NOTE ON NON-COMMUTATIVE SEMI-LOCAL RINGS

YUKITOSHI HINOHARA

Our aim in this note is to generalize some topological results of commutative noetherian rings to non-commutative rings. As a supplemental remark of [2] we prove in §1 that any right ideal of a complete right semi-local ring is closed, and that

$$\bigcap_{s=1}^{\infty} MJ^s = (0)$$

for any finitely generated right module M over a complete right semi-local ring Λ where J is the Jacobson radical of Λ.

In §2 we are concerned with the flatness of modules. C. Lech gave in [7] an ideal theoretical criterion of the flatness of modules over a commutative ring. We notice that his criterion of the flatness is valid for non-commutative rings.

§1. Non-commutative semi-local rings

DEFINITION. *Let Λ be a ring with a unit element* 1 *and J its Jacobson radical; Λ is said to be right semi-local if the following conditions are satisfied:*

(a) $\bigcap_{s=1}^{\infty} J^s = 0$,

(b) *Λ is right noetherian,*

(c) *Λ/J satisfies the minimum condition on right ideals.*

This definition is due to E. H. Batho who studied the basic property of this class of rings in [2].

By virtue of the condition (a), we may introduce a Hausdorff topology (called the J-adic topology) in Λ and construct the completion $\hat{\Lambda}$ of Λ with respect to this topology.

For brevity, we call an ideal I of Λ a nucleus if $\bigcap_{s=1}^{\infty} I^s = 0$ and denote the Jacobson radical of the ring Λ by $J(\Lambda)$.

LEMMA 1. *Let Λ be a right semi-local ring and $\hat{\Lambda}$ the completion of Λ with respect to the $J(\Lambda)$-adic topology. Then we have*

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$$J(\hat{\Lambda})^s = J(\Lambda)^s\hat{\Lambda}, \qquad \bigcap_{s=1}^{\infty} J(\hat{\Lambda})^s = 0,$$
$$J(\hat{\Lambda})^s \cap \Lambda = J(\Lambda)^s, \qquad \Lambda/J(\Lambda)^s \cong \hat{\Lambda}/J(\hat{\Lambda})^s$$

for any positive integer s.

For the proof we refer to [Theorem 2.3 of 2] and [Theorem 2 of 3].

LEMMA 2. *Let Λ be a ring with a unit element* 1. *Then we have the relation $NJ(\Lambda) \neq N$ for any finitely generated right Λ-module $N(\neq 0)$.*

This is [Proposition 2 of 8, p. 200].

LEMMA 3. *Let Λ be a ring and J its Jacobson radical. Assume the following conditions for Λ:*

(a) *Λ/J satisfies the minimum condition on right ideals,*

(b) *J is a nucleus and has a finite number of right Λ-basis,*

(c) *Λ is complete with respect to the J-adic topology.*

Then we have the relation

$$\bigcap_{s=1}^{\infty} (M + FJ^s) = M$$

for any finitely generated Λ-submodule M of a free right Λ-module F.

We notice that $\bigcap_{s=1}^{\infty} FJ^s = (0)$ since F is a free Λ-module and J is a nucleus. Therefore we can define a Hausdorff topology in F by taking F, FJ, FJ^2, ... to be neighbourhoods of zero. Then the closure $\bar{N}$ of any submodule N of F is equal to $\bigcap_{s=1}^{\infty} (N + FJ^s)$.

Before proving Lemma 3 we prove

LEMMA 4. *If a submodule M of F is finitely generated we have $\bar{M} = M + \bar{M}J$.*

Proof. We consider the residue class module $F/\bar{M}J$ of which $(\bar{M} + MJ)/\bar{M}J$ is a submodule. Since $\bar{M}J \subseteq \bar{M}J$, we have $((\bar{M} + MJ)/\bar{M}J)J = (0)$. Let $\bar{m}$ be any element of $\bar{M}$. Then $\bar{m}$ can be written in the following form, for any positive integer t,

$$\bar{m} = \sum_{i=1}^{n} m_i \lambda_i + j_t$$

where $\{m_1, \ldots, m_n\}$ is a Λ-basis of M, $\lambda_i \in \Lambda$ and $j_t \in FJ^t$. Now $\mathfrak{m} = (\bar{m}\Lambda + M + \bar{M}J)/\bar{M}J$ is a finitely generated module and $\mathfrak{m}J = 0$. Therefore $\mathfrak{m}$ is considered as a finitely generated Λ/J-module. Since Λ/J satisfies the minimum condition on right ideals, the module $\mathfrak{m}$ satisfies the minimum condition on submodules. Consider the descending sequence of submodules

$$\mathfrak{m} \supseteq \mathfrak{m} \cap (FJ+MJ)/MJ \supseteq \mathfrak{m} \cap (FJ^2+MJ)/MJ \supseteq \cdots,$$

and there exists an integer u such that

$$\mathfrak{m} \cap (FJ^u+MJ)/MJ = \mathfrak{m} \cap (FJ^{u+1}+MJ)/MJ = \cdots.$$

The fact that MJ is closed implies

$$\mathfrak{m} \cap (FJ^u+MJ)/MJ = (0).$$

Let $\bar{m} = \sum_i m_i \lambda_i' + j_u$, $j_u \in FJ^u$. Then we have

$$j_u = \bar{m} - \sum_i m_i \lambda_i' \in FJ^u \cap (\bar{m}\Lambda + M).$$

Therefore $j_u \in MJ$, i.e. $\bar{m} \in M+MJ$. Thus we have $\bar{M} \subseteq M+MJ$. The converse inclusion is obvious and we completes the proof of Lemma 4.

Proof of Lemma 3. Since M is a finitely generated module and the two-sided ideal J is finitely generated as a right ideal, MJ^s is finitely generated for any positive integer s. Thus we have

$$\overline{MJ^i} = MJ^i + \overline{MJ^{i+1}}.$$

Now we are in a position to prove $\bar{M} = M$. Let $\bar{m}$ be any element of $\bar{M}$. Then we have, by virtue of the above relation of submodules,

$$\begin{aligned} \bar{m} &= \sum m_i \lambda_i^{(0)} + \bar{m}', \ \lambda_i^{(0)} \in \Lambda, \ \bar{m}' \in \overline{MJ} \subseteq \overline{FJ}, \\ \bar{m}' &= \sum m_i \lambda_i^{(1)} + \bar{m}'', \ \lambda_i^{(1)} \in J, \ \bar{m}'' \in \overline{MJ^2} \subseteq \overline{FJ^2}, \\ \bar{m}'' &= \cdots. \end{aligned}$$

Let $\bar{\lambda}_i = \sum_{j=0}^{\infty} \lambda_i^{(j)}$. Then we have $\bar{m} = \sum_i m_i \bar{\lambda}_i \in M$. This completes the proof of Lemma 3.

As an immediate consequence of Lemma 1 and Lemma 3, we have

THEOREM 1. *Let Λ be a right semi-local ring and J its Jacobson radical. Then any finitely generated right ideal of the completion $\hat{\Lambda}$ of Λ is closed, and therefore there holds the relation $I\hat{\Lambda} \cap \Lambda = \bar{I}$ where $\bar{I}$ is the closure in Λ of a right ideal I* of Λ.

THEOREM 2. *Let Λ be a complete right semi-local ring. Then any right ideal of Λ is closed. Further, for any finitely generated right Λ-module M, we have*

$\bigcap_{s=1}^{\infty} MJ^s = 0$ *where* $J = J(\Lambda)$.

Proof. There exists an exact sequence of right Λ-modules

$$0 \to N \to F \to M \to 0$$

where F is a finitely generated free Λ-module. Since Λ is right noetherian, N is finitely generated. Thus we deduce $\bigcap_{s=1}^{\infty} (N + FJ^s) = N$, i.e. $\bigcap_{s=1}^{\infty} (F/N)J^s = 0$. This is the required result since $F/N \cong M$.

By combining this theorem with [Remark 2 of § 4 in 9] and [Theorem 3.4 of 2], we have

COROLLARY. *A complete right semi-local ring* Λ *is linearly compact as a right* Λ*-module in the discrete topology.*

Finally we have the following result:

THEOREM 3. *Let* Λ *be a right noetherian ring with a unit element* 1, *and* Q *a two-sided ideal of* Λ *which is a nucleus. If any right ideal* I *of* Λ *is closed with respect to the* Q*-adic topology, then we have* $\bigcap_{s=1}^{\infty} MQ^s = (0)$ *for any finitely generated right* Λ*-module* M.

Proof. We assume that $\bigcap_{s=1}^{\infty} MQ^s \neq 0$ and deduce a contradiction. We consider the set $\mathfrak{S}$ of all submodules S such that $\bigcap_{i=1}^{\infty} (M/S)Q^i \neq 0$. Let N' be a maximal element of $\mathfrak{S}$. Then by assumption we have $N = \bigcap_{s=1}^{\infty} (N' + MQ^s) \neq N'$. Let M' be any submodule of M properly containing N'. Then $M' \supseteq N$ by the maximality of N'. Let $\overline{m}$ be any element of M/N'. Then we have $\overline{m}\Lambda \cong \Lambda/0(\overline{m})$ where $0(\overline{m}) = \{\lambda \in \Lambda \mid \overline{m}\lambda = 0\}$. Since any submodule of $\overline{m}\Lambda$ contains the unique minimal submodule N/N' of M/N' (this implies that M/N' is sub-directly irreducible), $\Lambda/0(\overline{m})$ is subdirectly irreducible. Therefore there exists a positive integer t such that $Q^t \subseteq 0(\overline{m})$ since $0(\overline{m})$ is closed by assumption. This implies that $\overline{m}Q^t = 0$. Therefore there exists an integer s such that $(M/N')Q^s = 0$ since M/N' is finitely generated. This contradicts our assumption $N \neq N'$.

§ 2. Flatness of modules

Let Λ be a ring with a unit element 1 and M a (unitary) left Λ-module. Then the module M is said to be Λ-flat if $\mathrm{Tor}_n^{\Lambda}(C, M) = 0$ for all right Λ-modules C and all $n > 0$.

Let λ be an element of Λ, I a right ideal of Λ and M a left Λ-module. Then we use the following notations:

Z for the ring of all integers,
IM for the Z-submodule of M generated by the set IM,
$(I : \lambda)$ for $\{\mu \in \Lambda \mid \lambda\mu \in I\}$ and
$(IM : \lambda)_M$ for $\{m \in M \mid \lambda m \in IM\}$.

THEOREM 4. *Let Λ be a ring with a unit element* 1. *Then, for each left Λ-module M, the following conditions are equivalent to each other:*

(a) *M is Λ-flat,*

(b) $\mathrm{Tor}_1^{\Lambda}(\Lambda/I, M) = 0$ *for each right ideal I of Λ,*

(c) i) *For any right ideals I_1 and I_2 of Λ, there holds the relation* $(I_1 \cap I_2)M = I_1M \cap I_2M$, *and*

ii) *For each element λ of Λ, there holds the relation* $(0 : \lambda)M = (0 : \lambda)_M$.[1]

(d) $(I : \lambda)M = (IM : \lambda)_M$ *for each right ideal I and each element λ of Λ.*

The equivalence of the conditions (a) and (b) is an exercise of 4 (see p. 123 of [4]) and the implication (d) $\Rightarrow$ (a) is proved by the same way as in [7], so we prove only the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (d).

To deduce i) of (c) from (a), it suffices to prove

LEMMA 5. *Let I_1 and I_2 be right ideals of Λ and M a left Λ-module. If* $\mathrm{Tor}_1^{\Lambda}(\Lambda/I_1+I_2, M) = 0$ *we have the relation* $(I_1 \cap I_2)M = I_1M \cap I_2M$.[2]

This lemma follows immediately from the exact sequence:

$$\begin{array}{ccccccc}
 & & (I_1 \cap I_2)\otimes M & \longrightarrow & I_1 \otimes M & \to & I_1/I_1 \cap I_2 \otimes M \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & I_2 \otimes M & \longrightarrow & \Lambda \otimes M & \longrightarrow & \Lambda/I_2 \otimes M \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I_2/I_1 \cap I_2 \otimes M & \to & \Lambda/I_1 \otimes M & \to & \Lambda/I_1+I_2 \otimes M \to 0
\end{array}$$

where $\otimes$ means $\otimes_{\Lambda}$.

Proof of the implication (a) $\Rightarrow$ ii) *of* (c). From the flatness of the module M, we deduce a commutative exact diagram;

[1] A. Hattori called this property of a module torsion-free in [6].

[2] The proof of this lemma is a formal generalization of those of [Theorems 5 and 6 in 1, p. 111].

$$\begin{array}{ccccccccc} 0 \to & (0:\lambda)\otimes M & \to & \Lambda\otimes M & \to & \Lambda/(0:\lambda)\otimes M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & (0:\lambda)_M & \to & M & \longrightarrow & \lambda M & \to 0 \end{array}$$

where $\otimes$ means $\otimes_\Lambda$. From this diagram we have the required result $(0:\lambda)M = (0:\lambda)_M$ by virtue of the well known "five lemma".

Proof of the implication (c) $\Rightarrow$ (d). It suffices to prove $(I:\lambda)M \supseteq (IM:\lambda)_M$. Let m be any element of $(IM:\lambda)_M$. Then we have $\lambda m \in \lambda(IM:\lambda)_M = IM \cap \lambda M = (I \cap \lambda\Lambda)M = \lambda(I:\lambda)M$. Therefore there exists an element $m' \in (I:\lambda)M$ such that

$$\lambda m = \lambda m', \quad \text{i.e.} \quad \lambda(m-m') = 0.$$

This implies that $m - m' \in (0:\lambda)_M = (0:\lambda)M \subseteq (I:\lambda)M$. Thus we have $m \in (I:\lambda)M$, and this completes the proof of Theorem 4.

Remark. As an immediate consequence of Theorem 4, we have the following corollary by combining [Theorem 2 or Corollary of 5]:

A commutative integral domain Λ is a Prüfer ring if and only if there holds the relation

$$(I_1 \cap I_2)I = I_1 I \cap I_2 I,$$

for any ideals I_1, I_2, and I of Λ.[3)]

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Tokyo Metroporitan University

3) This result was suggested to the writer by T. Ishikawa.