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A CHARACTERIZATION OF THE VERONESE VARIETIES*

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Let $P^m(C)$ be the complex projective space of dimension m. In a previous paper [2] we have proved

THEOREM A. *Let f be a Kaehlerian immersion of a connected, complete Kaehler manifold M^n of dimension n into $P^m(C)$. If the image $f(\tau)$ of each geodesic τ in M^n lies in a complex projective line $P^1(C)$ of $P^m(C)$, then $f(M^n)$ is a complex projective subspace of $P^m(C)$, and f is totally geodesic.*

In the present note, we shall first provide a much simpler geometric proof of this result and then give a characterization of the Veronese varieties by means of the notion of circles in $P^m(C)$. Generally, a curve $x(t)$ with arc-length parameter t in a Riemannian manifold is called a circle if there exists a field of unit vectors Y_t along the curve, which, together with the unit tangent vectors X_t, satisfies the differential equations

$$\nabla_t X_t = kY_t \quad \text{and} \quad \nabla_t Y_t = -kX_t\,,$$

where k is a positive constant (see [4]).

By the Veronese variety we mean the imbedding of $P^n(C)$ into $P^m(C)$, where $m = n(n+3)/2$, which is defined as follows. Let S^{2n+1} be the unit sphere in the complex vector space C^{n+1} with the standard hermitian inner product (z, w) and corresponding real inner product $\langle z, w\rangle = \mathrm{Re}\,(z, w)$. On the other hand, the set of all complex symmetric matrices of degree $n+1$ can be considered as the vector space C^{m+1}, where $m = n(n+3)/2$, in which the standard hermitian inner product can be expressed by

$$(A, B) = \mathrm{trace}\, A\bar{B}\,, \qquad A, B \in C^{m+1}\,.$$

The mapping v which takes $x \in C^{n+1}$ into $x\,{}^t x \in C^{m+1}$ maps S^{2n+1} into the

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unit sphere S^{2m+1} of C^{m+1}, and induces a holomorphic imbedding of $P^n(C)$ into $P^m(C)$. If we choose the Fubini-Study metrics of constant holomorphic curvature $c\,(>0)$ for $P^m(C)$ and $c/2$ for $P^n(C)$, then the imbedding is isometric. This is what we call the Veronese imbedding.

We now state our new result.

THEOREM B. *Let f be a Kaehlerian immersion of a connected, complete Kaehler manifold M^n of dimension n into $P^m(C)$ with Fubini-Study metric. The image $f(\tau)$ of each geodesic τ in M^n is a circle in $P^m(C)$ if and only if f is congruent (by a holomorphic isometry of $P^m(C)$) to $i \circ v$, where v is the Veronese imbedding of $P^n(C)$ into $P^{m'}(C)$, with $m' = n(n+3)/2$, and i is the totally geodesic inclusion of $P^{m'}(C)$ into $P^m(C)$.*

1. Simpler proof of Theorem A.

Let x_0 be a point of M^n and let M^* be the complete totally geodesic complex submanifold (namely, n-dimensional projective subspace $P^n(C)$) through the point $f(x_0)$ and tangent to $f(M^n)$, that is, the tangent space $T_{f(x_0)}(M^*)$ equals $f_*(T_{x_0}(M^n))$, where f_* denotes the differential of f.

Let τ be an arbitrary geodesic in M^n starting at x_0. By assumption, there is a complex projective line $P^1(C)$ which contains $f(\tau)$. If X denotes the initial tangent vector of τ at x_0, then $f_*(X)$ is tangent to $P^1(C)$. If we denote by J the complex structure of $P^m(C)$ as well as that of M^n, then the vector $Jf_*(X) = f_*(JX)$ is tangent to $P^1(C)$. It follows that $T_{f(x_0)}(P^1(C))$ is spanned by $f_*(X)$ and $f_*(JX)$. On the other hand, these two vectors are contained in $f_*(T_{x_0}(M^n)) = T_{f(x_0)}(M^*)$. Thus $T_{f(x_0)}(P^1(C)) \subset T_{f(x_0)}(M^*)$. Since $P^1(C)$ and M^* are totally geodesic in $P^m(C)$, it follows that $P^1(C)$ is contained in M^*; thus $f(\tau)$ is contained in M^*. Since τ is an arbitrary geodesic in M, we have $f(M) = M^*$.

2. Veronese imbedding.

We shall show that the Veronese imbedding v of $P^n(C)$ into $P^m(C)$ with $m = n(n+3)/2$ has the property that the image of each geodesic in $P^n(C)$ is a circle in $P^m(C)$. This property does not depend on the choice of a positive constant c which we choose for the holomorphic sectional curvature of $P^m(C)$ (and that of $P^n(C)$ will be $c/2$). We recall how geometry of $P^m(C)$ is related to that of S^{2m+1}. The standard fibration $\pi: S^{2m+1} \to P^m(C)$ is a principal S^1-bundle. It has a connection whose horizontal subspaces Q_x, $x \in S^{2m+1}$, are given by

$$Q_x = \{X \in \boldsymbol{C}^{m+1}; \langle X, x\rangle = \langle X, ix\rangle = 0\} .$$

The projection π_* maps Q_x isomorphically onto the tangent space $T_u(P^m(\boldsymbol{C}))$, where $u = \pi(x)$. If we let

$$g(\pi_* X, \pi_* Y) = (4/c)\langle X, Y\rangle , \qquad X, Y \in Q_x ,$$

then g is the Fubini-Study metric with holomorphic sectional curvature c for $P^m(\boldsymbol{C})$. We shall choose $c = 4$ (to simplify constant factors in the computations that follow). Let us denote by ∇' the Riemannian connection for S^{2m+1} and by $\tilde{\nabla}$ the Kaehlerian connection for $P^m(\boldsymbol{C})$. We formulate the relationship between ∇' and $\tilde{\nabla}$ (see [3], Proposition 3) in the following form. A curve in S^{2m+1} is said to be horizontal if its tangent vectors are horizontal.

LEMMA 1. *Let x_t be a horizontal curve in S^{2m+1} and $u_t = \pi(x_t)$. If Z_t is a horizontal vector field along x_t and if $W_t = \pi_*(Z_t)$, then $\tilde{\nabla}_t W_t = \pi_*(\nabla'_t Z_t)$.*

LEMMA 2. *If x_t is a horizontal curve in S^{2m+1} with arc-length parameter t, then $\nabla'_t X_t$, where X_t denotes the tangent vector, is horizontal.*

Proof. We have

$$\nabla'_t X_t = dX/dt + x_t .$$

Since x_t is horizontal, we have $\langle X_t, ix_t\rangle = 0$ and hence

$$\langle dX/dt, ix_t\rangle + \langle X_t, iX_t\rangle = 0 .$$

But $\langle X_t, iX_t\rangle = 0$ so that $\langle dX/dt, ix_t\rangle = 0$. Thus we obtain

$$\langle \nabla'_t X_t, ix_t\rangle = \langle dX/dt, ix_t\rangle + \langle x_t, ix_t\rangle = 0 .$$

LEMMA 3. *If x_t is a circle in S^{2m+1} which is furthermore a horizontal curve, then $u_t = \pi(x_t)$ is a circle in $P^m(\boldsymbol{C})$.*

Proof. We have a field of unit vectors Y_t along x_t such that

$$\nabla'_t X_t = kY_t \quad \text{and} \quad \nabla'_t Y_t = -kX_t ,$$

where k is a positive constant and X_t is the tangent vector. By Lemma 2, $\nabla'_t X_t$ and hence Y_t are horizontal. The tangent vector of u_t is given by $U_t = \pi_*(X_t)$. Consider the field of unit normal vectors $V_t = \pi_*(Y_t)$;

note that π_* is isometric from Q_x to $T_{\pi(x)}(P^m(C))$. By Lemma 1, we have

$$\tilde{\nabla}_t U_t = \pi_*(\nabla'_t X_t) = \pi_*(kY_t) = kV_t$$

and, similarly,

$$\tilde{\nabla}_t V_t = \pi_*(\nabla'_t Y_t) = \pi_*(-kX_t) = -kU_t .$$

Thus u_t is a circle in $P^m(C)$.

Now we shall prove our assertion about the Veronese imbedding. We observe that the unitary group $U(n+1)$ acts naturally on S^{2n+1} and $P^n(C)$ as a group of isometries. Each geodesic τ in $P^n(C)$ is congruent by a transformation belonging to $U(n+1)$ to the curve with homogeneous coordinates $(\cos t, \sin t, 0, \cdots, 0)$. On the other hand, we can let $U(n+1)$ act on the space C^{m+1} of all complex symmetric matrices of degree $n+1$ by $Z \to AZ\,{}^tA$, where $Z \in C^{m+1}$ and $A \in U(n+1)$. This action preserves inner product in C^{m+1} and thus induces the action of $U(n+1)$ on S^{2m+1} and $P^m(C)$ as a group of isometries. Now the Veronese imbedding v is equivariant relative to the actions of $U(n+1)$ on $P^n(C)$ and on $P^m(C)$.

It is thus sufficient to prove the following. Let τ be the geodesic w_t in $P^n(C)$ given by $w_t = \pi(z_t)$, where $z_t = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), 0, \cdots, 0)$ is a curve on S^{2n+1}. Since the holomorphic sectional curvature of $P^n(C)$ has been chosen to be 2, we have

$$\|dw/dt\|^2 = 2\,\|dz/dt\|^2 = 1 ,$$

which shows that t is the arc-length parameter for the geodesic w_t. Let

$$x_t = v(z_t) , \qquad u_t = v(w_t) \quad \text{so that} \quad u_t = \pi(x_t) .$$

We wish to show that u_t is a circle in $P^m(C)$. The curve x_t on S^{2m+1} can be represented simply by the first 2×2 block of the form

$$\begin{bmatrix} \cos^2(t/\sqrt{2}) & \sin(t/\sqrt{2})\cos(t/\sqrt{2}) \\ \sin(t/\sqrt{2})\cos(t/\sqrt{2}) & \sin^2(t/\sqrt{2}) \end{bmatrix}$$

since the other components are all 0. The tangent vectors X_t of the curve x_t are represented in the same sense by

$$X_t = (1/\sqrt{2})\begin{bmatrix} -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \\ \cos(\sqrt{2}\,t) & \sin(\sqrt{2}\,t) \end{bmatrix} .$$

Since $\langle X_t, ix_t\rangle = 0$, x_t is a horizontal curve in S^{2m+1}. If we show that it is a circle in S^{2m+1}, then Lemma 3 implies that $u_t = \pi(x_t)$ is a circle in $P^m(\boldsymbol{C})$.

We have

$$dX/dt = \begin{bmatrix} -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \\ -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \end{bmatrix} .$$

The vector

$$\nabla'_t X_t = dX/dt + x_t$$

is also horizontal (since its components are real) and has length 1, because

$$\begin{aligned} &\langle dX/dt + x_t, dX/dt + x_t\rangle \\ &\quad = \langle dX/dt, dX/dt\rangle + 2\langle x_t, dX/dt\rangle + \langle x_t, x_t\rangle \\ &\quad = 2 + 2(-1) + 1 = 1 , \end{aligned}$$

by virtue of $\langle x_t, dX/dt\rangle = -\langle dx/dt, X_t\rangle = -1$.

We thus set $Y_t = dX/dt + x_t$, namely, $\nabla'_t X_t = Y_t$. Since $\langle Y_t, X_t\rangle = 0$, we have

$$\begin{aligned} \nabla'_t Y_t &= dY/dt = d^2X/dt^2 + X_t \\ &= \sqrt{2}\begin{bmatrix} \sin(\sqrt{2}\,t) & -\cos(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \end{bmatrix} \\ &\quad + (1/\sqrt{2})\begin{bmatrix} -\sin(\sqrt{2}\,t) & \cos(\sqrt{2}\,t) \\ \cos(\sqrt{2}\,t) & \sin(\sqrt{2}\,t) \end{bmatrix} \\ &= (1/\sqrt{2})\begin{bmatrix} \sin(\sqrt{2}\,t) & -\cos(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) & -\sin(\sqrt{2}\,t) \end{bmatrix} = -X_t . \end{aligned}$$

Thus we have shown that x_t is a circle of curvature $k = 1$.

3. Proof of Theorem B.

We now finish the proof of Theorem B. Let f be a Kaehlerian immersion of a complete Kaehler manifold M^n into $P^m(\boldsymbol{C})$ with the property that for each geodesic τ in M^n the image $f(\tau)$ is a circle in $P^m(\boldsymbol{C})$. We shall first show that

(i) the second fundamental form α is parallel;

(ii) f is isotropic, that is, $\|\alpha(X, X)\|$ is equal to a constant for all unit tangent vectors X to M^n at each point;

(iii) M^n has constant holomorphic curvature.

Let x_t be a geodesic on M^n with tangent vectors X_t of length 1. Denoting by $\tilde{\nabla}$ and ∇ the Kaehlerian connections of $P^m(C)$ and M^n, respectively, we have

$$\tilde{\nabla}_t X_t = \nabla_t X_t + \alpha(X_t, X_t) = \alpha(X_t, X_t) ,$$

where α is the second fundamental form. We obtain

$$(1) \qquad \tilde{\nabla}_t^2 X_t = -A_{\alpha(X_t, X_t)} X_t + \nabla_t^{\perp} \alpha(X_t, X_t) ,$$

where A is the second fundamental tensor and $\nabla^{\perp}$ the normal connection. On the other hand, since $f(x_t)$ is a circle by assumption, there exists a field of unit tangent vectors Y_t along x_t and $k > 0$ such that

$$\tilde{\nabla}_t X_t = kY_t \quad \text{and} \quad \tilde{\nabla}_t Y_t = -kX_t ,$$

thus

$$(2) \qquad \tilde{\nabla}_t^2 X_t = -k^2 X_t .$$

From (1) and (2) we obtain

$$(3) \qquad A_{\alpha(X_t, X_t)} X_t = k^2 X_t$$

and

$$(4) \qquad \nabla_t^{\perp} \alpha(X_t, X_t) = 0 .$$

Since x_t is a geodesic in M^n, the covariant derivative

$$(\nabla_t^* \alpha)(X_t, X_t) = \nabla_t^{\perp} \alpha(X_t, X_t) - \alpha(\nabla_t X_t, X_t) - \alpha(X_t, \nabla_t X_t)$$

is equal to 0 by virtue of (4). Evaluating this at $t = 0$ and observing that X_0 can be an arbitrary unit tangent vector at an arbitrary point of M^n, we have

$$(5) \qquad (\nabla_X^* \alpha)(X, X) = 0 \qquad \text{for all tangent vectors } X \text{ to } M^n.$$

Since $(\nabla_X^* \alpha)(Y, Z)$ is symmetric in X, Y and Z, we conclude that $\nabla^* \alpha = 0$, that is, α is parallel.

From (3) it follows that for any unit tangent vector X to M^n there exists a certain constant $k > 0$ such that

$$A_{\alpha(X, X)} X = k^2 X .$$

If Y is a tangent vector perpendicular to X, then

$$\langle A_{\alpha(X,X)}X, Y\rangle = 0$$

so that

$$\langle \alpha(X,X), \alpha(X,Y)\rangle = 0 \quad \text{whenever} \quad \langle X, Y\rangle = 0\ . \tag{6}$$

This condition implies that f is isotropic, that is, $\|\alpha(X,X)\|$ is equal to a constant for all unit tangent vectors X at each point (see [6], Lemma 1). It also follows that M^n has constant holomorphic sectional curvature (see [6], Lemma 6).

We now wish to prove that f is essentially the Veronese imbedding. Since α is parallel, the first normal spaces (spanned by the range of α at each point) are obviously parallel relative to the normal connection. The (complex) dimension of the normal spaces, say, p, is at most $n(n+1)/2$. It is known [1], Proposition 9, that there is a totally geodesic $P^{n+p}(\boldsymbol{C})$ in $P^m(\boldsymbol{C})$ such that $f(M^n) \subset P^{n+p}(\boldsymbol{C})$. We shall see that this immersion f_0 of M^n into $P^{n+p}(\boldsymbol{C})$ is the Veronese imbedding (and indeed $p = n(n+1)/2$).

If $p < n(n+1)/2$, Theorem 2 of [6] says that f_0 is totally geodesic. This will mean that the image of a geodesic in M^n is a geodesic in $P^{n+p}(\boldsymbol{C})$ and hence a geodesic in $P^m(\boldsymbol{C})$, contrary to the assumption that it is a circle in $P^m(\boldsymbol{C})$. Hence we must have $p = n(n+1)/2$. We already know that M^n has constant holomorphic sectional curvature. Since the second fundamental form is parallel, it follows from [5], Theorem 4.4, that this constant is half the constant holomorphic sectional curvature of $P^{n+p}(\boldsymbol{C})$. Moreover, such an immersion f_0 is rigid. Thus M^n is $P^n(\boldsymbol{C})$ with holomorphic sectional curvature, say, 2, if we assume that $P^m(\boldsymbol{C})$ and hence $P^{n+p}(\boldsymbol{C})$ has holomorphic sectional curvature 4. Now the Veronese imbedding v is a Kaehlerian imbedding of $P^n(\boldsymbol{C})$ into $P^{n+p}(\boldsymbol{C})$. By rigidity, f_0 is congruent to v by a holomorphic isometry of $P^{n+p}(\boldsymbol{C})$. Since this holomorphic isometry can be extended to a holomorphic isometry of $P^m(\boldsymbol{C})$, we can now conclude that $f: M^n \to P^m(\boldsymbol{C})$ is in fact congruent to $i \circ v$, where v is the Veronese imbedding of $P^n(\boldsymbol{C})$ into $P^{n+p}(\boldsymbol{C})$, $p = n(n+1)/2$, and i is a totally geodesic inclusion of $P^{n+p}(\boldsymbol{C})$ into $P^m(\boldsymbol{C})$. We have thus completed the proof of Theorem B.

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