of $p+s$, where p is a potential and s a hyperharmonic function, then $\underline{s} \leq s$, since $\underline{s} - s$ is a hypoharmonic minorant of p. It follows that the sum of a finite number of potentials is again a potential. It follows further that if the sum of a series of potentials is finite it is also a potential. A non-negative locally bounded hyperharmonic function can be set in exactly one way as a sum of a potential and a harmonic function.

LEMMA 1. *Let p be a locally bounded potential on X. There exists for any $x \in X$ a non-negative hyperharmonic function p_x, finite at x, such that for any*

*) This axiom was introduced firstly by H. Bauer [1] (Axiom K_1).

filter $\mathfrak{F}$ on X with empty adherence for which

$$\lim \inf_{\mathfrak{F}} p > 0$$

we have

$$\lim_{\mathfrak{F}} p_x = \infty.$$

Let $\mathscr{S}$ be the smallest set of hyperharmonic functions which contains p and such that for any $s \in \mathscr{S}$ and $V \in \mathfrak{B}$, $s^V \in \mathscr{S}$. From Theorem 4 it follows that the greatest lower bound of $\mathscr{S}$ is a harmonic function. This function vanishes because it is non-negative and not greater than p. For any natural number n let $p_n \in \mathscr{S}$, $p_n(x) < \frac{1}{2^n}$. The function $p_x = \sum_{n=1}^{\infty} p_n$ fulfils the required condition.

2. The normed Dirichlet problem. Let U be an open set for which ∂U is non-empty and let f be a numerical function on ∂U. We denote by $\overline{\mathscr{S}}_f^{U,X} = \overline{\mathscr{S}}_f^U$ (resp $\underline{\mathscr{S}}_f^{U,X} = \underline{\mathscr{S}}_f^U$) the set of lower bounded hyperharmonic (resp. upper bounded hypoharmonic) functions s, such that

$$\liminf_{U \ni x \to y} s(x) \geq f(y) \qquad (\text{resp. } \limsup_{U \ni x \to y} s(x) \leq f(y))$$

for any $y \in \partial U$ and there exists a compact subset K_s of X such that $s \geq 0$ (resp. $s \leq 0$) on $U - K_s$. We denote by $\overline{H}_f^{U,X} = \overline{H}_f^U = \overline{H}_f$ (resp. $\underline{H}_f^{U,X} = \underline{H}_f^U = \underline{H}_f$) the greatest lower bound of $\overline{\mathscr{S}}_f^U$ (resp. the least upper bound of $\underline{\mathscr{S}}_f^U$). *The open set U is called an* **M.P$_0$.-set** *if $\overline{H}_0^{U,X} = 0$.* Obviously an M.P.-set for which ∂U is non-empty, is an M.P$_0$.-set. From the proof of corollary 1 it results that U is an M.P$_0$.-set if and only if for any open relatively compact subset U' of X such that $U \cap X_0 \cap \partial U'$ is empty $U' \cap U$ is an M.P.-set. *If U is an M.P$_0$.-set we have $\underline{H}_f^U \leq \overline{H}_f$.*

If $\{f_n\}$ is a decreasing sequence such that $\underline{H}_{f_n}$ are harmonic then

$$\lim_{n \to \infty} \underline{H}_{f_n} = \underline{H}_{\lim f_n}.$$

Indeed let x be a point of U and for any n, $s_n \in \underline{\mathscr{S}}_{f_n}^U$ such that $s_n(x) > \underline{H}_{f_n}(x) - \frac{1}{2^n}$. The functions

$$s = \lim_{n \to \infty} \underline{H}_{f_n}, \qquad s'_m = \sum_{n=m}^{\infty} (s_n - \underline{H}_{f_n})$$

are hypoharmonic (Theorem 3) and $s + s'_m \in \underline{\mathscr{S}}_{f_n}^U$ for any n. Hence $s + s'_m \in \underline{\mathscr{S}}_{\lim_{n \to \infty} f_n}^U$,

$$s(x)-\frac{1}{2^{m-1}}<s(x)+s'_m(x)\le \underline{H}_{\lim\limits_{n\to\infty} f_n}(x),$$

$$s(x)\le \underline{H}_{\lim\limits_{n\to\infty} f_n}(x)\le \lim_{n\to\infty}\underline{H}_{f_n}(x)=s(x).$$

We have

$$\overline{H}_{f_1+f_2}\le \overline{H}_{f_1}+\overline{H}_{f_2},\qquad \underline{H}_{f_1+f_2}\ge \underline{H}_{f_1}+\underline{H}_{f_2},$$

wherever the right side has a sense, where f_1+f_2 is defined arbitrarily on the set

$$\{y\in\partial U|f_1(y)=+\infty,\ f_2(y)=-\infty\}\cap\{y\in\partial U|f_1(y)=-\infty,\ f_2(y)=+\infty\},$$

and for $\alpha>0$

$$\overline{H}_{\alpha f}=\alpha\overline{H}_f,\qquad \underline{H}_{\alpha f}=\alpha\underline{H}_f,\qquad \overline{H}_{-f}=-\underline{H}_f.$$

If the functions $\overline{H}_f$, $\underline{H}_f$ are finite (resp. *harmonic*) *and equal the function f is called* **resolutive** (resp. **harmonic resolutive**) *and*

$$H_f^{U,X}=H_f^U=H_f=\overline{H}_f=\underline{H}_f$$

is called the **normed solution of Dirichlet problem** *with f as boundary function. If U is an $M.P_0$.-set, the set of resolutive (resp. harmonic resolutive) real functions on ∂U form a real vector space. If f_1, f_2 are non-negative harmonic resolutive functions then* max (f_1, f_2) *is resolutive and $H_{\max(f_1,f_2)}$ is the least harmonic majorant of H_{f_1}, H_{f_2}.*

LEMMA 2. *Let U be an open set, $y\in\partial U$ and $\mathfrak{F}$ be a filter on U converging to y. We suppose that there exists a fundamental system $\mathfrak{V}$ of regular neighbourhoods of y such that for any $V\in\mathfrak{V}$ there exists a non-negative hyperharmonic function s_V on $V\cap U$ such that*

$$\lim_{\mathfrak{F}} s_V=0$$

and for any $z\in U\cap\partial V$

$$\liminf_{U\cap V\ni x\to z} s_V(x)>0.$$

Let s be a hyperharmonic (resp. hypoharmonic) function on the intersection of U with a neighbourhood of y for which

$$\liminf_{U\ni x\to y} s(x)=1\qquad (\text{resp. }\limsup_{U\ni x\to y} s(x)=1).$$

Let f be a non-negative numerical function on ∂U, for which $\overline{H}_f^U$ (resp. $\underline{H}_f^U$) *is harmonic on U and bounded on a neighbourhood of y. Then*

$$\limsup_{\mathfrak{F}} \overline{H}_f^U \leq (\limsup_{\partial U \ni x \to y} f(x))\,(\limsup_{\mathfrak{F}} s(x))$$

$$(\text{resp. } \liminf_{\mathfrak{F}} \underline{H}_f^U \geq (\liminf_{\partial U \ni x \to y} f(x))\,(\liminf_{\mathfrak{F}} s(x)))^{*)}.$$

We shall prove this lemma following the proof of Theorem 22 [3] Part IV. We suppose that

$$\limsup_{\partial U \ni x \to y} f(x) < \infty \qquad (\text{resp. } \liminf_{\partial U \ni x \to y} f(x) > 0)$$

since on the contrary case the assertion is trivial. Let α be a positive number

$$\alpha > \limsup_{\partial U \ni x \to y} f(x) \qquad (\text{resp. } \alpha < \liminf_{\partial U \ni x \to y} f(x)),$$

ε be a positive number smaller than 1, and $V \in \mathfrak{V}$ such that $\overline{H}_f^U$ (resp. $\underline{H}_f^U$) is bounded on $\overline{V} \cap U$, $f < \alpha$ (resp. $f > \alpha$) on $V \cap \partial U$, and $s > 1-\varepsilon$ (resp. $s < 1+\varepsilon$) on $\overline{V} \cap U$. Let further K be a compact set in $U \cap \partial V$ for which

$$\omega_y^V(U \cap \partial V - K) < \varepsilon.$$

There exists a positive number β such that for any $z \in K$

$$\liminf_{U \cap V \ni x \to z} \left(\frac{\alpha}{1-\varepsilon} s(x) + \beta s_V(x)\right) > \overline{H}_f^U(z)$$

$$\left(\text{resp. } \limsup_{U \cap V \ni x \to z} \left(\frac{\alpha}{1+\varepsilon} s(x) - \beta s_V(x)\right) < \underline{H}_f^U(z)\right).$$

We denote by h the harmonic function on V

$$x \to \omega_x^V(U \cap \partial V - K),$$

by γ the number $\sup_{x \in \overline{V} \cap U} \overline{H}_f^U(x)$ (resp. $\sup_{x \in \overline{V} \cap U} \underline{H}_f^U(x)$), and by s_0 the function defined on U, equal to $\overline{H}_f^U$ (resp. $\underline{H}_f^U$) on $U - V$ and equal to

$$\min\left(\overline{H}_f^U, \frac{\alpha}{1-\varepsilon} s + \beta s_V + \gamma h\right)$$

$$\left(\text{resp. } \max\left(\underline{H}_f^U, \frac{\alpha}{1+\varepsilon} s - \beta s_V - (\alpha+\gamma) h\right)\right)$$

on $U \cap V$. From Theorem 2 we see that s_0 is a hyperharmonic (resp. hypohar-

*) With the convention $0 \cdot \infty = \infty$, $\infty \cdot 0 = 0$.

monic) function. For any $s' \in \overline{\mathscr{S}}_f^U$ (resp. $\underline{\mathscr{S}}_f^U$) it can be verified that $s' - \overline{H}_f^U + s_0 \in \overline{\mathscr{S}}_f^U$ (resp. $s' - \underline{H}_f^U + s_0 \in \underline{\mathscr{S}}_f^U$). Hence

$$s' - \overline{H}_f^U + s_0 \geq \overline{H}_f^U \qquad (\text{resp. } s' - \underline{H}_f^U + s_0 \leq \underline{H}_f^U),$$
$$s_0 \geq \overline{H}_f^U \qquad (\text{resp. } s_0 \leq \underline{H}_f^U),$$
$$\lim\,\sup_{\mathfrak{F}} \overline{H}_f^U \leq \lim\,\sup_{\mathfrak{F}} s_0 \leq \lim\,\sup_{\mathfrak{F}} \left(\frac{\alpha}{1-\varepsilon} s + \beta s_V + \gamma h\right) \leq$$
$$\leq \frac{\alpha}{1-\varepsilon} \lim\,\sup_{\mathfrak{F}} s + \gamma\varepsilon$$
$$\left(\text{resp. } \lim\,\inf_{\mathfrak{F}} \underline{H}_f^U \geq \lim\,\inf_{\mathfrak{F}} s_0 \geq \lim\,\inf_{\mathfrak{F}} \left(\frac{\alpha}{1+\varepsilon} s - \beta s_V - (\alpha+\gamma) h\right) = \right.$$
$$\left. \geq \frac{\alpha}{1+\varepsilon} \lim\,\inf_{\mathfrak{F}} s - (\alpha+\gamma)\varepsilon\right).$$

ε and α being arbitrary, we get

$$\lim\,\sup_{\mathfrak{F}} \overline{H}_f^U \leq (\limsup_{\partial U \ni x \to y} f(x))\,(\lim\,\sup_{\mathfrak{F}} s)$$
$$(\text{resp. } \lim\,\inf_{\mathfrak{F}} \underline{H}_f^U \geq (\liminf_{\partial U \ni x \to y} f(x))\,(\lim\,\inf_{\mathfrak{F}} s)).$$

REMARK. If in this lemma s is harmonic and $\lim_{U \ni x \to y} s(x) = 1$, then it can be proved in the same way that

$$\lim\,\sup_{\mathfrak{F}} \overline{H}_f^U \leq \limsup_{\partial U \ni y \to x} f(x),$$
$$\lim\,\inf_{\mathfrak{F}} \underline{H}_f^U \geq \liminf_{\partial U \ni x \to y} f(x)$$

for any f (not necessarily non-negative) provided that $\overline{H}_f^U$, $\underline{H}_f^U$ are bounded. This remark will be not used in the sequel.

Let U be an open set and $y \in \partial U$. For any non-negative hyperharmonic (resp. hypoharmonic) function s, defined on the intersection of U with a neighbourhood of y for which

$$\liminf_{U \ni x \to y} s(x) = 1 \qquad (\text{resp. } \limsup_{U \ni x \to y} s(x) = 1),$$

we set

$$\sigma_U^*(y, s) = \limsup_{U \ni x \to y} s(x) \qquad (\text{resp. } \sigma_{*U}(y, s) = \liminf_{U \ni x \to y} s(x)),$$
$$\sigma_U^*(y) = \sigma^*(y) = \inf_s \sigma_U^*(y, s) \qquad (\text{resp. } \sigma_{*U}(y) = \sigma_*(y) = \sup_s \sigma_{*U}(y, s)).$$

If there does not exist an s with the required conditions we set $\sigma^*(y) = \infty$ (resp. $\sigma_*(y) = 0$). We observe that the existence of a harmonic function on a neighbourhood of y which is different from zero in y implies $\sigma^*(y) = \sigma_*(y) = 1$.

Hence if $\sigma^*(y)>1$ or if $\sigma_*(y)<1$, y is a zero point. The function σ^* (resp. σ_*) is upper (resp. lower) semi-continuous as can be easily verified. We set

$$A_U^* = A^* = \{y \in \partial U \mid \sigma^*(y) = 1\}, \qquad A_{*U} = A_* = \{y \in \partial U \mid \sigma_*(y) = 1\}.$$

Of course $\partial U - A^* \cap A_* \subset X_0$.

LEMMA 3. *Let U be an open set with non-empty boundary, p be a locally bounded potential on U, positive on a neighbourhood of any point of $\partial U - X_0$, and f be a non-negative function on ∂U. a) If f is lower semi-continuous and $\underline{H}_f$ harmonic on U bounded in a neighbourhood of any boundary point of U then*

$$\overline{H}_{f\sigma_*} \leq \underline{H}_f\ ^{*)}.$$

b) If f is upper semi-continuous, $\overline{H}_f$ harmonic on U and bounded in a neighbourhood of any boundary point of U, and if U is either relatively compact or an M.P$_0$.-set then

$$\overline{H}_f \leq \underline{H}_{f\sigma^*}\ ^{*)}.$$

Let x be a point of U and p_x be the hyperharmonic function associated to x and p by Lemma 1. Let ε be a positive number and $y \in \partial U$. We want to prove that

$$\limsup_{U \ni z \to y} (\overline{H}_f(z) - \varepsilon p_x(z)) \leq f(y)\sigma^*(y)$$

$$(\text{resp. } \liminf_{U \ni z \to y} (\underline{H}_f(z) + \varepsilon p_x(z)) \geq f(y)\sigma_*(y)).$$

Let $y \notin X_0$ and $\mathfrak{U}$ be an ultrafilter on U converging to y for which

$$\lim_{\mathfrak{U}} (\overline{H}_f - \varepsilon p_x) = \limsup_{U \ni z \to y} (\overline{H}_f(z) - \varepsilon p_x(z))$$

$$(\text{resp. } \lim_{\mathfrak{U}} (\underline{H}_f + \varepsilon p_x) = \liminf_{U \ni z \to y} (\underline{H}_f(z) + \varepsilon p_x(z))).$$

If

$$\lim_{\mathfrak{U}} p \neq 0$$

then

$$\lim_{\mathfrak{U}} p_x = \infty$$

and the required inequality is proved. In the opposite case the inequality follows from Lemma 2 taking $\mathfrak{V}$ equal to the set of all regular neighbourhoods

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*) With the convention $\infty \cdot 0 = 0$, $0 \cdot \infty = \infty$.

of y and for any $V\in\mathfrak{B}$, $s_V=p$; in the role of the function s we can take here any harmonic function in a neighbourhood of y equal to 1 at y. Suppose now $y\in X_0$. Then for any regular neighbourhood V of y we have $H_1^V(y)=0$. Let s be a hyperharmonic (resp. hypoharmonic) function on the intersection of U with a neighbourhood of y with

$$\liminf_{U\ni z\to y} s(z)=1 \qquad (\text{resp. } \limsup_{U\ni z\to y} s(z)=1).$$

Taking in Lemma 2, $\mathfrak{B}$ the set of all regular neighbourhoods of y, for any $V\in\mathfrak{B}$ H_1^V as s_V, and the trace of $\mathfrak{B}$ on U as $\mathfrak{F}$ we get from this lemma

$$\limsup_{U\ni z\to y} \overline{H}_f(z)\leq f(y)\,\sigma^*(y, s)$$

$$(\text{resp. } \liminf_{U\ni z\to y} \underline{H}_f(z)\geq f(y)\,\sigma_*(y, s)).$$

s being arbitrary the assertion is proved also in this case.

a) $\underline{H}_f+\varepsilon p_x\in\overline{\mathscr{S}}^U_{f\sigma_*}$. Indeed the above proof shows that the condition at the points of ∂U is fulfilled. Since $\underline{H}_f$, p_x are non-negative, $\underline{H}_f+\varepsilon p_x$ is non-negative on U. From this relation we get

$$\overline{H}_{f\sigma_*}\leq \underline{H}_f+\varepsilon p_x.$$

p_x being finite at x and ε and x being arbitrary we obtain

$$\overline{H}_{f\sigma_*}\leq \underline{H}_f.$$

b) Suppose now that U is relatively compact. Then from the first part of the proof we have $\overline{H}_f-\varepsilon p_x\in\underline{\mathscr{S}}^U_{f\sigma^*}$,

$$\overline{H}_f-\varepsilon p_x\leq \underline{H}_{f\sigma^*}.$$

p_x being finite at x and ε and x being arbitrary we obtain

$$\overline{H}_f\leq \underline{H}_{f\sigma^*}.$$

Suppose now that U is a non-relatively compact M.P$_0$.-set. Let G be a relatively compact open set whose boundary does not intersect X_0. From the proof of Corollary 1 it follows that $G\cap U$ is an M.P.-set. Let f_G be the function defined on $\partial(G\cap U)$ equal to 0 on $\overline{G}\cap\partial U$ and equal to $\overline{H}^U_f$ on $U\cap\partial G$. Since U is an M.P$_0$.-set $\overline{H}^U_f$ is non-negative. From $\overline{H}^U_f\in\overline{\mathscr{S}}^{G\cap U}_{f_G}$ and the fact that $G\cap U$ is an M.P$_0$.-set we deduce that $\overline{H}^{G\cap U}_{f_G}$ is non-negative harmonic and bounded. The function s_G on U equal to $\overline{H}_f^U$ on $U-\overline{G}$ equal to $\overline{H}_{fG}^{G\cap U}$ on $G\cap U$ and equal to

$$\liminf_{G\cap U\ni z\to y} \overline{H}_{fG}^{G\cap U}(z)$$

at any $y\in U\cap\partial G$ is hyperharmonic. Indeed s_G fulfils the conditions a) and b) from the definition of hyperharmonic functions. Let $\overline{s}\in\overline{\mathscr{S}}_{fG}^{G\cap U}$. The function $\overline{s}_G$ on U equal to $\overline{H}_f^U$ on $U-G$ and equal to min $(\overline{s}, \overline{H}_f^U)$ on $G\cap U$ is hyperharmonic by Theorem 2. Let V be a regular set in U and $f\in\mathscr{C}(\partial V)$, $f\leq s_G$. We have on V

$$\overline{s}_G\geq H_f^V.$$

From this inequality we get on $V-\partial G$

$$s_G=\inf_{\overline{s}\in\overline{\mathscr{S}}_{fG}^{G\cap U}}\overline{s}_G\geq H_f^V.$$

For any $y\in V\cap\partial G$ we have

$$s_G(y)=\liminf_{G\cap U\ni z\to y}s_G(z)\geq H_f^V(y).$$

s_G is therefore hyperharmonic.

Let $\mathfrak{G}$ denote the set of open relatively compact sets G for which $X_0\cap\partial G=\phi$. By Theorem 1 X_0 is closed and totally disconnected. Hence for any compact set K, there exists an open set $G\in\mathfrak{G}$ containing K. The family $(s_G)_{G\in\mathfrak{G}}$ is contained in a Perron set with the same greatest lower bound since from G_1, G_2, $G\in\mathfrak{G}$, $G\supset\overline{G_1\cup G_2}$, it follows

$$s_G\leq s_{G_1},\qquad s_G\leq s_{G_2}.$$

All functions s_G being non-negative and dominated by $\overline{H}_f^U$ the greatest lower bound u of this family is harmonic by Theorem 4 and non-negative. Let η be a positive number, and $y\in\partial U$. Taking a $G\in\mathfrak{G}$, which contains y we get from the first part of the proof

$$\limsup_{U\ni z\to y}(u(z)-\eta p_x(z))\leq\limsup_{G\cap U\ni z\to y}(\overline{H}_{fG}^{G\cap U}(z)-\eta p_x(z))\leq 0$$

if $y\notin X_0$. If $y\in X_0$, let V be a regular neighbourhood of y, $\overline{V}\subset G$, and α a sufficiently great number such that

$$\overline{H}_f^U<\alpha$$

on $\overline{V}\cap U$. We denote by s_0 the function on $U\cap G$ equal to $\overline{H}_f^U$ on $U\cap G-V$

and equal to min $(\overline{H}_f^U, \alpha H_1^V)$ on $V \cap U$. From Theorem 2 it follows that s_0 is hyperharmonic. s_0 belongs to $\overline{\mathscr{S}}_{f_G}^{U\cap G}$ and we get

$$\lim_{U\ni z\to y} \sup (u(z) - \eta p_x(z)) \leq \lim_{U\cap G\ni z\to y} \sup \overline{H}_{f_G}^{U\cap G}(z) \leq \lim_{U\cap G\ni z\to y} \sup s_0(z) \leq \alpha H_1^V(y) = 0.$$

Hence for any $\bar{s} \in \overline{\mathscr{S}}_f^U$ and $y \in \partial U$ we have

$$\lim_{U\ni z\to y} \inf (\bar{s}(z) - u(z) + \eta p_x(z)) \geq f(y).$$

From this inequality and from $\bar{s} \geq u$ it follows $\bar{s} - u + \eta p_x \in \overline{\mathscr{S}}_f^U$,

$$\bar{s}(x) - u(x) + \eta p_x(x) \geq \overline{H}_f^U(x).$$

$\bar{s}$ and η being arbitrary and $p_x(x)$ finite we deduce $u(x) = 0$. Let $\{G_n\}$ be a sequence from $\mathfrak{G}$ for which

$$\sum_{n=0}^{\infty} s_{G_n}(x) < \infty$$

and let us denote by s_x the hyperharmonic function

$$s_x = \sum_{n=0}^{\infty} s_{G_n}.$$

It is easy to verify that $\overline{H}_f - \varepsilon s_x$ is non-positive outside a compact set of X. From this and from the first part of the proof it follows

$$\overline{H}_f - \varepsilon p_x - \varepsilon s_x \in \underline{\mathscr{S}}_{f\sigma^*}^U, \qquad \overline{H}_f - \varepsilon p_x - \varepsilon s_x \leq \underline{H}_{f\sigma^*}.$$

p_x and s_x being finite at x and ε and x being arbitrary we obtain

$$\overline{H}_f \leq \underline{H}_{f\sigma^*}.$$

THEOREM 5. *Let U be an $M.P_0$.-set, p be a locally bounded potential on U, positive on a neighbourhood of any point of $\partial U - X_0$, and s be a non-negative hyperharmonic function on U bounded in the neighbourhood of any point of $\overline{U}$ and for which*

$$\lim_{U\ni x\to y} \inf s(x) \geq 1$$

for any $y \in A_*$. If f is a bounded lower semi-continuous function on ∂U, then $f\chi_{A_*}$ is harmonic resolutive, where χ_{A_*} is the characteristic function of $\dot{A}_*$.

Let us denote

$$B = \left\{y \in \partial U \mid \lim_{U\ni x\to y} \inf s(x) > \frac{1}{2}\right\}.$$

B is an open set on ∂U and $A_* \subset B$. Suppose firstly $f \geq 0$. Then, for a suitable positive number α, $\alpha s \in \overline{\mathscr{S}}^U_{f\chi_B\sigma_*^n}$, where χ_B is the characteristic function of B and n a natural number. Hence $\underline{H}_{f\chi_B\sigma_*^n}$ is harmonic and bounded in a neighbourhood of any boundary point of U. We get by the preceding lemma

$$\overline{H}_{f\chi_{A_*}} \leq \overline{H}_{f\chi_B\sigma_*^{n+1}} \leq \underline{H}_{f\chi_B\sigma_*^n}.$$

It follows

$$\underline{H}_{f\chi_{A_*}} \leq \overline{H}_{f\chi_{A_*}} \leq \lim_{n\to\infty} \underline{H}_{f\chi_B\sigma_*^n} = \underline{H}_{f\chi_{A_*}}.$$

For a general f denote

$$\alpha = \inf f.$$

The functions χ_{A_*}, $(f-\alpha)\chi_{A_*}$ being resolutive, the function

$$f\chi_{A_*} = (f-\alpha)\chi_{A_*} + \alpha\chi_{A_*}$$

is also resolutive.

If U is an MP_0-set, a set $M \subset \partial U$ is called **negligible** *if $\overline{H}_{\chi_M} = 0$, where χ_M denotes the characteristic function of M. A set M is negligible if and only if for any $x \in U$ there exists a non-negative hyperharmonic function s_x finite at x for which*

$$\liminf_{U \ni z \to y} s_x(z) = \infty$$

for any $y \in M$. The condition is obviously sufficient. If M is negligible there exists for any natural number n an $s_n \in \overline{\mathscr{S}}^U_{\chi_M}$ such that

$$s_n(x) < \frac{1}{2^n}.$$

and we can take

$$s_x = \sum_{n=1}^{\infty} s_n.$$

If two functions f, g on ∂X differ only on a negligible set, then

$$\underline{H}_f = \underline{H}_g, \qquad \overline{H}_f = \overline{H}_g.$$

Indeed we have for $x \in U$ and $\varepsilon > 0$, $\varepsilon s_x \in \overline{\mathscr{S}}^U_{|f-g|}$, and therefore

$$\overline{H}_{|f-g|} = 0,$$
$$\overline{H}_f \leq \overline{H}_{|f-g|} + \overline{H}_g = \overline{H}_g.$$

THEOREM 6. *Let U be an $M.P_0$.-set, p be a locally bounded potential on U, positive on a neighbourhood of any point of $\partial U - X_0$, and f_0 be a non-negative upper (resp. lower) semi-continuous function on ∂U for which $\overline{H}_{f_0}$ (resp. $\underline{H}_{f_0}$) is harmonic on U and bounded in a neighbourhood of any boundary point of U. If $\partial U - A^*$ (resp. $\partial U - A_*$) is negligible then any Borel function**) f, $|f| \leq f_0$, *is harmonic resolutive.*

Let f be a non-negative upper (resp. lower) semi-continuous function on ∂U, $f \leq f_0$. $\overline{H}_f$ (resp. $\underline{H}_f$) is harmonic and bounded in a neighbourhood of any boundary point of U. This is obvious for $\underline{H}_f$. Let $\mathscr{S}$ be the smallest Perron set containing the set $\{\min(\overline{s}, \overline{H}_{f_0}) \mid \overline{s} \in \overline{\mathscr{S}}^U_{f_0}\}$. For any $s' \in \overline{\mathscr{S}}^U_{f_0}$ and any $s \in \mathscr{S}$ we have $s + s' - \overline{H}^U_{f_0} \in \overline{\mathscr{S}}^U_f$. Hence

$$\overline{H}^U_f \leq s + s' - \overline{H}^U_{f_0}, \qquad \overline{H}^U_{f_0} \leq \inf_{s \in \mathscr{S}} s.$$

The converse inequality being trivial, $\overline{H}^U_{f_0}$ is the greatest lower bound of a Perron set and therefore harmonic.

From Lemma 3 and from the fact that f and $f\sigma^*$ (resp. $f\sigma_*$) differ only on a negligible set we have

$$\underline{H}_f \leq \overline{H}_f \leq \underline{H}_{f\sigma^*} = \overline{H}_f$$
$$(\text{resp. } \underline{H}_f \leq \overline{H}_f = \overline{H}_{f\sigma_*} \leq \underline{H}_f),$$

and f is resolutive.

Let $\mathfrak{B}$ be the class of Borel-sets $M \subset \partial U$ for which $f_0\chi_M$ is resolutive. $\mathfrak{B}$ contains the closed (resp. open) sets and from $M \in \mathfrak{B}$ it follows $\partial U - M \in \mathfrak{B}$. Let $M_1, M_2 \in \mathfrak{B}$. From

$$f_0\chi_{M_1 \cup M_2} = \max(f_0\chi_{M_1}, f_0\chi_{M_2})$$

it follows that $M_1 \cup M_2 \in \mathfrak{B}$. Let $\{M_n\}$ be an increasing sequence of $\mathfrak{B}$. Then $f_0\chi_{M_n} \uparrow f_0\chi_{\bigcup_{n=1}^{\infty} M_n}$ and $\bigcup_{n=1}^{\infty} M_n \in \mathfrak{B}$. $\mathfrak{B}$ coincides therefore with the class of all Borel sets.

Let f be a Borel function on ∂U, $0 \leq f \leq f_0$, and n a natural number. For any natural number i, $0 \leq i \leq 2^n$, we design

*) A function is called a Borel function if it is Borel measurable. The class of Borel sets is the smallest class of sets which contains the open sets and is closed with respect to countable union and contians together with a set its complement.

$$A_i = \left\{ y \in \partial U \mid \frac{i}{2^n} f_0(y) \leq f(y) < \frac{i+1}{2^n} f_0(y) \right\}.$$

$$f_n = f_0 \sum_{i=0}^{n} \frac{i}{2^n} \chi_{A_i}.$$

f_n is resolutive and $f_n \uparrow f$. Hence f is resolutive. This result can be extended immediately to a Borel function f, $|f| \leq f_0$.

COROLLARY 3. *Let s be a positive finite and continuous hyperharmonic function on X, U be an $M.P_0$.-set*[*)], *and p be a locally bounded potential on U positive in a neighbourhood of any boundary point of U. Then any Borel function f, $|f| \leq s$, is resolutive.*

In this case $A^* = \partial U$ and $\overline{H}_s$ is harmonic and bounded in the neighbourhood of any boundary point of U.

Let U be an open set and U° be the set of points $x \in U$ for which any locally bounded potential on U vanishes at x. U° is closed in U. If U is σ-compact and U° compact, then there exists a locally bounded potential on U positive in the neighbourhood of ∂U.

LEMMA 4. *Let $x \in U$ and U_x denote the set of points $y \in U - \{x\}$ such that if s_1, s_2 are locally bounded non-negative hyperharmonic functions on U then*

$$s_1(x) s_2(y) - s_1(y) s_2(x) = 0.$$

If $x \notin X_0 \cup \overline{U}_x$ then $x \notin U^\circ$.

Let $\mathscr{C}$ denote the set of restrictions on U° of the set of non-negative harmonic functions on U. If $f_1, f_2 \in \mathscr{C}$ then $\min(f_1, f_2) \in \mathscr{C}$. Indeed let u_i ($i = 1, 2$) be a non-negative harmonic function on U whose restriction on U° coincides with f_i. Then $\min(u_1, u_2)$ is a locally bounded non-negative hyperharmonic function on U. Denote by u the greatest harmonic minorant of $\min(u_1, u_2)$. Since $\min(u_1, u_2) - u$ is a potential on U

$$\min(f_1, f_2) = \min(u_1, u_2) = u$$

on U°.

Let V be a regular set in U, $x \in V$, $\overline{V} \cap U_x = \phi$, F be the carrier of ω_x^V, and p be a locally bounded potential on U. Since $x \notin X_0$, $F \neq \phi$, for a sufficiently small V. Suppose $x \in U^\circ$. From

*) If $\mathscr{X}$ satisfies Brelot's axioms, the existence of s implies that any open set is an $M.P_0$.-set ([3], Part IV, Theorem 3 (ii)).

$$0 = p(x) \geq \int p\, d\omega_x^V$$

it follows that p vanishes on F and $F \subset U^\circ$.

Let $y \in F$ and s_1, s_2 be two locally bounded non-negative hyperharmonic functions on U such that

$$s_1(x)\, s_2(y) - s_2(x)\, s_1(y) \neq 0.$$

Let u_i $(i=1, 2)$ be the greatest harmonic minorant of s_i. Since $s_i - u_i$ is a locally bounded potential on U, $s_i = u_i$ on U° and

$$u_1(x)\, u_2(y) - u_2(x)\, u_1(y) \neq 0.$$

If $u_i(x) = 0$ then from

$$u_i(x) = \int u_i\, d\omega_x^V$$

it would result $u_i(y) = 0$ which contradicts the above inequality. We may suppose therefore

$$u_1(x) = u_2(x) = 1, \qquad u_1(y) < u_2(y).$$

Since $x \in U^\circ$, $F \subset U^\circ$,

$$\int \min\,(u_1, u_2)\, d\omega_x^V = \min\left(\int u_1 d\omega_x^V, \int u_2 d\omega_x^V\right)$$

and we get the contradictory inequality

$$0 < \int (u_2 - \min\,(u_1, u_2))\, d\omega_x^V = u_2(x) - \min\,(u_1(x), u_2(x)) = 0.$$

It follows from this lemma that if $\mathscr{H}$ satisfies Bauer's Trennungsaxiom, then U° is empty.

THEOREM 7. *Let U be an $M.P_0$.-set with σ-compact boundary for which either $(\overline{U}^\circ \cap \partial U) \cup (\partial U - A^*)$ or $(\overline{U}^\circ \cap \partial U) \cup (\partial U - A_*)$ is negligible. If f is a real continuous function on ∂U for which $\overline{H}_{|f|}$ is harmonic and bounded in the neighbourhood of any boundary point of U then f is harmonic resolutive.*

It is sufficient to prove this theorem for a non-negative f. Replacing X by $X - \{x \in \partial U | f(x) = 0\}$, we may suppose further f positive. Let $x \in U$ and s_x be a non-negative hyperharmonic function finite at x and such that

$$\liminf_{U \ni z \to y} s_x(z) = \infty$$

for any $y \in (\overline{U}^{\circ} \cap \partial U) \cup (\partial U - A^*)$ (resp. $y \in (\overline{U}^{\circ} \cap \partial U) \cup (\partial U - A_*)$). For any natural number n we denote

$$A_n = \{y \in \partial U \mid \liminf_{U \ni z \to y} s_x(z) > n \max (f(y), \limsup_{U \ni z \to y} \overline{H}_f(z))\}.$$

A_n is an open set on ∂U which contains the set $(\overline{U}^{\circ} \cup \partial U) \cup (\partial U - A^*)$ (resp. $(\overline{U}^{\circ} \cap \partial U) \cup (\partial U - A_*)$).

Let ε be a positive number. For any $y \in \partial U - A_n$ we take a regular neighbourhood V_y of y and a hyperharmonic (resp. hypoharmonic) function s_y on $U \cap V_y$ which satisfy the following conditions: a) $\overline{V}_y \cap U^{\circ} = \phi$; b) for any $z \in V_y \cap \partial U$ we have $|f(z) - f(y)| < \varepsilon f(z)$; c) $1 - \varepsilon < s_y < 1 + \varepsilon$. There exists a compact set K_y on $U \cap \partial V_y$ and an open set W_y on $V_y \cap \partial U$ containing y such that

$$f(y) + \sup_{z' \in U \cap V_y} (\overline{H}_f(z')) \, \omega_z^{V_y}(U \cap \partial V_y - K_y) < \varepsilon f(z)$$

for any $z \in W_y$. Since $\partial U - A_n$ is σ-compact there exists a sequence $\{y_i\}$ in $\partial U - A_n$ such that

$$\partial U - A_n \subset \bigcup_{i=1}^{\infty} W_{y_i}.$$

For any i there exists a potential p_i on U, finite at x and positive on K_{y_i} because $K_{y_i} \cap U^{\circ} = \phi$. Let p_{ix} denote the hyperharmonic function associated to p_i and x by Lemma 1, with

$$p_{ix}(x) < \frac{1}{2^i},$$

and let us denote

$$p_x = \sum_{i=1}^{\infty} p_{ix}.$$

Let $\eta > 0$, $y \in \partial U$ and $\mathfrak{U}$ be an ultrafilter on U converging to y such that

$$\lim_{\mathfrak{U}} \left(\overline{H}_f - \frac{s_x}{n} - \eta p_x\right) = \limsup_{U \ni z \to y} \left(\overline{H}_f(z) - \frac{s_x(z)}{n} - \eta p_x(z)\right)$$

$$\left(\text{resp. } \lim_{\mathfrak{U}} \left(\underline{H}_f + \frac{s_x}{n} + \eta p_x\right) = \liminf_{U \ni z \to y} \left(\underline{H}_f(z) + \frac{s_x(z)}{n} + \eta p_x(z)\right)\right).$$

If $y \in A_n$ then

$$\lim_{\mathfrak{U}}\left(\overline{H}_f-\frac{s_x}{n}-\eta p_x\right)\leq 0$$

$$\left(\text{resp. } \lim_{\mathfrak{U}}\left(\underline{H}_f+\frac{s_x}{n}+\eta p_x\right)\geq f(y)\right).$$

If $y\in A^*-A_n$ (resp. $y\in A_*-A_n$) then there exists an i such that $y\in W_{y_i}$. If

$$\lim_{\mathfrak{U}} p_i \neq 0$$

then

$$\lim_{\mathfrak{U}} p_x=\infty,$$

$$\lim_{\mathfrak{U}}\left(\overline{H}_f-\frac{s_x}{n}-\eta p_x\right)=-\infty,$$

$$\left(\text{resp. } \lim_{\mathfrak{U}}\left(\underline{H}_f+\frac{s_x}{n}+\eta p_x\right)=+\infty\right).$$

If

$$\lim_{\mathfrak{U}} p_i=0$$

then it can be proved like in Lemma 2 that

$$\lim_{\mathfrak{U}} \overline{H}_f\leq\theta(\varepsilon)f(y),\qquad \theta(\varepsilon)=\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2+\varepsilon$$

$$(\text{resp. } \lim_{\mathfrak{U}} \underline{H}_f\geq\theta(-\varepsilon)f(y)).$$

For any $y\in\partial U$ we have therefore

$$\limsup_{U\ni z\to y}\left(\overline{H}_f(z)-\frac{s_x(z)}{n}-\eta p_x(z)\right)\leq\theta(\varepsilon)f(y)$$

$$\left(\text{resp. } \liminf_{U\ni z\to y}\left(\underline{H}_f(z)+\frac{s_x(z)}{n}+\eta p_x(z)\right)\geq\theta(-\varepsilon)f(y)\right).$$

If $(\overline{U}^\circ\cap\partial U)\cup(\partial U-A_*)$ is negligible then $\underline{H}_f+\dfrac{s_x}{n}+\eta p_x\in\mathscr{S}^U_{\theta(-\varepsilon)f}$ and we get, for a sufficiently small ε,

$$\underline{H}_f+\frac{s_x}{n}+\eta p_x\geq\overline{H}_{\theta(-\varepsilon)f}=\theta(-\varepsilon)\overline{H}_f.$$

p_x and s_x being finite at x we get

$$\underline{H}_f(x)\geq\overline{H}_f(x)$$

making successively $\eta\downarrow 0$, $\varepsilon\downarrow 0$, $n\uparrow\infty$. f is therefore resolutive.

Let now $(\overline{U}^\circ\cap\partial U)\cup(\partial U-A^*)$ be negligible and G be an open relatively compact set for which $X_0\cap\partial G=\phi$. We denote by f_G the function on $\partial(U\cap G)$ equal to 0 on $\overline{G}\cap\partial U$ and equal to $\overline{H}_f$ on $U\cap\partial G$ and by s_G the function on U equal to $\overline{H}_f$ on $U-\overline{G}$ equal to $\overline{H}^{G\cap U}_{f_G}$ on $G\cap U$ and equal to

$$\liminf_{G\cap U\ni z\to z'} \overline{H}_{fG}^{G\cap U}(z)$$

for any $z'\in U\cap\partial G$. It has been shown in the proof of Lemma 3 that s_G is a hyperharmonic function. Let n be a natural number, η' be a positive number, $y\in G\cap\partial U$, and $\mathfrak{U}$ be an ultrafilter on $G\cap U$ converging to y such that

$$\lim_{\mathfrak{U}}\left(s_G-\frac{s_x}{n}-\eta' p_x\right)=\limsup_{U\ni z\to y}\left(s_G(z)-\frac{s_x(z)}{n}-\eta p_x(z)\right).$$

If $y\in A_n$ then

$$\lim_{\mathfrak{U}}\left(s_G-\frac{s_x}{n}\right)\leq 0.$$

If $y\in\partial U-A_n$ then there exists an i such that $y\in W_{y_i}$. If

$$\lim_{\mathfrak{U}} p_i \neq 0$$

then

$$\lim_{\mathfrak{U}}(s_G-\eta' p_x)=-\infty.$$

If

$$\lim_{\mathfrak{U}} p_i=0$$

then it can be proved like in Lemma 2 that

$$\lim_{\mathfrak{U}} s_G\leq \varepsilon f(y).$$

For any case we get

$$\limsup_{U\ni z\to y}\left(s_G(z)-\frac{s_x(z)}{n}-\eta' p_x(z)\right)\leq \varepsilon f(y).$$

Let us denote by $\mathfrak{G}$ the set of relatively compact open sets G for which $X_0\cap\partial G=\phi$ and by u the greatest lower bound of the family $\{s_G\}_{G\in\mathfrak{G}}$. u is harmonic. Let $s\in\overline{\mathscr{S}}_f^U$. Then $s\geq u$ and for any $y\in\partial U$

$$\liminf_{U\ni z\to y}\left(s(z)-u(z)+\frac{s_x(z)}{n}+\eta' p_x(z)\right)\geq(1-\varepsilon)f(y).$$

It follows

$$s-u+\frac{s_x}{n}+\eta' p_x\geq(1-\varepsilon)\overline{H}_f.$$

p_x and s_x being finite at x we get $u(x)=0$ making succesively $\eta'\downarrow 0$, $\varepsilon\downarrow 0$, $n\uparrow\infty$, $s(x)\downarrow\overline{H}_f(x)$. Let $\{G_i\}$ be a sequence in $\mathfrak{G}$ such that

$$s_0 = \sum_{i=1}^{\infty} s_{G_i}$$

is finite at x. Then $\overline{H}_f - \eta s_0$ is non-positive outside a compact set of X. It follows

$$\overline{H}_f - \frac{s_x}{n} - \eta p_x - \eta s_0 \in \underline{\mathscr{S}}^U_{\theta(\varepsilon)f},$$

$$\overline{H}_f - \frac{s_x}{n} - \eta p_x - \eta s_0 \leq \underline{H}_{\theta(\varepsilon)f}.$$

Since s_x, p_x, s_0 are finite at x we get

$$\overline{H}_f(x) \leq \underline{H}_f(x)$$

making succesively $\eta \downarrow 0$, $\varepsilon \downarrow 0$, $n \uparrow \infty$. f is therefore resolutive.

If $\mathscr{H}$ satisfies the axioms of Brelot's or Bauer's theory then $X_0 = U^\circ = \phi$ and $A^* = A_* = \partial U$. Therefore Theorem 7 contains Hervé's [5] and Bauer's [1] (Satz 24) results about the resolutivity of continuous functions on relatively compact open sets. On the other hand, in Brelot's axiomatic there exists always a positive potential on U and the same is true in Bauer's axiomatic if U is σ-compact. Hence Theorem 6 proves the resolutivity of bounded Borel-measurable functions in these cases. This gives the possibility to prove, without the condition that X has a countable basis, that if $\mathscr{H}$ satisfies Brelot's axioms and the axiom D [3] the limit of a decreasing sequence of non-negative hyperharmonic functions differs from a hyperharmonic function on a polar set.

The condition $A^* = A_* = \partial U$ and the fact that the constants are harmonic is not sufficient in order that any continuous function on ∂U is resolutive, even if U is relatively compact. An example is given by a region on a compact Riemann surface, whose boundary consists of more than one point and is of capacity zero.

3. Relation between the normed and the usual Dirichlet problem. The study of the Dirichlet problem on X is interesting only in the case when X is a non-compact M.P.-set. We shall suppose from now on that X satisfies this condition. Let Y be a compactification of X, i.e., a compact Hausdorff space which contains X as a dense subspace and $\Delta = Y - X$. Since X is locally compact, Δ is compact. Let f be a numerical function on Δ. We denote by $\overline{\mathscr{S}}^{X,Y}_f = \overline{\mathscr{S}}^X_f$ (resp. $\underline{\mathscr{S}}^{X,Y}_f = \underline{\mathscr{S}}^X_f$) the set of lower bounded hyperharmonic (resp. upper bounded hypoharmonic) functions s, such that

$$\liminf_{X \ni x \to y} s(x) \geq f(y) \qquad (\text{resp. } \limsup_{X \ni x \to y} s(x) \leq f(y))$$

for any $y \in \Delta$. The greatest lower bound of $\overline{\mathscr{S}}_f^X$ (resp. the least upper bound of $\underline{\mathscr{S}}_f^X$) is denoted by $\overline{H}_f^{X,Y} = \overline{H}_f^X = \overline{H}_f$ (resp. $\underline{H}_f^{X,Y} = \underline{H}_f^X = \underline{H}_f$); since X is an M.P.-set $\underline{H}_f \leq \overline{H}_f$. If the functions $\overline{H}_f$ and $\underline{H}_f$ are finite and equal, the function f is called resolutive and

$$H_f^{X,Y} = H_f^X = H_f = \overline{H}_f = \underline{H}_f$$

is called the solution of the Dirichlet problem with f as the boundary function.

If any bounded continuous (resp. lower semi-continous) function on Δ is resolutive then Y is called a **Baire** (resp. **Borel**) **resolutive compactification of X.**

In the rest of this paper Y will be a fixed compactification of X.

The normed Dirichlet problem and the Dirichlet problem formulated above are closely related. Indeed let U be an open set for which ∂U is not empty and f be a numerical function defined on ∂U. Let $\overline{U}^Y$ (resp. $\partial_Y U$) denote the closure (resp. the boundary) of U in Y. Taking U instead of X and $\overline{U}^Y$ instead of Y in the preceding considerations, and defining f_0 equal to f on $\partial_X U$ and equal to zero on $\partial_Y U - \partial_X U$ it is clear that

$$\overline{\mathscr{S}}_f^{U,X} \subset \overline{\mathscr{S}}_{f_0}^{U,\overline{U}^Y}, \qquad \underline{\mathscr{S}}_f^{U,X} \subset \underline{\mathscr{S}}_{f_0}^{U,\overline{U}^Y}.$$

Therefore we have

$$\overline{H}_f^{U,X} \geqq \overline{H}_{f_0}^{U,\overline{U}^Y}, \qquad \underline{H}_f^{U,X} \leqq \underline{H}_{f_0}^{U,\overline{U}^Y}.$$

If any bounded Borel function on $\partial_X U$ is resolutive then U is called a **Borel-resolutive set.**

LEMMA 5. *Let Y be a compactification of X, U be a Borel resolutive set, and f be a non-negative resolutive function on Δ. Then the function g on $\partial_Y U$, equal to f on $\Delta \cap \partial_Y U$ and equal to zero on $X \cap \partial_Y U = \partial_X U$, is resolutive.*

We set $u = H_f^{X,Y}$. Let $\bar{s}$ (resp. $\underline{s} \geq 0$) belong to $\overline{\mathscr{S}}_f^{X,Y}$ (resp. $\underline{\mathscr{S}}_f^{X,Y}$) and $\underline{s}' \geq 0$ (resp. $\bar{s}'$) belong to $\underline{\mathscr{S}}_{\underline{s}}^{U,X}$ (resp. $\overline{\mathscr{S}}_{\underline{s}}^{U,X}$). Then $\bar{s} - \underline{s}'$ (resp. $\underline{s} - \bar{s}'$) belongs to $\overline{\mathscr{S}}_g^{U,\overline{U}^Y}$ (resp. $\underline{\mathscr{S}}_g^{U,\overline{U}^Y}$) and therefore $\overline{H}_g^{U,\overline{U}^Y}$ is finite and

$$\overline{H}_g^{U,\overline{U}^Y} - \underline{H}_g^{U,\overline{U}^Y} \le (\bar{s} - \underline{s}') - (\underline{s} - \bar{s}').$$

Since $\underline{s}'$, $\bar{s}'$ are arbitrary we get

$$\overline{H}_g^{U,\overline{U}^Y} - \underline{H}_g^{U,\overline{U}^Y} \le (\bar{s} - \underline{H}_{\bar{s}}^{U,X}) - (\underline{s} - \overline{H}_{\underline{s}}^{U,X}).$$

U being a Borel resolutive set and $\underline{s}$ a bounded upper semi-continuous function on $\partial_Y U$ we have

$$\overline{H}_{\underline{s}}^{U,X} = \underline{H}_{\underline{s}}^{U,X} \le \underline{H}_{\bar{s}}^{U,X},$$

$$\overline{H}_g^{U,\overline{U}^Y} - \underline{H}_g^{U,\overline{U}^Y} \le \bar{s} - \underline{s}.$$

$\bar{s}$, $\underline{s}$ being arbitrary it follows that g is resolutive.

THEOREM 8. *If Y is a Baire (resp. Borel) resolutive compactification of X and U is a Borel resolutive open subset of X, then $\overline{U}^Y$ is a Baire (resp. Borel) resolutive compactification of U.*

Since the resolutive functions form a real vector space it is sufficient to prove that any continuous (resp. lower semi-continuous) bounded non-negative function f' on $\partial_Y U$ is resolutive. Let f be a continuous (resp. lower semi-continuous) bounded non-negative function on Δ equal to f' on $\Delta \cap \overline{U}^Y$ and g be equal to $f = f'$ on $\Delta \cap \overline{U}^Y$ and equal to zero on $\partial_X U$. From the preceding lemma it follows that g is resolutive. Since U is Borel resolutive the function $f' - g$ is also resolutive, hence f' is resolutive.

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