A FOUNDATION OF TORSION THEORY FOR MODULES OVER GENERAL RINGS

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When we consider modules A over a ring R which is not a commutative integral domain, the usual torsion theory becomes somewhat inadequate, since zero-divisors of R are disregarded and since the torsion elements of A do not in general form a submodule. In this paper we shall try to remedy such defects by modifying the fundamental notions such as torsion modules, divisible modules, etc.

We define in §1 torsion-free modules and divisible modules. It turns out that they are well maniable by using the functors Tor_1 and Ext^1. Torsion [resp. reduced] modules are defined in §4 by the property that they have no torsion-free [resp. divisible] parts in a certain sense. Now two fundamental problems arise: Is the torsion-freeness of a (right, say) module equivalent to the vanishing of its torsion part? Is it possible to divide any (right) module into torsion-free and torsion parts? The main proposition of this paper (Prop. 13) states that in order that either one of these problems be answered affirmatively it is necessary and sufficient that all the principal (left) ideals be projective. A ring satisfying this condition will be called a (left) PP ring in this paper. Integral domains and semi-hereditary rings constitute two important classes of PP rings. In §5 we show that our modified theory reduces to the usual theory in two special types of PP rings, namely commutative, Noetherian, PP rings and non-commutative integral domains which have full quotient rings.[1]

It is true that the most properties concerning non-principal ideals can not be touched by torsion theory in any sense, but only principal ideals have some remarkable properties. For instance $\mathrm{Tor}_1(R/\lambda R,\quad)$ and $\mathrm{Ext}^1(R/R\lambda,\quad)$ are put into relation in connection with the purity of module extensions, and this fact yields a new approach to a theorem of Harada (§2).

Received April 4, 1960.

1) (Added in proof) See also a paper of S. Endo, to appear in this volume of this Journal.

§ 1. Torsion-free modules and divisible modules

Let R be a ring with a unit element 1, A an R-left module on which 1 acts as the identity. If $r(\lambda)$ denotes the right ideal of R consisting of right annihilators of $\lambda \in R$, then the subset $r(\lambda)A$ is so to speak *a priori* torsion with respect to λ. Now we call A *torsion-free* if, for every $\lambda \in R$, we have $a \in r(\lambda)A$ whenever $\lambda a = 0$. Similarly we call A *divisible* if we have $a \in \lambda A$ whenever $l(\lambda)a = 0$, for every $\lambda \in R$, where $l(\lambda)$ is the left ideal of left annihilators of λ. We define similarly a torsion-free right module and a divisible right module. Obviously our definitions coincide with the usual ones when R is an integral domain.

In the following, a module means a left R-module unless otherwise stated. We also omit the subscript R or the superscript R in Hom, $\otimes$, Ext, Tor.

Let us consider the sequence

$$R \xrightarrow{\lambda} \lambda R \xrightarrow{\iota} R$$

where the first arrow is the left multiplication by λ, the second the natural injection. Tensoring with A over R, we have

$$A \xrightarrow{\lambda \otimes 1} \lambda R \otimes A \xrightarrow{\iota \otimes 1} A$$

where $\lambda \otimes 1$ is an epimorphism with kernel $r(\lambda)A$, the kernel of $\iota \otimes 1$ is $\mathrm{Tor}_1(R/\lambda R, A)$ and the composed map $(\iota \otimes 1)(\lambda \otimes 1)$ is identified with the left multiplication by λ in A. So we see immediately that

$$\mathrm{Tor}_1(R/\lambda R, A) \approx \{a \in A \ ; \ \lambda a = 0\}/r(\lambda)A,$$

hence

PROPOSITION 1. *A is torsion-free if and only if* $\mathrm{Tor}_1(R/\lambda R, A) = 0$ *for every* $\lambda \in R$.

Similarly we see that

$$\mathrm{Ext}^1(R/R\lambda, A) \approx \{a \in A \ ; \ l(\lambda)a = 0\}/\lambda A,$$

and we have

PROPOSITION 1′. *A is divisible if and only if* $\mathrm{Ext}^1(R/R\lambda, A) = 0$ *for every* $\lambda \in R$.

PROPOSITION 2. *A flat module is always torsion-free; if every finitely generated right ideal of R is principal, then a torsion-free module is flat.*

An injective module is always divisible; if every left ideal of R is principal, then a divisible module is injective.

Proof. The first assertion is a simple corollary of Prop. 1. To prove the second, let A be torsion-free. Since Tor_1 is half exact and commutes with direct limits, we have only to show that $\mathrm{Tor}_1(R/\mathfrak{r}, A)=0$ for all right ideals $\mathfrak{r}$ (see Cartan-Eilenberg [3] VI, Exerc. 6). By a similar argument we may restrict ourselves to finitely generated $\mathfrak{r}$. On assuming that such an $\mathfrak{r}$ be principal, we have $\mathrm{Tor}_1(R/\mathfrak{r}, A)=0$ by Prop. 1. Assertions concerning the divisibility are proved similarly and more simply.

§ 2. Connections with purity of extensions

An extension of R-modules

$$(*)\qquad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad \text{(exact)}$$

is said to be *pure* if it has one of the following two equivalent properties:

(i) $A \cap \lambda B = \lambda A$ for every $\lambda \in R$.

(ii) If $\lambda c = 0$ for a $c \in C$, then there exists $b \in B$ satisfying $\beta b = c$ and $\lambda b = 0$.

(In (i) A is identified with $\alpha A \subset B$.) It is easy to see that these are equivalent respectively to

(i′) $R/\lambda R \otimes A \to R/\lambda R \otimes B$ is a monomorphism for every $\lambda \in R$.

(ii′) $\mathrm{Hom}(R/R\lambda, B) \to \mathrm{Hom}(R/R\lambda, C)$ is an epimorphism for every $\lambda \in R$.

Now we have

PROPOSITION 3. *C is torsion-free if and only if every extension $(*)$ with C as the factor module is pure. A is divisible if and only if every extension $(*)$ with A as the kernel is pure.*

Proof for the first statement. Consider the exact sequence

$$\mathrm{Tor}_1(R/\lambda R, B) \to \mathrm{Tor}_1(R/\lambda R, C) \to R/\lambda R \otimes A \to R/\lambda R \otimes B.$$

Then the 'only if' part is clear, and the 'if' part is seen, taking especially B to be projective. Proof for the second statement is quite similar.

As a corollary we have the equivalence of the following three statements;

(a) *Every extension $(*)$ is pure.*

(b) *Every module is torsion-free.*

(c) *Every module is divisible.*

Now (c) is equivalent to the statement that every $R/R\lambda$ ($\lambda \in R$) is projective. But this last property characterizes the regular rings in the sense of J. von Neumann, as is easily seen. Similarly (b) is equivalent to that every $R/\lambda R$ ($\lambda \in R$) is flat. If $\mathrm{w.gl.dim}\, R = 0$, namely if every module is flat, then so is $R/\lambda R$, and we see that R is regular from the equivalence of (b) and (c). Conversely let R be regular. Then a finitely generated right ideal is principal (see Neumann [10] or Nakayama-Azumaya [9] p. 90). Hence a torsion-free module is flat by Prop. 2, and so (b) states that $\mathrm{w.gl.dim}\, R = 0$. Accordingly we can reformulate the equivalence of (a), (b), (c) as follows:

PROPOSITION 4. *The following three statements on R are equivalent:*

(a) *Every extension* (*) *is pure.*

(b′) $\mathrm{w.gl.dim}\, R = 0$.

(c) *R is a regular ring.*

Thus we have obtained a new approach to Harada's theorem [4] which states the equivalence of (b′) and (c).

Notice also that (b′) and (c) are left-right symmetrical.

§ 3. Fundamental Properties

In this section we study submodules, factor modules etc. of torsion free or divisible modules.

PROPOSITION 5. *An extension of a torsion-free module by a torsion free module yields always a torsion-free module. Similarly for divisible modules.*

This is clear from the half-exactness of Tor and Ext.

If C [resp. A] is a factor [resp. sub] module of B, we have an exact sequence of type (*), with α the natural injection, β the natural projection and A the kernel of α [resp. C the cokernel of β]. Such a sequence will be called *associated* with $B \rightarrow C$ [resp. $A \rightarrow B$].

PROPOSITION 6. *A factor module C of a torsion-free module B is torsion-free if and only if the associated exact sequence* (*) *is pure. Similarly, a submodule A of a divisible module B is divisible if and only if the associated sequence* (*) *is pure.*

This follows immediately from the exactness of the sequence appeared in the proof of Prop. 3, as well as its counterpart for Ext.

We shall call a ring R a *left PP* [resp. *PF*] *ring* if every principal left ideal of R is projective [resp. flat]. A right PP [resp. PF] ring is defined similarly. Obviously a PP ring is a PF ring. Main examples of PP rings are furnished by (not necessarily commutative) integral domains and semi-hereditary rings.

PROPOSITION 7. *In order that any submodule of a torsion-free left module be again torsion-free, it is necessary and sufficient that R be a right PF-ring. In order that any factor module of a divisible left module be again divisible, it is necessary and sufficient that R be a left PP-ring.*

Proof. Let B be torsion-free, and A its submodule. Consider the exact sequence

$$\mathrm{Tor}_2(R/\lambda R, B) \to \mathrm{Tor}_2(R/\lambda R, C) \to \mathrm{Tor}_1(R/\lambda R, A) \to \mathrm{Tor}_1(R/\lambda R, B) = 0$$

belonging to the associated exact sequence $(*)$. If every λR is flat, we have $\mathrm{Tor}_2(R/\lambda R, C) \cong \mathrm{Tor}_1(\lambda R, C) = 0$, whence $\mathrm{Tor}_1(R/\lambda R, A) = 0$, namely A is torsion-free. Conversely, for an arbitrary module C, there exists an exact sequence $(*)$ with B projective, for which obviously $\mathrm{Tor}_2(R/\lambda R, B) = 0$. Hence $\mathrm{Tor}_1(R/\lambda R, A) = 0$ implies $\mathrm{Tor}_1(\lambda R, C) \cong \mathrm{Tor}_2(R/\lambda R, C) = 0$. C being arbitrary, λR is flat. Proof for the divisibility is given similarly.

PROPOSITION 8. i) *A direct sum of torsion-free modules is torsion free.*

ii) *In order that a direct product of torsion-free modules be always torsion-free, it is necessary and sufficient that for every $\lambda \in R$ the right annihilator $r(\lambda)$ be finitely generated.*

Proof. i) is clear from the fact that Tor commutes with a direct sum.

ii) Let $\{A_\alpha\}$ be torsion-free modules, and $A = \prod A_\alpha$ its direct product. For $a = (a_\alpha) \in A$ $(a_\alpha \in A_\alpha)$, $\lambda a = 0$ means $\lambda a_\alpha = 0$ for every α. If $r(\lambda)$ is generated by a finite number of elements, say $\mu_1, \dots, \mu_r$, then there exist $a_{i\alpha} \in A_\alpha$ such that $a_\alpha = \sum_i \mu_i a_{i\alpha}$. Put $a_i = (a_{i\alpha}) \in A$, then we have $a = \sum \mu_i a_i$. Hence A is torsion-free. To prove the converse, we take, for every λ, the direct product $A_\lambda = \prod_{\alpha \in r(\lambda)} R_\alpha$ of isomorphic copies R_α of R over the index set $r(\lambda)$. Let a_λ be the 'diagonal' element of A_λ having the α-th component α for every $\alpha \in r(\lambda)$. Then $\lambda a_\lambda = 0$. Hence, if A_λ is torsion-free, there exist a finite number of elements μ_i of R and $a_i = (\alpha_{i\alpha})$ of A_λ $(i = 1, \dots, r)$ such that $a_\lambda = \sum \mu_i a_i$, namely

$\alpha = \sum \mu_i \alpha_{i\alpha}$ for every $\alpha \in r(\lambda)$. Hence $r(\lambda)$ is finitely generated.

The corresponding proposition for divisible modules is simpler, namely:

PROPOSITION 8′. *A direct product as well as a direct sum of divisible modules is divisible.*

PROPOSITION 9. *If R is a left PP-ring, then every right R-module possesses the largest torsion-free factor module, and every left R-module possesses the largest divisible submodule.*

Proof. The assertion concerning the divisible submodule is an immediate consequence of Prop. 7 and 8′, while the assertion concerning the torsion-free factor module follows from (the mirror statements of) Prop. 7 and 8, if one observes the elementary fact that $l(\lambda)$ is finitely generated when the principal left ideal $R\lambda$ is projective. For the sake of the later use we shall prove in this occasion the following more general Lemma: We call a module A to be *of finite relations* if there exists an epimorphism $\pi : P \to A$ with P free such that the kernel of π is finitely generated. Then we have

LEMMA 1. *A finitely generated module A is projective if and only if A is flat and is of finite relations.*

Proof. A may be expressed as the factor module of a free module F on finite basis by the relation module B. If A is projective, B is a direct summand of F and therefore is finitely generated. Conversely, let $\pi : P_0 \to A$ be an epimorphism with a finitely generated kernel, say B, where P_0 is free. Then P_0 is likewise finitely generated. B admits an epimorphism $\pi_1 : P_1 \to B$ from a finitely generated free module P_1. Applying the standard arguments to the sequence $\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$, we have the isomorphism

$$\mathrm{Tor}_1(\mathrm{Hom}_Z(X, C), A) \cong \mathrm{Hom}_Z(\mathrm{Ext}^1(A, X), C)$$

for any R-module X and any injective Z-module C, where Z denotes the ring of rational integers. It follows from this isomorphism that A is flat if and only if A is projective (cf. Cartan-Eilenberg [3], VI, §5 and Exerc. 3). This proves our Lemma.

It is an open question whether the converse of Prop. 9 holds or not. Cf. Prop. 13 below.

When R is a PP ring we may have actually more explicit treatment in virtue of

LEMMA 2. *A principal left ideal $R\lambda$ is projective if and only if there exists an idempotent e_λ such that $\lambda = e_\lambda \cdot \lambda$ and $l(\lambda) = l(e_\lambda) = Re'_\lambda$, where $e_\lambda + e'_\lambda = 1$.*

Proof. If $R\lambda$ is projective, the sequence $0 \to l(\lambda) \to R \to R\lambda \to 0$ splits. Hence there exists $e_\lambda \in R$ such that $e_\lambda \lambda = \lambda$ and $l(e_\lambda) = l(\lambda)$. As the former identity implies $(1 - e_\lambda)\lambda = 0$, the latter shows $(1 - e_\lambda)e_\lambda = 0$, i.e. e_λ is an idempotent. The converse is clear.

From the definition of torsion-free resp. divisible modules follows

PROPOSITION 10. *Let R be a left PP ring. A right module A is torsion-free if and only if $\lambda_r : Ae_\lambda \to A$, induced by the right operation of λ, is monomorphic for every $\lambda \in R$; while a left module A is divisible if and only if $\lambda_l : A \to e_\lambda A$, induced by the left operation of λ, is epimorphic for every $\lambda \in R$.*

Both mappings λ_r, λ_l indicated above are not R-homomorphisms in general. But if S is another operator domain of A commuting with R, they are S-homomorphisms. Examples of such S that may be taken universally for all R-modules are the ring of rational integers Z, the center of R and, in case R is commutative, R itself. Now, we have

PROPOSITION 11. *Let R be a left PP ring.*

i) *In case $({}_RA_S, {}_SC)$, if A is R-divisible, then $A \otimes_S C$ is also R-divisible.*

ii) *In case $({}_RA_S, C_S)$, if A is R-divisible, then* $\mathrm{Hom}_S(A, C)$ *is R-torsion-free; and in case $({}_SA, {}_SC_R)$, if C is R-torsion-free, then* $\mathrm{Hom}_S(A, C)$ *is R-torsion-free.*

We omit the simple proof, and instead, remark that it is a special case of a proposition concerning additive functors of S-modules (cf. Cartan-Eilenberg [3] VII, §1).

This proposition will be utilized later to study a problem which involves only an operator ring R (Prop. 14).

§4. Torsion modules and reduced modules

A left module A will be called a *torsion* module if $\mathrm{Hom}(A, C) = 0$ for every torsion-free module C. From the standard properties of Hom we deduce readily

PROPOSITION 12. i) *The direct sum $\sum A_\alpha$ is a torsion module if and only if every summand A_α is torsion module.*

ii) *If A is a torsion module, then so is any homomorphic image of A.*

iii) *Any extension of a torsion module by a torsion module yields again a torsion module.*

COROLLARY. *A module A has the largest torsion submodule.*

We call the largest torsion submodule of A *the torsion submodule* of A, and denote it by $T(A)$.

A *reduced* module C is defined by the property that $\mathrm{Hom}(A, C)=0$ for every divisible module A. We have similarly as above

PROPOSITION 12′. i) *The direct product $\prod C_\alpha$ is reduced if and only if every C_α is reduced.*

ii) *If C is reduced, then so is any submodule of C.*

iii) *Any extension of a reduced module by a reduced module yields again a reduced module.*

COROLLARY. *Among submodules B of A with divisible factor modules there exists the smallest one, which we denote by $D(A)$.*

If R is a PF ring, A is a torsion module if and only if it has only the trivial torsion-free factor A/A (Prop. 7). Similarly, if R is a PP ring, C is reduced if and only if it has only the trivial divisible submodule 0.

If R is a commutative integral domain, our definitions of torsion modules and reduced modules coincide with the usual ones. Now, in this case, A is torsion-free if and only if $T(A)=0$. In our general setting, the 'only if' part still holds evidently, but not necessarily the 'if' part. So we shall call A *weakly torsion-free* when $T(A)=0$. Similarly we call A *weakly divisible* when $D(A)=A$. Then a divisible module is weakly divisible. A fundamental question asking when the weak torsion-freeness [resp. weak divisibility] reduces to the torsion-freeness [resp. divisibility] hitherto considered is answered by the following

PROPOSITION 13. *The following statements for a ring R are equivalent:*

i) *R is a left PP ring.*

ii) *For every right module B, $B/T(B)$ is torsion-free.*

iii) *Any weakly torsion-free right module is torsion-free.*

iv) *If* $\mathrm{Hom}(A, C)=0$ *for all torsion right modules A, then C is torsion-free.*

ii′) *For every left module B, $D(B)$ is divisible.*

iii′) *Any weakly divisible left module is divisible.*

iv$'$) *If* Hom $(A, C) = 0$ *for all reduced left modules* C, *then* A *is divisible.*

Proof. i) $\Rightarrow$ ii) By Prop. 9, B has the smallest submodule A such that B/A is torsion-free. Since A can not have a non-trivial torsion-free factor by Prop. 5, A is a torsion module. Hence $A \subset T(B)$. But it is then clear that $A = T(B)$, and hence $B/T(B)$ is torsion-free.

ii) $\Rightarrow$ iii) Evident.

iii) $\Rightarrow$ iv) Putting $A = T(C)$, we have $T(C) = 0$.

Assume iv), then submodules and direct products of torsion-free modules are always torsion-free. Hence Prop. 7 and 8 together with Lemma 1 following the Prop. 9 show us that R is in fact a PP-ring.

i) $\Rightarrow$ ii$'$) $\Rightarrow$ iii$'$) $\Rightarrow$ iv$'$) $\Rightarrow$ i) can be proved similarly.

Proposition 14. *If* A *is a torsion right module and* B *a divisible left module, we have* $A \otimes B = 0$.

Proof. For any Z-module C we have a natural isomorphism

$$\mathrm{Hom}_Z(A \underset{R}{\otimes} B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_Z(B, C)).$$

Since, by Prop. 11, Hom (B, C) is R-torsion-free, the right hand side reduces to 0. C being arbitrary, we have $A \otimes B = 0$.

Concerning submodules of a torsion module we have not so large information. Let B be a torsion module and A its submodule. Then it is clear that A is also a torsion module if and only if $\mathrm{Ext}^1(A', C) \to \mathrm{Ext}^1(B, C)$ is monomorphic for every torsion-free module C, where $A' = B/A$. This is certainly the case if C admits a monomorphism into an injective, torsion-free module Q, since then we have Hom $(B, Q) = 0$ as well as $\mathrm{Ext}^1(A', Q) = 0$. Hence

Proposition 15. *If every torsion-free module admits a monomorphism into an injective, torsion-free module, then any submodule of a torsion module is a torsion module.*

The condition of the Proposition is certainly satisfied by a (not necessarily commutative) integral domain R possessing the full ring of quotients. In this case, however, our torsion theory reduces to the usual one (see Prop. 18). Beyond this trivial case we have no examples of R satisfying that condition.

Proposition 16. *In case* $(A_S, {}_SB_R)$, *if* B *is a torsion* R-*module, then so is*

$A \otimes_S B$.

Proof. For every torsion-free R-module C, we have $\mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)) = 0$.

COROLLARY. *Under the assumptions of Prop.* 15, $\mathrm{Tor}_n^S(A, B)$, $n = 0, 1, 2, \ldots$, *are torsion modules whenever B is a torsion module.* $(A_S, {}_SB_R)$.

Indeed, with a suitable S-module A' we have $\mathrm{Tor}_n^S(A, B) \cong \mathrm{Tor}_1^S(A', B)$, the right hand side in turn is a submodule of certain R-torsion module $A'' \otimes_S B$.

§ 5. Comparison with the usual torsion theory

a) Commutative ring R.

LEMMA 3. *A commutative ring satisfying the maximum condition for ring direct summands is a PP ring if and only if it is a direct sum of a finite number of integral domains.*

Proof. For a commutative ring R, an idempotent $e \neq 1$ yields a ring direct sum decomposition $R = Re \oplus R(1-e)$. Hence a PP ring is indecomposable if and only if it is an integral domain by Lemma 2. This being said, our Lemma is easily proved by familiar arguments.

Let in general $R = R_1 \oplus \cdots \oplus R_r$ be a ring direct sum $(r < \infty)$. Then an R-module A is decomposed into the direct sum $A = A_1 \oplus \cdots \oplus A_r$, $A_i = R_i A$. In particular for $\lambda = \sum \lambda_i$, $\lambda_i \in R_i$, $R/R\lambda \cong R_1/R_1\lambda_1 \oplus \cdots \oplus R_r/R_r\lambda_r$. On the other hand we have

$$\mathrm{Tor}_n^R(A, B) \cong \sum \mathrm{Tor}_n^{R_i}(A_i, B_i),$$
$$\mathrm{Ext}_R^n(A, B) \cong \sum \mathrm{Ext}_{R_i}^n(A_i, B_i).$$

It follows readily that the torsion theory for R-modules can be reduced completely to those for R_i-modules.

Applying this to our case, we have immediately

PROPOSITION 17. *The torsion theory for modules over a commutative PP ring satisfying the ascending chain condition for ring direct summands coincides with the usual torsion theory.*

Remark. As another simple consequence of the above Lemma 3, we have a theorem of Auslander-Buchsbaum [2] stating that a commutative Noetherian ring is hereditary, if and only if it is a direct sum of a finite number of Dedekind domains.

b) Non-commutative integral domain R.

For a moment let R be arbitrary. Denote the semigroup of regular elements (i.e. non zero-divisors) of R by R^*. We call as usual an element a of an R-module A to be a *torsion element* if there exists $\lambda \in R^*$ such that $\lambda a = 0$. We denote the set of torsion elements of A by $t(A)$. We see easily $t(A) \subset T(A)$.

Now we know that R has the left full ring of quotients, which we shall call simply the left quotient ring and denote by Q_l, if and only if for any $\lambda \in R^*$ and $\mu \in R$ there exist $\alpha \in R^*$ and $\beta \in R$ such that $\alpha\mu = \beta\lambda$; similarly for the existence of the right quotient ring Q_r (see Asano [1] or Jacobson [6]).

LEMMA 4. *If R has the left quotient ring Q_l, $t(A)$ is a submodule of A and coincides with the kernel of the natural mapping $A \to Q_l \otimes A$.*

Proof is easy and we omit it.

PROPOSITION 18. *For a (not necessarily commutative) integral domain R, the following statements are equivalent:*

i) *R has the left quotient ring Q_l.*

ii) *For any left module A, we have $T(A) = t(A)$.*

iii) *For any $\lambda \neq 0$, every element of $R/R\lambda$ is a torsion element.*

Proof. i) $\Rightarrow$ ii) By Lemma 4, $t(A)$ is a submodule of $T(A)$. On the other hand $A/t(A)$ is torsion-free in the usual sense, hence also in our sense since R has no zero-divisor. Hence $t(A) = T(A)$.

ii) $\Rightarrow$ iii) It is clear that $\mathrm{Hom}(R/R\lambda, C) = 0$ for $\lambda \in R^*$ and for any torsion-free C, i.e. $R/R\lambda$ is a torsion module. Hence iii) follows from ii) taking $A = R/R\lambda$.

Finally iii) states the necessary and sufficient condition above mentioned for the existence of Q_l.

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