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Nagoya Math. J.
Vol. 40 (1970), 213–220

INTERSECTIONS OF ARC-CLUSTER SETS FOR MEROMORPHIC FUNCTIONS

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1. Introduction

Let D and C denote the open unit disk and the unit circle in the complex plane, respectively; and let f be a function from D into the Riemann sphere Ω. An arc $\gamma \subset D$ is said to be an *arc at* $p \in C$ if $\gamma \cup \{p\}$ is a Jordan arc; and, for each t $(0 < t < 1)$, the component of $\gamma \cap \{z: t \leq |z| < 1\}$ which has p as a limit point is said to be a *terminal subarc of* γ. If γ is an arc at p, the *arc-cluster set* $C(f, p, \gamma)$ is the set of all points $a \in \Omega$ for which there exists a sequence $\{z_k\} \subset \gamma$ with $z_k \to p$ and $f(z_k) \to a$.

We say that the function f has the *n-arc property at* $p \in C$, for some integer n $(n \geq 2)$, if there exist n arcs $\gamma_1, \cdots, \gamma_n$ at p for which the intersection of all n of the sets $C(f, p, \gamma_j)$ $(j = 1, \cdots, n)$ is empty; if, in addition, the arcs $\gamma_1, \cdots, \gamma_n$ can be chosen to be mutually disjoint, we say that f has the *n-separated-arc property at* p.

A point $p \in C$ at which f has the 2-arc property is called an *ambiguous point of* f. Bagemihl's ambiguous point theorem ([1], p. 380, Theorem 2) states that the set of ambiguous points of an arbitrary function from D into Ω is countable.

Gresser ([3], p. 145, Theorem 2) has proved the existence of a meromorphic function in D having the 3-separated-arc property at each point of a perfect subset of C. Hence, in view of Bagemihl's ambiguous point theorem, a meromorphic function in D having the 3-arc property (or even the 3-separated-arc property) at a point $p \in C$ need not have the 2-arc property at p. However, we show that if a meromorphic function f in D has the 3-arc property at a point p, then in a certain sense f is very close

Received July 31, 1969.

* This paper is based on part of the author's doctoral dissertation submitted to Michigan State University in March, 1969. The author wishes to thank Professor Peter Lappan who directed the research.

to having the 2-arc property at p. The other main result of this paper states that a meromorphic function f in D has the n-arc property (resp., the n-separated-arc property) at p for some integer n $(n>3)$ if and only if f has the 3-arc property (resp., the 3-separated-arc property) at p.

2. Preliminary Results

A *region* is any non-empty open connected subset of Ω; and a region G is a *Jordan region* if the boundary of G, denoted ∂G, is the union of a finite number (>0) of mutually disjoint Jordan curves. A *continuum* is any non-empty closed connected subset of Ω; and a *proper continuum* is any continuum properly contained in Ω.

THEOREM 1. *Let T be a countable subset of Ω and let $K_1, \cdots, K_n$ be n $(n \geq 3)$ proper continua with $\cap_{j=1}^{n} K_j = \phi$. Then there exist n Jordan regions $G_1, \cdots, G_n$ for which the following conditions hold:*

(1) $K_j \subset G_j$ $(j=1, \cdots, n)$,

(2) $\cap_{j=1}^{n} \bar{G}_j = \phi$ *(the bar denotes closure)*,

(3) $\partial G_j \cap T = \phi$ $(j=1, \cdots, n)$,

(4) *card* $[\partial G_1 \cap \partial G_2] < \aleph_0$

and

(5) $\partial G_1 \cap \partial G_2 \cap \partial G_3 = \phi$.

Proof. Let χ denote the chordal metric on Ω. There clearly exists a number $\varepsilon>0$ for which $\cap_{j=1}^{n} [K_j]_\varepsilon = \phi$ and $[K_j]_\varepsilon \neq \Omega$ $(j=1, \cdots, n)$, where $[K_j]_\varepsilon$ denotes the set of all points $a \in \Omega$ satisfying $\chi(a, K_j) < \varepsilon$. Using the Heine-Borel Theorem, for each $j=1, \cdots, n$ we obtain finitely many open spherical caps $S(j,1), \cdots, S(j, n_j)$ having centers in K_j with

$$K_j \subset \cup_{k=1}^{n_j} S(j,k) \subset \cup_{k=1}^{n_j} \bar{S}(j,k) \subset [K_j]_\varepsilon.$$

Since K_j is connected, each set

$$0_j = \cup_{k=1}^{n_j} S(j,k) \quad (j=1, \cdots, n)$$

is a region. Since for each $j=1, \cdots, n$ the set $\{S(j,k)\}_{k=1}^{n_j}$ is finite, we can choose open spherical caps $S_*(j,k)$ $(k=1, \cdots, n_j)$ with

$$S(j,k) \subset S_*(j,k) \subset \bar{S}_*(j,k) \subset [K_j]_\varepsilon$$

such that the region

$$0_j^* = \cup_{k=1}^{n_j} S_*(j,k)$$

is a Jordan region and

$$\{\text{radius } S_*(1,k)\}_{k=1}^{n_1} \cap \{\text{radius } S_*(2,k)\}_{k=1}^{n_2} = \phi.$$

From the latter condition it follows that

$$\text{card } [\partial 0_1^* \cap \partial 0_2^*] < \aleph_0.$$

Consequently, we can rechoose (if necessary) some of the caps $S_*(3,k)$ $(k = 1, \cdots, n_3)$ so that

$$\partial S_*(3,k) \cap [\partial 0_1^* \cap \partial 0_2^*] = \phi \quad (k = 1, \cdots, n_3).$$

Furthermore, since T is countable, we can rechoose (if necessary) some of the caps $S_*(j,k)$ so that

$$\partial S_*(j,k) \cap T = \phi \quad (j = 1, \cdots, n;\ k = 1, \cdots, n_j).$$

If we now set $G_j = 0_j^*$ $(j = 1, \cdots, n)$ all five conditions of the theorem can readily be verified and the proof is complete.

We say that the arcs $\gamma_1, \cdots, \gamma_n$ at $p \in C$ are *ordered arcs* if for each $j = 1, \cdots, n-1$ there exist an arc $\tau_j \subset D$ and a point $q \in C$ $(q \neq p)$ such that (1) $\tau_j \cup \{p, q\}$ is a Jordan arc and (2) γ_j and γ_{j+1} are, relative to an observer at p, contained in the left and right components of $D - \tau_j$, respectively. Then we say that the arc γ at p is *between the ordered arcs* γ_1 and γ_2 at p provided: if α is an arc in D for which $\alpha \cup \gamma_1 \cup \gamma_2 \cup \{p\}$ is a Jordan curve with interior domain Δ, then there exists a terminal subarc γ' of γ with $\gamma' \subset \Delta \cup \gamma_2$.

THEOREM 2. *Let f be meromorphic in D, let G_1 and G_2 be Jordan regions with*

$$\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi \quad (j = 1, 2),$$

and let γ_1, γ_2 be a pair of ordered arcs at $p \in C$ with $C(f, p, \gamma_j) \subset G_j$ $(j = 1, 2)$. Then either p is an ambiguous point of f or there exists an arc γ at p between γ_1 and γ_2 with

$$C(f, p, \gamma) \subset (G_1 \cap G_2) \cup \partial G_1.$$

Proof. There is no loss of generality in assuming that $\overline{f(\gamma_j)} \subset G_j$ $(j = 1, 2)$. Choose a Jordan arc $\alpha \subset D$ for which $\alpha \cup \gamma_1 \cup \gamma_2 \cup \{p\}$ is a Jordan curve Γ, and let Δ be the interior domain of Γ. Denote by Λ the set of components λ of $\Delta\cap f^{-1}(\partial G_1)$ which satisfy $\bar{\lambda}\cap\gamma_2\neq\phi$. Then each component $\lambda\in\Lambda$ is a homeomorphic image of the open interval $(0,1)$.

Consider the following cases: (I) There exists a terminal subarc γ_2' of γ_2 such that $\gamma_2'\cap\bar{\lambda}=\phi$ for each $\lambda\in\Lambda$.

(Ia) $f(\gamma_2')\subset\bar{G}_1$. Then for $\gamma=\gamma_2$ we have

$$C(f,p,\gamma)\subset\bar{G}_1\cap G_2\subset(G_1\cap G_2)\cup\partial G_1.$$

(Ib) $f(\gamma_2')\subset\Omega-G_1$. Then

$$C(f,p,\gamma_1)\cap C(f,p,\gamma_2)=\phi$$

and p is an ambiguous point of f. (II) For each terminal subarc γ_2' of γ_2 there exists a component $\lambda\in\Lambda$ with $\gamma_2'\cap\bar{\lambda}\neq\phi$. Then, since f is a local homeomorphism on $f^{-1}(\partial G_1)$, $\bar{\lambda}\cap\alpha\neq\phi$ for at most finitely many $\lambda\in\Lambda$. Consequently, there exists an arc γ at p with

$$\gamma\subset[\gamma_2\cap f^{-1}(G_1)]\cup(\bigcup_{\lambda\in\Lambda}\bar{\lambda}),$$

and it follows that

$$C(f,p,\gamma)\subset(G_2\cap\bar{G}_1)\cup\partial G_1=(G_1\cap G_2)\cup\partial G_1.$$

Thus we have established the theorem in both cases (I) and (II), and the theorem is proved.

We say that the arcs γ_1, γ_2 at $p\in C$ are *intersecting arcs* if every neighborhood of p contains a point of the intersection $\gamma_1\cap\gamma_2$. We now give an analogue of Theorem 2 for intersecting arcs.

THEOREM 2*. *Let f be meromorphic in D, let G_1 and G_2 be Jordan regions with*

$$\partial G_j\cap[\{f(z):f'(z)=0\}\cup\{\infty\}]=\phi\quad(j=1,2),$$

and let γ_1, γ_2 be a pair of intersecting arcs at $p\in C$ with $C(f,p,\gamma_j)\subset G_j$ $(j=1,2)$. Then there exists an arc γ at p with

$$C(f,p,\gamma)\subset(G_1\cap G_2)\cup\partial G_1.$$

Proof. As in the proof of Theorem 2, we assume that $\overline{f(\gamma_j)}\subset G_j$ $(j=1,2)$. Set $Q=\gamma_1\cap\gamma_2$ and note that $\overline{f(Q)}\subset G_1\cap G_2$. Let z,z' be a pair of points in Q for which the open subarc τ of γ_2 between z and z' satisfies $\tau\cap Q=\phi$.

Let τ_* be the closed subarc of γ_1 joining z to z'. Then $\tau \cup \tau_*$ is a Jordan curve, and we let Δ denote its interior domain.

Let Λ denote the set of components λ of $\Delta \cap f^{-1}(\partial G_1)$ satisfying $\bar{\lambda} \cap \tau \neq \phi$. Then, since $z, z' \in f^{-1}(G_1 \cap G_2)$, it is easy to see that there exists a Jordan arc $\rho_{z,z'}$ joining z to z' such that

$$\rho_{z,z'} \subset [\tau \cap f^{-1}(G_1)] \cup (\bigcup_{\lambda \in \Lambda} \bar{\lambda}).$$

It follows that

$$\overline{f(\rho_{z,z'})} \subset (G_2 \cap \bar{G}_1) \cup \partial G_1 = (G_1 \cap G_2) \cup \partial G_1.$$

Set $M = \cup \rho_{z,z'}$ where the union is taken over all pairs $z, z' \in Q$ for which the open subarc τ of γ_2 between z and z' satisfies $\tau \cap Q = \phi$. Since $Q \cup M \cup \{p\}$ is locally connected, it follows ([4], p. 27, Theorem 4.1) that there exists an arc γ at p with $\gamma \subset Q \cup M$. Then, since

$$\overline{f(\gamma)} \subset (G_1 \cap G_2) \cup \partial G_1,$$

the proof is complete.

3. The n-Separated-Arc Property

THEOREM 3. *If f is meromorphic in D, then f has the n-separated-arc property ($n > 3$) at $p \in C$ if and only if f has the 3-separated-arc property at p.*

Proof. If f has the 3-separated-arc property at p, then it is obvious that f has the n-separated-arc property at p for all n ($n > 3$). Thus, we need only prove that if f has the n-separated-arc property ($n > 3$) at p, then f has the $(n-1)$-separated-arc property at p.

Suppose $\gamma_1, \cdots, \gamma_n$ are n ordered arcs at p for which the intersection of all n of the sets $C(f, p, \gamma_j)$ $(j = 1, \cdots, n)$ is empty; and, to avoid the trivial case, assume that the intersection of any $n-1$ of them is non-empty. By Theorem 1 there exist Jordan regions G_j $(j = 1, \cdots, n)$ for which

(1) $C(f, p, \gamma_j) \subset G_j$ $(j = 1, \cdots, n)$,

(2) $\cap_{j=1}^{n} \bar{G}_j = \phi$,

(3) $\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi$ $(j = 1, \cdots, n)$

and

(4) $\partial G_1 \cap \partial G_2 \cap \partial G_3 = \phi$.

We assume that p is not an ambiguous point of f, in which case there would be nothing to prove. Due to conditions (1) and (3) we can apply Theorem 2 to obtain arcs σ_j $(j = 1, \cdots, n-1)$ at p between the corresponding arcs γ_j and γ_{j+1} such that

$$C(f, p, \sigma_j) \subset (G_j \cap G_{j+1}) \cup \partial G_j.$$

Since the arcs $\gamma_1, \cdots, \gamma_n$ are ordered, for each $j = 1, \cdots, n-1$ we can choose a terminal subarc σ_j^* of σ_j in such a way that the arcs $\sigma_1^*, \cdots, \sigma_{n-1}^*$ are mutually disjoint. Then with the aid of conditions (2) and (4) we obtain the relations

$$\begin{aligned} \cap_{j=1}^{n-1} C(f, p, \sigma_j^*) &\subset \cap_{j=1}^{n-1} [(G_j \cap G_{j+1}) \cup \partial G_j] \\ &= \cap_{j=1}^{n-1} \partial G_j = \phi. \end{aligned}$$

That is, f has the $(n-1)$-separated-arc property at p as was to be shown.

THEOREM 4. *Let f be meromorphic in D. If f has the 3-separated-arc property at $p \in C$, then there exist disjoint arcs σ_1 and σ_2 at p for which*

$$\textit{card}\ [C(f, p, \sigma_1) \cap C(f, p, \sigma_2)] < \aleph_0.$$

Proof. Suppose $\gamma_1, \gamma_2, \gamma_3$ are ordered arcs at p with

$$C(f, p, \gamma_1) \cap C(f, p, \gamma_2) \cap C(f, p, \gamma_3) = \phi.$$

If p is an ambiguous point of f, we are finished; hence we assume that p is not an ambiguous point of f. By Theorem 1 there exist Jordan regions G_1, G_2, G_3 for which

(1) $C(f, p, \gamma_j) \subset G_j$ $(j = 1, 2, 3)$,

(2) $\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3 = \phi$,

(3) $\partial G_j \cap [\{f(z) : f'(z) = 0\} \cup \{\infty\}] = \phi$ $(j = 1, 2, 3)$

and

(4) card $[\partial G_1 \cap \partial G_2] < \aleph_0$.

By Theorem 2 there exist arcs σ_j $(j = 1, 2)$ at p between the corresponding arcs γ_j and γ_{j+1} such that

$$C(f, p, \sigma_j) \subset (G_j \cap G_{j+1}) \cup \partial G_j.$$

Since the arcs γ_1, γ_2, γ_3 are ordered, we may assume that $\sigma_1 \cap \sigma_2 = \phi$. Then, using condition (2) we obtain the relations

$$C(f,p,\sigma_1)\cap C(f,p,\sigma_2)\subset[(G_1\cap G_2)\cup\partial G_1]\cap[(G_2\cap G_3)\cup\partial G_2]$$
$$=\partial G_1\cap\partial G_2;$$

and, in view of condition (4), the proof is complete.

Remark. In effect, Gresser ([3], p. 145, proof of Theorem 2) has proved the existence of a meromorphic function μ in D with the following property: there exists a triangle in Ω with sides s_1, s_2, s_3 and a perfect subset C' of C such that for each point $p\in C'$ there exist three mutually disjoint chords ρ_1, ρ_2, ρ_3 at p with

$$C(\mu,p,\rho_j)=s_j \ \ (j=1,2,3).$$

The function μ serves as an illustrative example of Theorem 4 in that

$$C(\mu,p,\rho_1)\cap C(\mu,p,\rho_2)\cap C(\mu,p,\rho_3)=\phi$$

and, for $i\neq j$,

$$\text{card }[C(\mu,p,\rho_i)\cap C(\mu,p,\rho_j)]=1.$$

4. The n-Arc Property

By following the same line of proof as in the proofs of Theorems 3 and 4 with Theorem 2* playing the role of Theorem 2, we establish the following results.

THEOREM 5. *If f is meromorphic in D, then f has the n-arc property $(n>3)$ at $p\in C$ if and only if f has the 3-arc property at p.*

THEOREM 6. *Let f be meromorphic in D. If f has the 3-arc property at $p\in C$, then there exist arcs σ_1 and σ_2 at p for which*

$$\textit{card }[C(f,p,\sigma_1)\cap C(f,p,\sigma_2)]<\aleph_0.$$

Remark. Theorem 6 is exemplified by the modular function m mapping D onto the universal covering surface of $\Omega-\{0,1,\infty\}$. Bagemihl, Piranian and Young ([2], p. 30, proof of Theorem 3) have shown that for each $p\in C$ there exist three arcs (any two of which are intersecting arcs) γ_1, γ_2, γ_3 at p such that

$$C(m,p,\gamma_1)\cap C(m,p,\gamma_2)\cap C(m,p,\gamma_3)=\phi$$

and, for $i\neq j$,

$$\text{card}\,[C(m, p, \gamma_i) \cap C(m, p, \gamma_j)] \leq 4.$$

If we set $\Pi(f, p) = \cap C(f, p, \gamma)$ where the intersection is taken over all arcs γ at p, the next result follows from Theorems 5 and 6 and the fact that $\Pi(f, p) = \phi$ implies that f has the n-arc property at p for some integer n $(n \geq 2)$.

THEOREM 7. *Let f be meromorphic in D. If $\Pi(f, p) = \phi$, then f has the 3-arc property at p and there exist arcs σ_1 and σ_2 at p for which*

$$\textit{card}\,[C(f, p, \sigma_1) \cap C(f, p, \sigma_2)] < \aleph_0.$$

5. Open Questions

1. Does there exist a meromorphic function in D which has the 3-arc property at a point $p \in C$ but does not have the 3-separated-arc property at p?

2. Does the modular function m have the 3-separated-arc property at each point of C?

3. If the answer to Question 2 is in the negative, does there exist a meromorphic function in D having the 3-separated-arc property at each point of C?

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