A GLOBAL FORMULATION OF THE FUNDAMENTAL THEOREM OF THE THEORY OF SURFACES IN THREE DIMENSIONAL EUCLIDEAN SPACE

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§ 1. Introduction

Let us consider a surface S of class C^3 in Euclidean space E_3 and

(1.1) $$x^i = x^i(u^1, u^2) \qquad (i, j, k = 1, 2, 3)$$

be its parametric representation of class C^3. Then

(1.2) $$g_{\beta\gamma} = \sum_i \frac{\partial x^i}{\partial u^\beta}\frac{\partial x^i}{\partial u^\gamma} \qquad (\alpha, \beta, \gamma = 1, 2)$$

(1.3) $$h_{\beta\gamma} = \left|\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} \quad \frac{\partial x^i}{\partial u^1} \quad \frac{\partial x^i}{\partial u^2}\right| \Big/ (g_{11}g_{22} - g_{12}^2)$$

are the first and second fundamental tensors of the surface S. If we put

(1.4) $$\frac{\partial x^i}{\partial u^\alpha} = x^i_\alpha$$

and denote the unit normal vector of S by e^i, then the functions x^i satisfy the derived equations

(1.5) $$\begin{aligned} \frac{\partial x^i_\beta}{\partial u^\gamma} &= \left\{ {\alpha \atop \beta\gamma} \right\} x^i_\alpha + h_{\beta\gamma} e^i, && \text{(Gauss)} \\ \frac{\partial e^i}{\partial u^\beta} &= -h^\alpha_\beta x^i_\alpha, && \text{(Weingarten)} \end{aligned}$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$'s are Christoffel's symbols constructed from $g_{\beta\gamma}$ and

(1.6) $$h^\alpha_\beta = g^{\alpha\gamma} h_{\beta\gamma}.$$

By virtue of the relation

(1.7) $$\frac{\partial^2 x^i_\beta}{\partial u^\gamma \partial u^\delta} = \frac{\partial^2 x^i_\beta}{\partial u^\delta \partial u^\gamma}, \qquad \frac{\partial^2 e^i}{\partial u^\beta \partial u^\gamma} = \frac{\partial^2 e^i}{\partial u^\gamma \partial u^\beta},$$

we see that $g_{\beta\gamma}$ and $h_{\beta\gamma}$ satisfy the following relations:

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$$
\begin{aligned}
&R^{\alpha}_{\beta\gamma\delta} = h_{\beta\gamma} h^{\alpha}_{\delta} - h_{\beta\delta} h^{\alpha}_{\gamma}, && \text{(Gauss)} \\
&h_{\beta\gamma,\delta} = h_{\beta\delta,\gamma}, && \text{(Codazzi)}
\end{aligned} \tag{1.8}
$$

where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann's curvature tensor constructed from $g_{\beta\gamma}$ and $h_{\beta\gamma,\delta}$ is the covariant derivative of $h_{\beta\gamma}$ with respect to $g_{\beta\gamma}$.

The converse of this fact is well-known as the fundamental theorem of the theory of surfaces in E_3 and is stated as follows:

I. Given a symmetric positive definite tensor $g_{\beta\gamma}(u^1, u^2)$ of class C^2 and a symmetric tensor $h_{\beta\gamma}(u^1, u^2)$ of class C^1 so that they satisfy the Gauss and Codazzi relations (1.8), then there exists a surface S in E_3 which has the given tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ as its first and second fundamental tensors respectively.

II. If two surfaces S and $\overline{S}$ in E_3 have the same tensors $g_{\beta\gamma}(u^1, u^2)$ and $h_{\beta\gamma}(u^1, u^2)$ as their first and second fundamental tensors, then they are congruent under the group of motions of E_3.

Usually the fundamental theorem is stated as a theorem and is not separated in two. However, we divided the contents in two parts, Theorems I and II above correspond to the first and second main theorems of the classical theory of invariants.

We shall take up the Theorem I. In all classical books on Euclidean differential geometry, as far as I know, the domains of definition of the tensors $g_{\beta\gamma}$, $h_{\beta\gamma}$ and the surface S are not given. In other words, all the authors treated the theorem as a theorem in the small. In this paper, we shall formulate Theorem I in a global form which can be done as an application of the theory of distributions and the theory of fiber bundles. Our main results are Theorem 1 of § 4 and Theorems 3 of § 8.

A global formulation of Theorem II is given as Theorem 2 of § 4 and Theorem 4 of § 8.

Part I. The case where the domain of definition of the tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ is a connected and simply connected domain in u^1u^2-plane

§ 2. The involutive distribution in $D \times F$

Let us assume that a symmetric positive definite tensor $g_{\beta\gamma}$ and a symmetric tensor $h_{\beta\gamma}$ are given on a connected and simply connected domain D in u^1u^2-plane and they satisfy the Gauss and Codazzi relations (1.8) on D.

Let R_{12} be a 12-dimensional real number space with the coordinates (x^i, x^i_1, x^i_2, e^i). Consider in R_{12} a submanifold K defined by

$$\frac{x^1_1}{x^1_2} = \frac{x^2_1}{x^2_2} = \frac{x^3_1}{x^3_3} \tag{2.1}$$

and denote by F the complement of K in R_{12}, i.e.

$$F = R_{12} - K. \tag{2.2}$$

Let us consider the product space $D \times F$ with the natural topology and denote the natural projection $D \times F \to D$ by π. $D \times F$ is a 14-dimensional manifold. At every point $(u^\alpha, x^i, x^i_1, x^i_2, e^i)$ of $D \times F$ we can associate two vectors ξ and η with the following components:

$$\begin{aligned} \xi &: \left(1,\ 0,\ x^i_1,\ \left\{ {\alpha \atop 11} \right\} x^i_\alpha + h_{11} e^i,\ \left\{ {\alpha \atop 12} \right\} x^i_\alpha + h_{12} e^i,\ - h^\alpha_1 x^i_\alpha \right), \\ \eta &: \left(0,\ 1,\ x^i_2,\ \left\{ {\alpha \atop 12} \right\} x^i_\alpha + h_{12} e^i,\ \left\{ {\alpha \atop 22} \right\} x^i_\alpha + h_{22} e^i,\ - h^\alpha_2 x^i_\alpha \right). \end{aligned} \tag{2.3}$$

They constitute two vector fields over $D \times F$. As ξ and η are linearly independent at every point of $D \times F$, they span a two dimensional plane in the tangent space at every point of $D \times F$. Hence, the vector fields ξ and η determine a 2-dimensional distribution over $D \times F$.

LEMMA 1. *The distribution determined by ξ and η in $D \times F$ is involutive.*

Proof. In order to show that the distribution is involutive, it is sufficient to show that the Poisson bracket $[\xi, \eta]$ of ξ and η is a linear combination of ξ and η. Here we can prove that $[\xi, \eta] = 0$.

Let us denote the components of vectors ξ, η and $[\xi, \eta]$ in $D \times F$ by $(\xi^\alpha, \xi^i, \xi^i_1, \xi^i_2, \xi^i_\Delta)$ and so on.

First, it is evident that

$$[\xi, \eta]^\alpha = 0,$$

because ξ^α and η^α are constants. We shall note explicitly that

$$\begin{aligned} [\xi, \eta]^\alpha = \xi^\beta \frac{\partial \eta^\alpha}{\partial u^\beta} + \xi^j \frac{\partial \eta^\alpha}{\partial x^j} + \xi^j_1 \frac{\partial \eta^\alpha}{\partial x^j_1} + \xi^j_2 \frac{\partial \eta^\alpha}{\partial x^j_2} + \xi^j_\Delta \frac{\partial \eta^\alpha}{\partial e^j} \\ - \eta^\beta \frac{\partial \xi^\alpha}{\partial u^\beta} - \eta^j \frac{\partial \xi^\alpha}{\partial x^j} - \eta^j_1 \frac{\partial \xi^\alpha}{\partial x^j_1} - \eta^j_2 \frac{\partial \xi^\alpha}{\partial x^j_2} - \eta^j_\Delta \frac{\partial \xi^\alpha}{\partial e^j} \end{aligned} \tag{2.4}$$

by definition.

Secondly, replacing α by i in the last equation and noting that $\xi^i = x_1^i$ and $\eta^i = x_2^i$, we get

$$[\xi, \eta]^i = \xi_2^i - \eta_1^i = 0,$$

because $\xi_2^i = \eta_1^i = \left\{ {\alpha \atop 12} \right\} x_\alpha^i + h_{12} e^i$.

Thirdly, replacing α in (2.4) by the index $\overset{i}{1}$ and noting that

$$\xi_1^i = \left\{ {\alpha \atop 11} \right\} x_\alpha^i + h_{11} e^i, \qquad \eta_1^i = \left\{ {\alpha \atop 12} \right\} x_\alpha^i + h_{12} e^i$$

we get

$$\begin{aligned}[\xi, \eta]_1^i = \xi^\beta\Bigg(\frac{\partial \left\{ {\alpha \atop 12} \right\}}{\partial u^\beta} x_\alpha^i + \frac{\partial h_{21}}{\partial u^\beta} e^i\Bigg) + \xi_1^i \left\{ {1 \atop 12} \right\} + \xi_2^i \left\{ {2 \atop 12} \right\} + \xi_\Delta^i h_{21} \\ - \eta^\beta\Bigg(\frac{\partial \left\{ {\alpha \atop 11} \right\}}{\partial u^\beta} x_\alpha^i + \frac{\partial h_{11}}{\partial u^\beta} e^i\Bigg) - \eta_1^i \left\{ {1 \atop 11} \right\} - \eta_2^i \left\{ {2 \atop 11} \right\} - \eta_\Delta^i h_{11}.\end{aligned}$$

Putting the components (2.3) of ξ and η in the right hand side of the last equation, we can easily transform it into the following form:

$$[\xi, \eta]_1^i = (R_{121}^\alpha - h_{11} h_2^\alpha - h_{12} h_1^\alpha) x_\alpha^i + (h_{12,1} - h_{11,2}) e^i.$$

Hence, by virtue of (1.8), we see that

$$[\xi, \eta]_1^i = 0.$$

In the same way, we can verify without difficulty that

$$[\xi, \eta]_2^i = 0, \qquad [\xi, \eta]_\Delta^i = 0.$$

Summing up these equations we get

$$[\xi, \eta] = 0. \tag{2.5}$$

Q.E.D.

§ 3. Homeomorphism of the domain D and the maximal integral manifold D^*

As the distribution in consideration is involutive, there exists a maximal integral manifold of the distribution through every point of $D \times F$. Let us take the maximal integral manifold passing through the point A^* $(u_0^\alpha, x_0^i, x_{10}^i, x_{20}^i, e_0^i)$ $(A(u_0^\alpha) \in D)$ such that

$$(3.1)\qquad \begin{cases} \sum_i x^i_{\beta 0} x^i_{\gamma 0} = g_{\beta\gamma}(u_0), \\ \sum_i x^i_{\beta 0} e^i_0 = 0, \\ \sum_i (e^i_0)^2 = 1 \end{cases}$$

and denote it by D^*. We shall show that D^* is homeomorphic to D under the natural projection π.

First we shall prove that D^* is mapped onto D by π. To do so we take a differentiable curve Γ starting from the point $A(u^\alpha_0)$:

$$(3.2)\qquad u^\alpha = u^\alpha(t), \quad 0 \leqq t \leqq 1 \quad (u^\alpha_0 = u^\alpha(0))$$

and consider the solution of simultaneous differential equations

$$(3.3)\qquad \begin{aligned} \frac{dx^i}{dt} &= x^i_\alpha \dot{u}^\alpha, \qquad \left(\dot{u}^\alpha = \frac{du^\alpha}{dt}\right) \\ \frac{dx^i_\beta}{dt} &= \left[\left\{ {\alpha \atop \beta\gamma} \right\} x^i_\alpha + h_{\beta\gamma} e^i\right] \dot{u}^\gamma, \\ \frac{de^i}{dt} &= -h^\alpha_\beta x^i_\alpha \dot{u}^\beta \end{aligned}$$

under the initial condition "for $t=0$, $x^i = x^i_0$, $x^i_\beta = x^i_{\beta 0}$ and $e^i = e^i_0$." If we denote the solution by $x^i(t)$, $x^i_\beta(t)$ and $e^i(t)$ $(0 \leqq t \leqq 1)$, then $(u^\alpha(t), x^i(t), x^i_1(t), x^i_2(t), e^i(t))$ $(0 \leqq t \leqq 1)$ determines a curve Γ^* in $D \times R_{12}$. we assert that the curve Γ^* lies in $D \times F$. To show it, we first note that for the solution in consideration the following equations hold good:

$$\begin{aligned} \frac{d(\sum x^i_\beta x^i_\gamma)}{dt} &= \left[\left\{ {\alpha \atop \beta\delta} \right\}(\sum x^i_\alpha x^i_\gamma) + \left\{ {\alpha \atop \gamma\delta} \right\}(\sum x^i_\beta x^i_\alpha) + h_{\beta\gamma}(\sum x^i_\gamma e^i) + h_{\gamma\delta}(\sum x^i_\beta e^i)\right]\dot{u}^\delta, \\ \frac{d(\sum x^i_\beta e^i)}{dt} &= \left[-h^\alpha_\gamma(\sum x^i_\alpha x^i_\beta) + \left\{ {\alpha \atop \beta\gamma} \right\}(\sum x^i_\alpha e^i) + h_{\beta\gamma}(\sum e^i e^i)\right]\dot{u}^\gamma, \\ \frac{d(\sum e^i e^i)}{dt} &= -2h^\alpha_\beta(\sum x^i_\alpha e^i)\dot{u}^\beta. \end{aligned}$$

Accordingly, $\sum x^i_\beta x^i_\gamma$, $\sum x^i_\beta e^i$ and $\sum (e^i)^2$ are a set of solutions of the following differential equations on $P_{\alpha\beta} = P_{\beta\alpha}$, Q_β and R:

$$\begin{aligned} \frac{dP_{\beta\gamma}}{dt} &= \left[\left\{ {\alpha \atop \beta\delta} \right\} P_{\alpha\gamma} + \left\{ {\alpha \atop \gamma\delta} \right\} P_{\alpha\beta} + h_{\beta\delta} Q_\gamma + h_{\gamma\delta} Q_\beta\right]\dot{u}^\delta, \\ \frac{dQ_\beta}{dt} &= \left[-h^\alpha_\gamma P_{\alpha\beta} + \left\{ {\alpha \atop \beta\gamma} \right\} Q_\alpha + h_{\beta\gamma} R\right]\dot{u}^\delta, \\ \frac{dR}{dt} &= -2h^\alpha_\beta Q_\alpha \dot{u}^\beta, \end{aligned}$$

On the other hand, if we replace $P_{\beta\gamma}$, Q_β and R by $g_{\beta\gamma}(u(t))$, 0 and 1 respectively, the last equations are satisfied identically. So if we note that $\sum x^i_\beta x^i_\gamma$, $\sum x^i_\beta e^i$, $\sum (e^i)^2$ have same values as $g_{\beta\gamma}(u_0)$, 0 and 1 at u^α_0 by (3.1), we can see that the relations

$$
\begin{aligned}
&\textstyle\sum x^i_\beta x^i_\gamma = g_{\beta\gamma}(u(t)),\\
&\textstyle\sum x^i_\beta e^i = 0, \qquad\qquad (3.3)\\
&\textstyle\sum (e^i)^2 = 1
\end{aligned}
$$

hold identically by virtue of the uniqueness of solutions of differential equations.

From (3.3) we see that on Γ^*

$$
\begin{vmatrix} x^2_1 & x^3_1 \\ x^2_2 & x^3_2 \end{vmatrix}^2 + \begin{vmatrix} x^3_1 & x^1_1 \\ x^3_2 & x^1_2 \end{vmatrix}^2 + \begin{vmatrix} x^1_1 & x^2_1 \\ x^1_2 & x^2_2 \end{vmatrix}^2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} \neq 0.
$$

Hence, Γ^* lies in $D \times F$.

Now, every tangent vector of Γ^* is easily seen to be of the form $\dot{u}^1\xi + \dot{u}^2\eta$, so the curve Γ^* lies on the integral manifold D^*. This shows that every point of the curve is the image of a point of Γ^* on D^* by π. As D is connected, every point of D is bound with A by a differentiable curve. Hence, D^* is mapped onto D by π. We shall say that the curve Γ^* is a *lift* of Γ through the point A.

Conversely, every point of D^* is the end point of the lift of a curve in D starting from A.

LEMMA 2. *Let Γ_0 and Γ_1 be two differentiable curves having the same end points A and B in the simply connected domain D. Then the lifts Γ^*_0 and Γ^*_1 of Γ_0 and Γ_1 respectively passing through the same point A^* over A have the same end point B^*.*

Proof. Let us denote vectors of natural frames of reference in D by e_1 and e_2 respectively. Then by the natural projection, it is clear that

$$\pi\xi = e_1, \qquad \pi\eta = e_2,$$

where ξ and η are tangent vectors of D^* defined by (2.3). Hence, by the projection π, the vector space spanned by ξ and η at $P^*(u^\alpha, x^i, x^i_1, x^i_2, e^i)$ on D^* is mapped onto the vector space spanned by e_1 and e_2 at $P(u^\alpha)$ in D, which shows that the projection π is locally homeomorphic.

Accordingly, if we note that Γ_0 is continuously deformable to Γ_1, we can easily show that Γ_0^* and Γ_1^* have the same end point B^*.

By Lemma 2, the inverse mapping $\pi^{-1} : D \to D^*$ is one valued, hence the natural projection $D^* \to D$ is one to one and onto. Moreover, as we have already seen, it is locally homeomorphic. Consequently, we can see that the natural projection is a homeomorphism of D^* onto D. Hence, we get the following

LEMMA 3. *For every maximal integral manifold D^* in $D \times F$ over a simply connected domain D, the natural projection π is a homeomorphism of D^* onto D.*

§ 4. The fundamental theorems

By Lemma 3, we may choose u^α as parameters of D^*, the point u^α on D^* being the inverse image of the point u^α in D by π. So the integral manifold D^* can be expressed by functions

$$x^i(u^1, u^2),\ x_1^i(u^1, u^2),\ x_2^i(u^1, u^2),\ e^i(u^1, u^2) \tag{4.1}$$

defined on D.

Now, the tangents of u^1-curves and u^2-curves on this integral manifold have components of the forms $(\delta_1^\alpha, \ldots)$ and $(\delta_2^\alpha, \ldots)$ respectively, we see that ξ, η on the integral manifold are nothing but tangent vectors of u^1-curves and u^2-curves. Hence the functions in (4.1) are solutions of the following differential equations:

$$\begin{aligned} \frac{\partial x^i}{\partial u^\alpha} &= x_\alpha^i, \\ \frac{\partial x_\beta^i}{\partial u^\gamma} &= \left\{ {\alpha \atop \beta\gamma} \right\} x_\alpha^i + h_{\beta\gamma} e^i, \\ \frac{\partial e^i}{\partial u^\beta} &= -h_\beta^\alpha x_\alpha^i. \end{aligned} \tag{4.2}$$

By virtue of (4.2), we may easily see, in the same way as in § 3, that

$$\begin{aligned} &\sum x_\beta^i(u) x_\gamma^i(u) = g_{\beta\gamma}(u), \\ &\sum x_\beta^i(u) e^i(u) = 0, \\ &\sum (e^i(u))^2 = 1 \end{aligned} \tag{4.3}$$

hold identically on D.

Now, let us consider the mapping f of D^* into E_3 defined by

$$(u^\alpha, x^i, x_1^i, x_2^i, e^i) \to x^i.$$

Then, the image $S=fD^*=f\circ\pi^{-1}D$:

$$x^i=x^i(u^1,\ u^2)\qquad (u^1,\ u^2)\in D \tag{4.4}$$

is a surface in E_3 defined on D in the sense that the functions $x^i(u^1,\ u^2)$ is differentiable and the Jacobians

$$\frac{\partial(x^2,\ x^3)}{\partial(u^1,\ u^2)},\qquad \frac{\partial(x^3,\ x^1)}{\partial(u^1,\ u^2)},\qquad \frac{\partial(x^1,\ x^2)}{\partial(u^1,\ u^2)}$$

do not vanish simultaneously at every point of D. This implies that there exist sufficiently small neibourhoods to every pair of corresponding points of D and S such that they are homeomorphic with each other under $f\circ\pi^{-1}$. We shall say that the surface S is the f-image of D^*.

By virtue of $(4.2)_1$, we see that x^i_α are tangent vectors of S and satisfy the relations $(4.3)_1$. Hence the given tensor $g_{\beta\gamma}$ is the first fundamental tensor of the surface S. And by $(4.3)_{2,3}$ we see that $e^i(u)$ is the unit normal vector of the surface S. Moreover, $(4.2)_2$ shows that the given tensor $h_{\beta\gamma}$ is the second fundamental tensor of S. Accordingly, we get the following

THEOREM 1. *Suppose that a positive definite tensor $g_{\beta\gamma}(u^1,\ u^2)$ of class C^2 and a tensor $h_{\beta\gamma}(u^1,\ u^2)$ of class C^1 are defined on a connected and simply connected domain D in the u^1u^2-plane and they satisfy the Gauss and Codazzi relations* (1.8). *Then, in E_3 there exists a surface S defined on D which is the f-image of D^* over D and have the given tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ as its first and second fundamental tensors.*

If the surface has no self intersections or self contact points, then S is homeomorphic with D. However, even if S has self intersections or self contact points, under suitable conventions, we may regard them as homeomorphic in an extended sense.

It is clear that the following theorem holds true as an immediate consequence of the fact that there exists one and only one maximal integral manifold passing through any given point A $(u^\alpha_0,\ x^i_0,\ x^i_{10},\ x^i_{20},\ e^i_0)$ in $D\times F$.

THEOREM 2. *If two surfaces S and $\bar{S}$ defined over a connected and simply connected domain D of the u^1u^2-plane have the same tensors $g_{\beta\gamma}(u^1,\ u^2)$ and $h_{\beta\gamma}(u^1,\ u^2)$ as their first and second fundamental tensors respectively, then they are congruent in the large under the group of motions in E_3.*

Part II. The case where the domain of definition D of the tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ is a two dimensional differentiable manifold

§5. Construction of a fibre bundle over D

Let $\{U_\lambda\}$ ($\lambda \in J$, J : a set of indices) be (simply connected) coordinate neighbourhoods of class C^3 which cover the given manifold D. Then for every U_λ we can construct a product space $U_\lambda \times F$ as we have defined in Part I. We shall show first that points of the product spaces $U_\lambda \times F$ ($\lambda \in J$) can be identified so that all $U_\lambda \times F$ constitute a fibre bundle over D.

To do so, let us consider linear transformations of F onto itself of the following type:

$$(5.1)\qquad \begin{cases} \bar{x}^i = x^i, \\ \bar{x}_1^i = a_{11}x_1^i + a_{12}x_2^i, \\ \bar{x}_2^i = a_{21}x_1^i + a_{22}x_2^i, \\ \bar{e}^i = e^i. \end{cases} \qquad \begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix} \neq 0$$

It is clear that all linear transformations of this type form a group which we shall call G.

Now, suppose that $U_\lambda \cap U_\mu$ is not empty. Denoting the coordinates in U_λ and U_μ by u^α and $\bar{u}^\alpha$ respectively, the coordinate transformation in $U_\lambda \cap U_\mu$ is given by

$$(5.2)\qquad \bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2), \qquad (u^1, u^2) \in U_\lambda \cap U_\mu \subset U_\lambda$$

or by

$$(5.2)'\qquad u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2), \qquad (\bar{u}^1, \bar{u}^2) \in U_\lambda \cap U_\mu \subset U_\mu$$

equivalent to the former. Put

$$(5.3)\qquad \begin{cases} \bar{x}^i = x^i, \\ \bar{x}_1^i = \dfrac{\partial u^\alpha}{\partial \bar{u}^1} x_\alpha^i, \\ \bar{x}_2^i = \dfrac{\partial u^\alpha}{\partial \bar{u}^2} x_\alpha^i, \\ \bar{e}^i = e^i, \end{cases}$$

and let us identify points $(u^\alpha, x^i, x_1^i, x_2^i, e^i)$ $(u^\alpha \in U_\lambda \cap U_\mu)$ of $U_\lambda \times F$ with points $(\bar{u}^\alpha, \bar{x}^i, \bar{x}_1^i, \bar{x}_2^i, \bar{e}^i)$ of $U_\mu \times F$, where $\bar{u}^\alpha$ is related to u^α by (5.2) and $\bar{x}^i$,

$\bar{x}_1^i$, $\bar{x}_2^i$, and $\bar{e}^i$ are related to x^i, x_1^i, x_2^i, e^i by (5.3). We shall denote the linear transformation (5.3) by $\psi_{\mu\lambda}(u^1, u^2)$. Then it is easily seen that for every set of three neighbourhoods U_λ, U_μ and U_ν such that $U_\lambda \cap U_\mu \cap U_\nu$ is not empty,

$$\psi_{\nu\mu}\psi_{\mu\lambda} = \psi_{\nu\lambda}.$$

Hence, all product spaces $U_\lambda \times F$ constitute a fibre bundle over D with the director space F and the structural group G. We shall denote this fibre bundle by $\mathfrak{B}$ and the projection $\mathfrak{B} \to D$ by π.

§ 6. Unification of distributions defined in $U_\lambda \times F$

Let us assume that a symmetric positive definite tensor $g_{\beta\gamma}$ and a symmetric tensor $h_{\beta\gamma}$ are given over the manifold D and they satisfy the Gauss and Codazzi equations (1.8) over D.

Now, every product bundle $U_\lambda \times F$ has a distribution defined in Part I. We shall show next the following

LEMMA 4. *All distributions in $U_\lambda \times F$ ($\lambda \in J$) are unified to a two dimensional distribution in $\mathfrak{B}$ which is identical in $U_\lambda \times F$ with that in $U_\lambda \times F$.*

Proof. Let us first denote the components of the vector ξ of $U_\lambda \times F$ in $\pi^{-1}(U_\lambda \cap U_\mu)$ with respect to $(u^\alpha, x^i, x_1^i, x_2^i, e^i)$ by ξ^A $\left(A, B = \alpha, i, \underset{1}{i}, \underset{2}{i}, \underset{\Delta}{i}\right)$ and the components of it with respect to $(\bar{u}^\alpha, \bar{x}^i, \bar{x}_1^i, \bar{x}_2^i, \bar{e}^i)$ by $\xi^{\bar{A}}$. Let us also denote the components of the vector ξ of $U_\mu \times F$ in $\pi^{-1}(U_\lambda \cap U_\mu)$ with respect to $(\bar{u}^\alpha, \bar{x}^i, \bar{x}_1^i, \bar{x}_2^i, \bar{e}^i)$ by $\bar{\xi}^{\bar{A}}$. And similarly we define η^A, $\eta^{\bar{A}}$ and $\bar{\eta}^{\bar{A}}$. Of course $\bar{\xi}^{\bar{A}}$ and $\bar{\eta}^{\bar{A}}$ have components given as follows:

$$(6.1)\qquad \begin{aligned} \bar{\xi} &: \left(\delta_1^\alpha,\ \bar{x}_1^i,\ \left\{ {\bar{\alpha} \atop 11} \right\} \bar{x}_\alpha^i + \bar{h}_{11}\bar{e}^i,\ \left\{ {\bar{\alpha} \atop 12} \right\} \bar{x}_\alpha^i + \bar{h}_{12}\bar{e}^i,\ -\bar{h}_1^\alpha \bar{x}_\alpha^i\right) \\ \bar{\eta} &: \left(\delta_2^\alpha,\ \bar{x}_2^i,\ \left\{ {\bar{\alpha} \atop 12} \right\} \bar{x}_\alpha^i + \bar{h}_{12}\bar{e}^i,\ \left\{ {\bar{\alpha} \atop 22} \right\} \bar{x}_\alpha^i + \bar{h}_{22}\bar{e}^i,\ -\bar{h}_2^\alpha \bar{x}_\alpha^i\right). \end{aligned}$$

Now, let us put

$$x^\alpha = u^\alpha, \qquad x_\Delta^i = e^i$$

for brevity. Then the transformations (5.2) and (5.3) can be written as

$$\bar{x}^A = \bar{x}^A(x^\alpha,\ x^i,\ x_1^i,\ x_2^i,\ x_\Delta^i).$$

By virtue of the above notations, we see that

$$\xi^{\bar{A}} = \frac{\partial \bar{x}^A}{\partial x^B}\xi^B. \tag{6.2}$$

We can easily verify that the matrix $\left(\dfrac{\partial \bar{x}^A}{\partial x^B}\right)$ is given as follows:

$$\begin{pmatrix} \dfrac{\partial \bar{u}^\alpha}{\partial u^\beta} & 0 & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 & 0 \\ x^i_\alpha \dfrac{\partial^2 u^\alpha}{\partial \bar{u}^1 \partial \bar{u}^\gamma}\dfrac{\partial \bar{u}^\gamma}{\partial u^\beta} & 0 & \dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^1} & 0 \\ x^i_\alpha \dfrac{\partial^2 u^\alpha}{\partial \bar{u}^2 \partial \bar{u}^\gamma}\dfrac{\partial \bar{u}^\gamma}{\partial u^\beta} & 0 & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^2} & 0 \\ 0 & 0 & 0 & 0 & \delta^i_j \end{pmatrix} \tag{6.3}$$

By virtue of the form of the matrix $\left(\dfrac{\partial \bar{x}^A}{\partial x^B}\right)$ we see first that

$$\xi^{\bar{\alpha}} = \frac{\partial \bar{u}^\alpha}{\partial u^\beta}\xi^\beta = \frac{\partial \bar{u}^\alpha}{\partial u^1},$$

$$\therefore \quad \xi^{\bar{\alpha}} = \frac{\partial \bar{u}^1}{\partial u^1}\bar{\xi}^{\bar{\alpha}} + \frac{\partial \bar{u}^2}{\partial u^1}\bar{\eta}^{\bar{\alpha}}.$$

Secondly, we get

$$\xi^{\bar{i}} = \xi^i = x^i_1 = \frac{\partial \bar{u}^1}{\partial u^1}\bar{x}^i_1 + \frac{\partial \bar{u}^2}{\partial u^1}\bar{x}^i_2,$$

$$\therefore \quad \xi^{\bar{i}} = \frac{\partial \bar{u}^1}{\partial u^1}\bar{\xi}^{\bar{i}} + \frac{\partial \bar{u}^2}{\partial u^1}\bar{\eta}^{\bar{i}}.$$

Thirdly, we get

$$\xi^{\bar{i}}_1 = x^i_\alpha \frac{\partial^2 u^\alpha}{\partial \bar{u}^1 \partial \bar{u}^\gamma}\frac{\partial \bar{u}^\gamma}{\partial u^\beta}\xi^\beta + \frac{\partial u^1}{\partial \bar{u}^1}\xi^i_1 + \frac{\partial u^2}{\partial \bar{u}^1}\xi^i_2.$$

Putting the components in the right hand side of the last equation by (2.3), we see

$$\begin{aligned} \xi^{\bar{i}}_1 &= \left(\frac{\partial^2 u^\alpha}{\partial \bar{u}^1 \partial \bar{u}^\gamma}\frac{\partial \bar{u}^\gamma}{\partial u^1} + \left\{ {\alpha \atop \beta 1} \right\}\frac{\partial u^\beta}{\partial \bar{u}^1}\right)x^i_\alpha + h_{\alpha 1}\frac{\partial u^\alpha}{\partial \bar{u}^1}e^i \\ &= \left\{ {\bar{\varepsilon} \atop 1\gamma} \right\}\frac{\partial u^\alpha}{\partial \bar{u}^\varepsilon}\frac{\partial \bar{u}^\gamma}{\partial u^1}x^i_\alpha + \bar{h}_{1\gamma}\frac{\partial \bar{u}^\gamma}{\partial u^1}e^i \\ &= \left[\left\{ {\bar{\varepsilon} \atop 1\gamma} \right\}\bar{x}^i_\varepsilon + \bar{h}_{1\gamma}\bar{e}^i\right]\frac{\partial \bar{u}^\gamma}{\partial u^1}, \end{aligned}$$

$$\therefore \quad \xi^{\bar{i}}_1 = \frac{\partial \bar{u}^1}{\partial u^1}\bar{\xi}^{\bar{i}}_1 + \frac{\partial \bar{u}^2}{\partial u^1}\bar{\eta}^{\bar{i}}_1.$$

In the same way, we get

$$\xi_2^{\bar{i}} = \frac{\partial \bar{u}^1}{\partial u^1} \bar{\xi}_2^{\bar{i}} + \frac{\partial \bar{u}^2}{\partial u^1} \bar{\eta}_2^{\bar{i}},$$

$$\xi_\Delta^{\bar{i}} = \frac{\partial \bar{u}^1}{\partial u^1} \bar{\xi}_\Delta^{\bar{i}} + \frac{\partial \bar{u}^2}{\partial u^1} \bar{\eta}_\Delta^{\bar{i}}.$$

Summing up these results, we finally get

$$\xi^{\bar{A}} = \frac{\partial \bar{u}^1}{\partial u^1} \bar{\xi}^{\bar{A}} + \frac{\partial \bar{u}^2}{\partial u^1} \bar{\eta}^{\bar{A}}. \tag{6.4}$$

Similarly, we get

$$\eta^{\bar{A}} = \frac{\partial \bar{u}^1}{\partial u^2} \bar{\xi}^{\bar{A}} + \frac{\partial \bar{u}^2}{\partial u^2} \bar{\eta}^{\bar{A}}. \tag{6.5}$$

The last two equations show that the two dimensional plane spanned by ξ and η of $U_\lambda \times F$ in $\pi^{-1}(U_\lambda \cap U_\mu)$ is identical with the plane spanned by ξ and η of $U_\mu \times F$ in $\pi^{-1}(U_\lambda \cap U_\mu)$. This shows that all distributions in $U_\lambda \times F$ $(\lambda \in J)$ are unified to a unique two dimensional distribution in D.

§7. Properties of the maximal integral manifolds

In Part I, we defined lifts of curves in a simply connected domain in u^1u^2-plane. However, the notion of lifts is easily extended to our fibre bundle $\mathfrak{B}$ over the manifold D by virtue of Lemma 4. Let us consider the maximal integral manifold D^* passing through a point A^* of $\mathfrak{B}$ over a point A of D. Then the end point B^* of the lift Γ^* passing through A^* of a curve Γ with the end points A and B lies on D^* and every point of D^* can be obtained in this way. We can easily prove the following

LEMMA 5. *Every maximal integral manifold D^* in $\mathfrak{B}$ over a two dimensional manifold D is a covering manifold of D.*

Proof. First it is evident that the projection $\pi : D^* \to D$ is onto. Now, let us consider the intersection of D^* and $\pi^{-1}(U_\lambda)$. In general, it is not arcwise connected and has many arcwise connected components. By virtue of Lemma 3, every such arcwise connected component is homeomorphic with U_λ by the projection π. Hence, if we denote the points on D^* over P on U_λ by $P_1, P_2, \ldots$ then $P_1, P_2, \ldots$ have homeomorphic neighbourhoods with a neighbourhood of P by π. Accordingly, D^* is a covering manifold of D.

Corollary. For every maximal integral manifold D^* in $\mathfrak{B}$ over a simply connected manifold D, the natural projection is a homeomorphism.

§ 8. The fundamental theorems

Now, we may introduce to every neighbourhood of D^* over U_λ the same parameters (u^1, u^2) as in § 4. Then we may apply the discussions given at § 4.

Analogous relations to (4.2) and (4.3) hold good over the whole D. Hence if we consider the mapping f of D^* into E_3 defined by

$$(u^\alpha,\ x^i,\ x_1^i,\ x_2^i,\ e^i) \to x^i$$

then, the image $S = f\ D^* = f \circ \pi^{-1} D$ is a surface in E_3 defined on D in the sense that every piece of it which is the f-image of a portion of D^* corresponding to U_λ of D is a surface defined on U_λ in the sense of § 4.

If the surface S has no self intersections or self contact points, then it is evident that S is homeomorphic with D^* and hence S is a covering manifold of D. (Especially, if D is simply connected, then S is homeomorphic with D.) However, even if S has self intersections or self contact points, under suitable conventions, we may regard S as a covering manifold of D in an extended sense.

We shall state the final results as follows:

THEOREM 3. *Suppose that a symmetric positive definite tensor $g_{\beta\gamma}$ of class C^2 and a symmetric tensor $h_{\beta\gamma}$ of class C^1 are defined on a two dimensional differentiable manifold D and they satisfy the Gauss and Codazzi relations* (1.8). *Then in E_3 there exists a surface S defined on D, which is the f-image of D^* over D such that any point $\hat{P}$ of S over an arbitrary point P of D has as its first and second fundamental tensors the tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ given first.*

Finally we shall give a theorem corresponding to Theorem 2 for the case where D is a manifold.

THEOREM 4. *If two surfaces S and $\bar{S}$ in E_3 which are differentiably homeomorphic with each other have, at corresponding points, the same tensors $g_{\beta\gamma}$ and $h_{\beta\gamma}$ as their first and second fundamental tensors respectively, then they are congruent in the large under the group of motions in E_3.*

Proof. Let D be a two dimensional differentiable manifold differentiably homeomorphic with S and $\bar{S}$. Then, Theorem 4 is an immediate consequence of the fact that there exists one and only one maximal integral manifold passing through any given point A $(u_0, x^i_0, x^i_{10}, x^i_{20}, e^i_0)$ in the fibre bundle $\mathfrak{B}$.

N.B. It is highly probable that analogous theories will hold good for subvarieties S_m $(m \geqq 2)$ in Euclidean space E_n $(n > m)$ and for the theory of subvarieties in affine, projective and conformal differential geometries.

References

[1] C. Chevalley, Theory of Lie groups, I, (1946).
[2] N. E. Steenrod, The topology of fibre bundles, (1951).

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