MODULE OF ANNULUS

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1. Let C and C' be two simple closed curves in the complex z-plane which have no point in common and surround the origin. Denote by D the annulus bounded by C and C'. Consider a family $\{\gamma\}$ of rectifiable curves γ in D and the family P of all non-negative lower semi-continuous functions $\rho = \rho(z)$ in D. Put

$$L_\rho\{\gamma\} = \inf_{\gamma \in \{\gamma\}} \int_\gamma \rho |dz|.$$

Understanding $\frac{0}{0} = \frac{\infty}{\infty} = 0$, we call the quantity

$$\lambda\{\gamma\} = \sup_{\rho \in P} \frac{(L_\rho\{\gamma\})^2}{\iint_D \rho^2 d\sigma}$$

the extremal length of the family $\{\gamma\}$, where $d\sigma$ denotes the
$\{\gamma'\}$ be the family of all rectifiable curves γ' in D joining C with C' and let $\{\gamma''\}$ be that of all rectifiable curves γ'' in D separating C from C'. Then it is known that

$$\lambda\{\gamma'\} = \frac{1}{\lambda\{\gamma''\}} \tag{1}$$

and that the quantity

$$\mu = 2\pi\lambda\{\gamma'\} \tag{2}$$

is the module of D. In this note, we give some estimates of μ.

2. Let D be an annulus stated in § **1.** We denote by l_θ the intersection of the half straight line $\arg z = \theta\ (0 \leqq \theta \leqq 2\pi)$ with D and by $l(\theta)$ the logarithmic length of l_θ, that is,

$$l(\theta) = \int_{l_\theta} \frac{dr}{r}, \qquad z = re^{i\theta}.$$

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The following was proved by Rengel [3].

THEOREM. *The module μ of D satisfies the inequality*

$$\mu \leqq \frac{1}{2\pi} \int_0^{2\pi} l(\theta)\, d\theta. \tag{3}$$

Now we shall prove the following which implies Rengel's theorem stated above.

THEOREM 1. *For the module μ of D, the inequality*

$$\mu \leqq \frac{2\pi}{\displaystyle\int_0^{2\pi} \frac{d\theta}{l(\theta)}} \tag{4}$$

holds.

Proof. Let $\{\gamma'\}$ be the family of all rectifiable curves γ' joining C with C' in D. Then it is obvious that

$$L_\rho\{\gamma'\} \leqq \int_{l_\theta} \rho\, dr$$

for any $\rho \in P$ and for any $\theta\,(0 \leqq \theta \leqq 2\pi)$. By the Schwarz inequality, we have

$$\begin{aligned}(L_\rho\{\gamma'\})^2 &\leqq \left(\int_{l_\theta} \rho\, dr\right)^2 \\ &\leqq \int_{l_\theta} \frac{dr}{r} \int_{l_\theta} \rho^2 r\, dr = l(\theta) \int_{l_\theta} \rho^2 r\, dr,\end{aligned}$$

or

$$(L_\rho\{\gamma'\})^2 \frac{1}{l(\theta)} \leqq \int_{l_\theta} \rho^2 r\, dr.$$

Integrating both sides with respect to θ, we get

$$\frac{(L_\rho\{\gamma'\})^2}{\displaystyle\iint_D \rho^2\, d\sigma} \leqq \frac{1}{\displaystyle\int_0^{2\pi} \frac{d\theta}{l(\theta)}},$$

which gives

$$\lambda\{\gamma'\} \leqq \frac{1}{\displaystyle\int_0^{2\pi} \frac{d\theta}{l(\theta)}}.$$

From (2), we obtain our theorem.

Remark. The Schwarz inequality yields

$$(2\pi)^2 = \left(\int_0^{2\pi} d\theta\right)^2 \leqq \int_0^{2\pi} \frac{d\theta}{l(\theta)} \int_0^{2\pi} l(\theta)\, d\theta, \tag{5}$$

from which Rengel's theorem is obtained immediately by using Theorem 1. In (5), the equality holds if and only if $l(\theta)$ is a constant. In this case, the curve C' is obtained as a set of points $\alpha z\,(z \in C)$, where α is a positive constant, and Rengel's inequality (3) and ours (4) are identical.

3. Here we give an estimate, from below, of the module of an annulus of a special type.

Let C be a simple closed curve in the z-plane surrounding the origin and let (C) be a domain bounded by C and containing the origin. If, for any point $z \in C$ and for any $t\,(0 \leqq t < 1)$, the point tz lies in (C), then we say that C is *strictly star-like* with respect to the origin.

Consider a curve C strictly star-like with respect to the origin. We assume that C consists of a finite number of arcs $C^k : r = r_k(\theta)$, $\theta_{k-1} \leqq \theta \leqq \theta_k$ $(k = 1, 2, \ldots, n)$, where $\theta_0 = 0$, $\theta_n = 2\pi$ and each $r_k(\theta)$ has a continuous derivative $r_k'(\theta)$ in $\theta_{k-1} \leqq \theta \leqq \theta_k$. The expression $r = r(\theta)\,(0 \leqq \theta \leqq 2\pi)$ defined by putting $r(\theta) = r_k(\theta)$ for $\theta_{k-1} \leqq \theta \leqq \theta_k$ is a representation of C in the polar form. In such a case, we say that the curve C is *piecewise smooth.*

Denote by C' the curve defined by $r = \alpha r(\theta)$, where α is a real constant such that $0 < \alpha < 1$.

We can prove

THEOREM 2. *Let C and C' be defined as above. Then the module μ of the annulus D bounded by C and C' is estimated from below as follows:*

$$\mu \geqq \frac{1}{K(C)} \log \frac{1}{\alpha},$$

where $K(C) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left\{1 + \left(\frac{r'(\theta)}{r(\theta)}\right)^2\right\} d\theta$ *is a constant depending only on the curve C.*

Proof. Let $C_t\,(0 \leqq t \leqq 1)$ be a curve having a representation in the polar form

$$C_t : r = R(\theta, t) = r(\theta)\{\alpha + (1-\alpha)t\}.$$

The curves C_1 and C_0 are identical with C and C' respectively. Consider the family $\{\gamma''\}$ of all rectifiable curves γ'' separating C from C' in D and the family P in § 1. It is easy to see that

$$ds = \sqrt{(R(\theta, t))^2 + \left(\frac{\partial R(\theta, t)}{\partial \theta}\right)^2}$$
$$= r(\theta)\{\alpha + (1-\alpha)t\}\sqrt{1 + \left(\frac{r'(\theta)}{r(\theta)}\right)^2}\, d\theta$$

is the line-element along $C_t (0 < t < 1)$ and that

$$d\sigma = (r(\theta))^2(1-\alpha)\{\alpha + (1-\alpha)t\} d\theta\, dt$$

is the area-element. It is evident that

$$L_\rho\{\gamma''\} \leqq \int_{C_t} \rho\, ds$$

for $0 < t < 1$. Hence, by the Schwarz inequality, we have

$$(L_\rho\{\gamma''\})^2 \leqq \left(\int_{C_t} \rho\, ds\right)^2$$
$$\leqq \frac{\alpha + (1-\alpha)t}{1-\alpha} \int_0^{2\pi} \left\{1 + \left(\frac{r'(\theta)}{r(\theta)}\right)^2\right\} d\theta \int_0^{2\pi} \rho^2 (r(\theta))^2 (1-\alpha)\{\alpha + (1-\alpha)t\} d\theta.$$

Therefore, using the same argument as in the proof of Theorem 1 and noting (1) and (2), we get

$$\frac{(L_\rho\{\gamma''\})^2}{\iint_D \rho^2 d\sigma} \leqq K_0(C) \frac{1}{\log \frac{1}{\alpha}},$$

where $K_0(C) = \int_0^{2\pi} \left\{1 + \left(\frac{r'(\theta)}{r(\theta)}\right)^2\right\} d\theta$. Putting $K(C) = \frac{1}{2\pi} K_0(C)$, we have our theorem.

Example. Let Π_n be a regular polygon of center at the origin and with n sides of equal length and let Π'_n be another regular polygon obtained from Π_n by a transformation $z = \alpha z'$ $(z' \in \Pi_n)$, where $0 < \alpha < 1$. If we denote by μ the module of the annulus bounded by Π_n and Π'_n, then, using Theorems 1 and 2, we get

$$\frac{1}{K(\Pi_n)} \log \frac{1}{\alpha} \leqq \mu \leqq \log \frac{1}{\alpha},$$

where

$$K(\Pi_n) = \frac{n}{\pi} \tan \frac{\pi}{n}.$$

4. Applying Theorem 2, we prove the following

THEOREM 3. *Let Δ be a domain in the z-plane whose boundary consists of the origin and of an enumerable number of sets E_k $(k = 1, 2, \ldots)$, where E_k lies on a simple closed curve C^k strictly star-like with respect to the origin and may consist of arcs and points. If there exists a simple closed curve C which is piecewise smooth in the sense stated in* § **3** *and surrounds the origin and if, for each k, there exists a positive number α_k such that the set of points $\alpha_k z$ $(z \in C)$ is contained in C_k and such that $\alpha_k > \alpha_{k+1}$, $\lim_{k\to\infty} \alpha_k = 0$, then the origin is a weak boundary component of Δ.*

Proof. Let us denote by D_k the annulus bounded by C^k and C^{k+1}. Then D_k is contained in Δ. Denoting by μ_k the module of D_k, we see by Theorem 2 that there exists a constant $K(C)$ depending only on C such that

$$\mu_k \geqq \frac{1}{K(C)} \log \frac{\alpha_k}{\alpha_{k+1}}.$$

Hence we get

$$\sum_{k=1}^{\infty} \mu_k \geqq \frac{1}{K(C)} \sum_{k=1}^{\infty} \log \frac{\alpha_k}{\alpha_{k+1}},$$

whose right hand side diverges. By Grötzsch's theorem [2] (Cf. Savage [4]), we have our assertion.

Remark. This theorem implies Theorem 1 in [1].

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