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STOCHASTIC INTEGRALS IN ABSTRUCT WIENER SPACE II: REGULARITY PROPERTIES

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Introduction

This paper continues the study of stochastic integrals in abstract Wiener space previously given in [14]. We will present, among other things, the detailed discussion and proofs of the results announced in [16]. Let $H \subset B$ be an abstract Wiener space. Consider the following stochastic integral equation in $H \subset B$,

$$X(t) = x + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds \ , \tag{1}$$

where $W(t)$ is a Wiener process in B. Under certain assumptions on A and σ we showed in [14] that (1) has a unique non-anticipating continuous solution and that this solution is a Markov process. If A and σ are differentiable in the second variable we can differentiate the above equation "formally" with respect to the starting point x to obtain the formal operator-valued stochastic integral equation

$$Y(t) = I + \int_0^t A_x(s, X(s))Y(s)dW(s) + \int_0^t \sigma_x(s, X(s))Y(s)ds \ , \tag{2}$$

where A_x and σ_x are derivatives of A and σ in the second variable, respectively. (2) is a linear integral equation and obviously has a unique solution which qualifies to be called the derivative of $X(t)$ in some sense. If A and σ are furthermore twice differentiable we can differentiate (2) formally in the same manner to obtain another stochastic integral equation whose solution is the second derivative of $X(t)$. Thus roughly speaking, the solution $X(t)$ of (1), regarded as a function of its starting point, is as smooth as A and σ.

Let f be a real-valued continuous function in B. Let $\theta(x) =$

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$E_x[f(X(t))]$. If f is differentiable then formally by the "chain rule" we have $\theta'(x) = E_x[Y(t)^*(f'(X(t)))]$, where $Y(t)$ is the solution of (2) and $*$ denotes the adjoint of operators of H. If f is twice differentiable then so is θ and a formal expression for $\theta''(x)$ can be written by using also the second derivative of $X(t)$. Thus if A and σ are C^∞-functions then θ is as smooth as f. Furthermore, if $f''(x)$ is a Hilbert-Schmidt operator then $\theta''(x)$ is also a Hilbert-Schmidt operator.

The above approach of discussing the regularity properties of $X(t)$ and $\theta(x)$ was first introduced by Gikhman [3; 4]. It was carried over to infinite dimensional Hilbert spaces by Dalec'kii [1; 2]. See also[18; 23]. We generalize it to Banach spaces (§ 2) and, furthermore, study the related operator-valued stochastic integrals and prove the corresponding versions of Ito's formula and Girsanov-Skorokhod-McKean's formula (§ 1). In case A and σ are time-independent we show in the end of the paper that $X(t)$ generates a semi-group on the Banach space of bounded continuous functions on B vanishing at infinity. The proof is due to K. Ito.

Recently, Kannan and Bharucha-Reid [10; 11] have defined several operator-valued stochastic integrals and proved some generalizations of Ito's formula. However, there is no apparent relation between their work and ours.

This paper is closely related to Piech's. In a series of papers [19; 20; 21; 22] she studies the corresponding parabolic equation of (1) with $\sigma \equiv 0$ and A satisfying stronger assumptions. In particular, A is non-degenerate. She constructs a fundamental solution $\{q_t(s, dy)\}$ which is related to the process $X(t)$ by $\int_B f(y)q_t(x, dy) = E_x[f(X(t))]$ for bounded Lip-1 functions f [17]. Her conclusions about the regularity properties of the function $\theta(x) = E_x[f(X(t))]$ are stronger than ours in this particular case.

Notation

1.	E	expectation
2.	$H \subset B$	abstract Wiener space
3.	$B^* \subset H \subset B$	(through identifications)
4.	$\vert \cdot \vert$	H-norm (*see* 7)
5.	$\Vert \cdot \Vert$	B-norm (*see* 8)
6.	$L^n(X; Y)$	continuous n-linear maps from $\underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ into Y

7.	$\lvert\cdot\rvert$	norm of $L^n(H;R)$
8.	$\lVert\cdot\rVert$	norm of $L^n(B;R)$
9.	$L_{(2)}(H;H)$	Hilbert-Schmidt operators of H (*see* 12)
10.	$\lvert\cdot\rvert_2, \langle\ ,\ \rangle_2$	norm, inner product of $L_{(2)}(H;H)$. (*see* 13)
11.	$\sim$	$\tilde{T}(x)=T(x,\cdot,\cdots,\cdot)$. $T\in L^n(X;R), \tilde{T}\in L(X;L^{n-1}(X;R))$ (*cf.* 33).
12.	$L^n_{(2)}(H;R)$	Hilbert-Schmidt type n-linear forms of H.
13.	$\lvert\cdot\rvert_2, \langle\ ,\ \rangle_2$	norm, inner product of $L^n_{(2)}(H;R)$
14.	$\circ_j$	$S\circ_j T$, $S\in L^n(X;R)$, $T\in L(X;X)$; $S\circ_j T\in L^n(X;R)$. $(S\circ_j T(x_1,\cdots,x_j,\cdots,x_n)=S(x_1,\cdots,Tx_j,\cdots,x_n))$
15.	$\lVert\cdot\rVert_X$	norm of $L(X;X)$.
16.	$W(t)$	Wiener process in B.
17.	$\mathscr{M}_t$	σ-field generated by $\{W(s); s\leq t\}$
18.	$\mathscr{L}[L^n_{(2)}(H;R)]$	non-anticipating stochastic processes ξ with state space $L^n_{(2)}(H;R)$ such that $\int_0^\tau E\lvert\xi(t)\rvert^2\,dt<\infty$ for each finite τ. (*see* 20)
19.	$\mathscr{L}[L^n(B;R)]$	
20.	$\mathscr{L}[X]$	non-anticipating stochastic processes ξ with state space X such that $\int_0^\tau E\lvert\xi(t)\rvert_X^2\,dt<\infty$ for each finite τ. (*cf.* 27)
21.	$\mathscr{S}(\mathscr{H};\mathscr{K})$	trace-class type bilinear form from $\mathscr{H}\times\mathscr{H}$ into $\mathscr{K}$.
22.	TRACE S	trace of $S\in\mathscr{S}(\mathscr{H};\mathscr{K})$.
23.	$\triangle$	a) $T\in L^n(H;R)$, $S\in L(L^{n-1}(H;R);L^{n-1}(H;R))$ $S\triangle T\in L^n(H;R)$. $((S\triangle T)^\sim=S\circ\tilde{T})$ c) $S\in L^n(H;R)$, $T\in L(L^{n-1}(H;R);R)$; $S\triangle T\in H$. $(\langle S\triangle T,h\rangle=T(\tilde{S}(h)))$.
24.	$L_1(H;H)$	trace class operators of H.
25.	$X_x(t)$	diffusion process starting at x.
26.	C^n_H	n-smooth functions in H-directions.
27.	$\mathscr{L}(D)$	square integrable random variables taking values in D. (*cf.* 20)
28.	$\delta\xi_x$	MS-H-derivative of a random variable ξ at x.
29.	MS-C^n_H	MS-n-smooth random variables in H-directions.
30.	$\delta Z, \delta^n Z$	MS-H-derivative of a diffusion process Z.

31. $\hat{}$ $\quad S \in L^n(H;R),\ \hat{S} \in L^{n-1}(H;H)$
$(\langle \hat{S}(h_1, \cdots, h_{n-1}), h \rangle = S(h, h_1, \cdots, h_{n-1}))$

32. $:$ $\quad T \in L^3(H;R),\ S \in L^n(H;R);\ S: T \in L^{n+1}(H;R).$
$(S: T(h_1, h_2, \cdots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \cdots, h_n), h_{n+1})).$

33. $\check{}$ $\quad S \in L^n(H;R),\ \check{S} \in L(H; L^{n-1}(H;R))$
$(\check{S}(h) = S(\cdot, \cdot, \cdots, h)).$ $(cf.\ 11)$

1. Operator-Valued Stochastic Integrals

Let $H \subset B$ be an abstract Wiener space. $|\cdot|$ and $\|\cdot\|$ denote the H-norm and B-norm, respectively. We will regard $B^* \subset H^* \approx H \subset B$ in the natural way. As in [14] we assume that there is a sequence Q_n of finite dimensional projections such that (i) $Q_n(B) \subset B^*$ and (ii) Q_n converges strongly to the identity both in B and in H. Furthermore, we will assume that there exists an orthonormal basis $\{e_n\}$ of H such that $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$. This additional assumption is satisfied by all of the presently known abstract Wiener spaces.

Notation:

(i) $L^n(X;Y) =$ the Banach space of all continuous n-linear maps from X^n into Y, where X and Y are Banach spaces. L^1 will be written as L.

(ii) $L^{n-1}(X;X^*) \approx L^n(X;R)$

(iii) $\|\cdot\|$ and $|\cdot|$ denote the norms of $L^n(B;R)$ and $L^n(H;R)$, respectively. Clearly $L^n(B;R) \subset L^n(H,R)$ and $|\cdot|$ is dominated by $\|\cdot\|$ with some constant depending on n.

(iv) $L_{(2)}(H;H)$ $(\equiv L^2_{(2)}(H;R))$ denotes the Hilbert space of all Hilbert-Schmidt operators of H with H-S-norm $|\cdot|_2 = \langle\ ,\ \rangle_2^{1/2}$. It can be shown easily that $|S|_2 \leq (\sum_{j=1}^{\infty} \|e_j\|^2)^{1/2} \|S\|$ for all $S \in L^2(B;R)$, where $\{e_j\}$ is given in the additional assumption. Thus we have $L^2(B;R) \subset L^2_{(2)}(H;R)$.

(v) Let $T \in L^n(X;R)$. Define $\tilde{T} \in L(X; L^{n-1}(X;R))$ by $\tilde{T}(x) = T(x, \cdot, \cdot, \cdots, \cdot)$.

Now we want to define inductively a sequence of Hilbert spaces $L^n_{(2)}(H;R)$, $n \geq 1$, with $L^1_{(2)}(H;R) = H$ by convention and $L^2_{(2)}(H;R)$ given above.

DEFINITION 1. Let $T \in L^n(H;R)$, $n \geq 3$. T is said to be of *Hilbert-Schmidt type* if (i) $\tilde{T}(H) \subset L^{n-1}_{(2)}(H;R)$ and (ii) $\tilde{T}$ is a Hilbert-Schmidt operator from H into $L^{n-1}_{(2)}(H;R)$.

Let $L^n_{(2)}(H;R)$ denote the space of all Hilbert-Schmidt type n-linear forms of H. It is a Hilbert space with the inner product $\langle S, T\rangle_2 =$ the Hilbert-Schmidt inner product of $\tilde{S}$ and $\tilde{T}$, $S, T \in L^n_{(2)}(H;R)$. Clearly,

$$\langle S, T\rangle_2 = \textstyle\sum_{i_1,i_2,\cdots,i_n} S(v_{i_1}, v_{i_2}, \cdots, v_{i_n}) T(v_{i_1}, v_{i_2}, \cdots, v_{i_n}) ,$$

where $\{v_k\}$ is any orthonormal basis of H. Let $|S|_2 = \langle S, S\rangle_2^{1/2}$. Note that we have used the same notation $|\cdot|_2$ and $\langle\,,\,\rangle_2$ to denote the norm and the inner product of $L^n_{(2)}(H;R)$ for all $n \geq 2$ since there is no confusion. For example, the meaning of the following equality is clear, when $S, T \in L^n_{(2)}(H;R)$,

$$\langle S, T\rangle_2 = \sum_k \langle \tilde{S}(v_k), \tilde{T}(v_k)\rangle_2 . \tag{3}$$

LEMMA 1.1. (a) $|S| \leq |S|_2$ *for all* S *in* $L^n_{(2)}(H;R)$.

(b) $|T|_2 \leq c^n \|T\|$ *for all* T *in* $L^n(B;R)$, *where* c *is a constant. Thus we have the relation* $L^n(B;R) \subset L^n_{(2)}(H;R) \subset L^n(H;R)$, $n \geq 1$.

(c) $L^n(B;R)$ *is dense in* $L^n_{(2)}(H;R)$.

Proof. Let $\{v_j\}$ be an orthonormal basis of H. Then

$$\begin{aligned} S(h_1, h_2, \cdots, h_n)^2 &= \{\textstyle\sum_j (h_1, v_j) S(v_j, h_2, \cdots, h_n)\}^2 \\ &\leq \{\textstyle\sum_j (h_1, v_j)^2\}\{\textstyle\sum_j S(v_j, h_2, \cdots, h_n)^2\} \\ &= |h_1|^2 \textstyle\sum_j S(v_j, h_2, \cdots, h_n)^2 \\ &\leq |h_1|^2 |h_2|^2 \cdots |h_n|^2 \textstyle\sum_{i_1,i_2,\cdots,i_n} S(v_{i_1,i_2}, \cdots, v_{i_n})^2 , \end{aligned}$$

whence (a) follows. To prove (b) and (c) let $\{Q_n\}$ and $\{e_k\}$ be given in the beginning of this section. Then

$$\begin{aligned} |T|_2^2 &= \textstyle\sum_{i_1,i_2,\cdots,i_n} T(e_{i_1}, e_{i_2}, \cdots, e_{i_n})^2 \\ &\leq \textstyle\sum_{i_1,i_2,\cdots,i_n} (\|T\| \, \|e_{i_1}\| \, \|e_{i_2}\| \cdots \|e_{i_n}\|)^2 \\ &= (\textstyle\sum_i \|e_i\|^2)^n \, \|T\|^2 . \end{aligned}$$

Moreover, if $U \in L^n_{(2)}(H;R)$, let $U_j = U(Q_j(\cdot), Q_j(\cdot), \cdots, Q_j(\cdot))$. Then $U_j \in L^n(B;R)$ and $|U_j - U|_2 \to 0$.

EXAMPLE 1. Let $H = L^2(0,1)$ (real-valued). Suppose ϕ is a measurable function on $(0,1)^n$ such that

$$\int_0^1\int_0^1 \cdots \int_0^1 |\phi(t_1, t_2, \cdots, t_n)|^2 \, dt_1 dt_2 \cdots dt_n < \infty .$$

Define $K\colon H^n \to R$ by

$$K(f_1, f_2, \cdots f_n) = \int_0^1\int_0^1\cdots\int_0^1 \phi(t_1, t_2, \cdots, t_n) f_1(t_1) f_2(t_2)\cdots f_n(t_n) dt_1 dt_2 \cdots dt_n \,.$$

Then K is a Hilbert-Schmidt type n-form on H and $|K|_2 = \left[\int_0^1\int_0^1\cdots\int_0^1 |\phi(t_1, t_2, \cdots, t_n)|^2\, dt_1 dt_2 \cdots dt_n\right]^{1/2}$.

EXAMPLE 2. Let C consist of all real-valued continuous functions on $[0, 1]$ which vanish at the origin. C is a Banach space with the sup norm. Let $C' = \{f \in C\,; f$ is absolutely continuous and $f' \in L^2(0, 1)\}$. C' is a Hilbert space with the inner product $\langle f, g\rangle = \int_0^1 f'(t)g'(t)dt$. $C' \subset C$ is an abstract Wiener space [5; 6 pp. 388–390]. Define $K\colon C'^n \to R$ by

$$K(f_1, f_2, \cdots, f_n) = \int_0^1 f_1'(t) f_2(t)\cdots f_n(t)dt \,.$$

Then K is a Hilbert-Schmidt n-form on C' and it can be checked easily that $|K|_2 = n^{-1/2}$. However, K can not be extended to C^n. This example shows that $L^n(C\,; R) \subsetneqq L^n_{(2)}(C'\,; R)$.

Notation. Let X be a Banach space. Let $S \in L^n(X\,; R)$ and $T \in L(X\,; X)$. Define the composition $S \circ_j T$ of S and T in the j-th factor by: $S \circ_j T(x_1, x_2, \cdots, x_j, \cdots, x_n) = S(x_1, x_2, \cdots, Tx_j, \cdots, x_n)$, $x_k \in X$, $k = 1, 2, \cdots, n$. Thus $S \circ_j T \in L^n(X\,; R)$. $\|T\|_X$ denotes the operator norm of T.

LEMMA 1.2. (a) $\|S \circ_j T\| \leq \|S\|\,\|T\|_B$, $S \in L^n(B\,; R)$, $T \in L(B\,; B)$.

(b) $|S \circ_j T| \leq |S|\,\|T\|_H$, $S \in L^n(H\,; R)$, $T \in L(H\,; H)$.

(c) *If* $S \in L^n_{(2)}(H\,; R)$ *and* $T \in L(H\,; H)$ *then* $S \circ_j T \in L^n_{(2)}(H\,; R)$ *and* $|S \circ_j T|_2 \leq |S|_2\,\|T\|_H$, $j = 1, 2, \cdots, n$.

Proof. (a) and (b) are trivial. We use induction to prove (c). The cases with $n = 1, 2$ are well-known. Assume we have the lemma for $n-1$. Let $S \in L^n_{(2)}(H\,; R)$ and $T \in L(H\,; H)$. Clearly $(S \circ_j T)^{\sim}(h) = S^{\sim}(h) \circ_{j-1} T$ for $j = 2, 3, \cdots n$. Hence by induction $S \circ_j T \in L^n_{(2)}(H\,; R)$, $j = 2, 3, \cdots, n$. Furthermore, let $\{v_k\}$ be an orthonormal basis of H,

$$\begin{aligned} |S \circ_j T|_2^2 &= \sum_k |(S \circ_j T)^{\sim}(v_k)|_2^2 \\ &= \sum_k |S^{\sim}(v_k) \circ_{j-1} T|_2^2 \\ &\leq \sum_k |S^{\sim}(v_k)|_2^2\, \|T\|_H^2 \qquad \text{by induction} \\ &= |S|_2^2\, \|T\|_H^2 \,. \end{aligned}$$

It remains to show the conclusion for $S\circ_1 T$. But $(S\circ_1 T)^{\sim} = S^{\sim}\circ T$. Thus $(S\circ_1 T)^{\sim}(H) \subset S^{\sim}(H) \subset L^{n-1}_{(2)}(H;R)$. Moreover by definition $|S\circ_1 T|_2 =$ the H-S-norm of $(S\circ_1 T)^{\sim}$ = the H-S-norm of $S^{\sim}\circ T \leq$ the product of H-S-norm of $S^{\sim}$ and $\|T\|_H = |S|_2 \|T\|_H$. Hence $|S\circ_1 T|_2 \leq |S|_2 \|T\|_H$.

We have now various spaces $L^n(B;R)$, $L^n_{(2)}(H;R)$ and $L^n(H;R)$, $n \geq 1$. Each such space has three topologies, namely, the uniform topology, strong topology and weak topology. However, it can be shown, by a similar argument used in [9], that these topologies generate the same Borel field. Thus we do not need to specify the Borel field corresponding to a particular topology when we talk about the measurability of a random variable with values in those spaces.

Let $W(t)$ be a Wiener process in B. Let $\mathscr{M}_t$ be the σ-field generated by $\{W(s); 0 \leq s \leq t\}$. A stochastic process $\zeta(t,\omega)$, $0 \leq t$ ann $\omega \in \Omega$, is *non-anticipating* if it is (t,ω)-jointly measurable and $\zeta(t,\cdot)$ is $\mathscr{M}_t$-measurable for each t. Let $\mathscr{L}[L^n_{(2)}(H;R)]$ denote the space consisting of all non-anticipating stochastic processes $\xi(t)$ with state space $L^n_{(2)}(H;R)$ such that $\int_0^\tau E|\xi(t)|_2^2\,dt < \infty$ for each $0 < \tau < \infty$. We will define a linear operator J from $\mathscr{L}[L^n_{(2)}(H;R)]$ into $\mathscr{L}[L^{n-1}_{(2)}(H;R)]$, $n \geq 3$. (The cases $n = 1, 2$ have been defined in [14], $L^0_{(2)}(H;R) = R$ by convention). In order to do this, we prove first a lemma about the space $\mathscr{L}[L^n(B;R)]$ consisting of all non-anticipating stochastic processes $\zeta(t)$ with state space $L^n(B;R)$ such that $\int_0^\tau E\|\zeta(t)\|^2\,dt < \infty$ for each $0 < \tau < \infty$. By Lemma 1.1 $\mathscr{L}[L^n(B;R)] \subset \mathscr{L}[L^n_{(2)}(H;R)]$. Moreover, $\mathscr{L}[L^n(B;R)]$ is dense in $\mathscr{L}[L^n_{(2)}(H;R)]$ in the following sense:

LEMMA 1.3. *If $\xi \in \mathscr{L}[L^n_{(2)}(H;R)]$ then there exists a sequence $\xi_n \in \mathscr{L}[L^n(B;R)]$ such that $\int_0^\tau E|\xi_n(t) - \xi(t)|_2^2\,dt \to 0$ as $n \to \infty$ for each $0 < \tau < \infty$.*

LEMMA 1.4. *If $\zeta \in \mathscr{L}[L^n(B;R)]$ then*
(a) *for $s < t$, $E|\tilde{\zeta}(s)(W(t) - W(s))|_2^2 = (t-s)E|\zeta(s)|_2^2$*
(b) *for $s < t < u < v$, $E\langle\tilde{\zeta}(s)(W(t) - W(s)), \tilde{\zeta}(u)(W(v) - W(u))\rangle_2 = 0$.*

Remark. The special cases $n = 1, 2$ appeared in [14].

Proof. Let $\{Q_k\}$ be the projections given in the beginning of this section. Let

$$\phi = |\tilde{\zeta}(s)(W(t) - W(s))|_2^2$$

and

$$\phi_k = |\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2 .$$

Since Q_k converges strongly to the identity in B, $\phi_k \to \phi$ almost surely. Furthermore,

$$\begin{aligned}\phi_k &\leq c^{2n} \|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))\|^2 \qquad \text{by Lemma 1.1,}\\ &\leq c^{2n} \|\zeta(s)\|^2 \|Q_k(W(t) - W(s))\|^2\\ &\leq c^{2n} \|\zeta(s)\|^2 \|Q_k\|_B^2 \|W(t) - W(s))\|^2\\ &\leq \text{constant} \|\zeta(s)\|^2 \|W(t) - W(s)\|^2 .\end{aligned}$$

Recall that $\sup_k \|Q_k\|_B^2 < \infty$ by the Uniform Boundedness Principle. But since ζ is non-anticipating,

$$\begin{aligned}E(\|\zeta(s)\|^2 \|W(t) - W(s)\|^2) &= E(\|\zeta(s)\|^2)E(\|W(t) - W(s)\|^2)\\ &= E(\|\zeta(s)\|^2)(t-s)\int_B \|x\|^2 p_1(dx) ,\end{aligned}$$

where p_1 is Wiener measure with variance parameter 1. Therefore, by the Lebesgue dominated convergence theorem,

$$(4) \qquad E|\tilde{\zeta}(s)(W(t) - W(s))|_2^2 = \lim_{k\to\infty} E|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2 .$$

Without loss of generality, we may assume that Q_k is the orthogonal projection onto the span of $\{f_j ; j = 1, 2, \cdots, k\}$, where $\{f_j\}$ is an orthonormal basis of H. Then

$$\begin{aligned}&|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2\\ &\quad = \langle\tilde{\zeta}(s)(Q_k(W(t) - W(s))), \tilde{\zeta}(s)(Q_k(W(t) - W(s)))\rangle_2\\ &\quad = \sum_{j,m=1}^{k} (W(t) - W(s), f_j)(W(t) - W(s), f_m)\langle\tilde{\zeta}(s)(f_j), \tilde{\zeta}(s)(f_m)\rangle_2 .\end{aligned}$$

Recall that ζ is non-anticipating and also that $E(W(t) - W(s), f_j)(W(t) - W(s), f_m) = (t - s)\delta_{jm}$. Hence we have

$$(5) \qquad E|\tilde{\zeta}(s)(Q_k(W(t) - W(s)))|_2^2 = \sum_{j=1}^{k} (t-s)E|\tilde{\zeta}(s)(f_j)|_2^2 .$$

It follows from (4) and (5) that

$$\begin{aligned}E|\tilde{\zeta}(s)(W(t) - W(s))|_2^2 &= \sum_{j=1}^{\infty} (t-s)E|\tilde{\zeta}(s)(f_j)|_2^2\\ &= (t-s)E|\zeta(s)|_2^2 \qquad \text{by (3).}\end{aligned}$$

Clearly, (b) can be shown in the same way.

Now, we are ready to define the linear operator J from $\mathscr{L}[L^n_{(2)}(H\,;R)]$ into $\mathscr{L}[L^{n-1}_{(2)}(H\,;R)]$. Let $\xi \in \mathscr{L}[L^n(B\,;R)]$ be simple with jumps at $0 < t_1 < t_2 < \cdots < t_k$. Define, if $t_j \leq t < t_{j+1}$, $0 \leq j \leq k$,

$$J_\xi(t) = \textstyle\sum_{i=0}^{j-1} \xi(t_i)(W(t_{i+1}) - W(t_i)) \\ + \xi(t_j)(W(t) - W(t_j)) \;.$$

Here $t_0 = 0$ and $t_{k+1} = \infty$ by convention. Clearly $J_\xi \in \mathscr{L}[L^{n-1}(B\,;R)] \subset \mathscr{L}[L^{n-1}_{(2)}(H\,;R)]$. Without loss of generality we may assume that $t = t_j$ for some j. Thus

$$J_\xi(t) = \textstyle\sum_{i=0}^{j-1} \xi(t_i)(W(t_{i+1}) - W(t_i)) \;.$$

Hence

$$|J_\xi(t)|_2^2 = \sum_{i,l=0}^{j-1} \langle \xi(t_i)(W(t_{i+1}) - W(t_i)), \xi(t_l)(W(t_{l+1}) - W(t_l))\rangle_2 \;.$$

It follows immediately from Lemma 1.4 that

$$E\,|J_\xi(t)|_2^2 = \textstyle\sum_{i=0}^{j-1} (t_{i+1} - t_i)E\,|\xi(t_i)|_2^2 \\ = E\int_0^t |\xi(s)|_2^2\, ds \;. \tag{6}$$

Moreover, it is easy to see that

$$E(J_\xi(t)\,|\,\mathscr{M}_s) = J_\xi(s) \qquad s \leq t \;. \tag{7}$$

From Lemma 1.3, (6), (7) and a standard argument in stochastic integral, we have

PROPOSITION 1.1. *There exists a linear operator J from $\mathscr{L}[L^n_{(2)}(H\,;R)]$ into $\mathscr{L}[L^{n-1}_{(2)}(H\,;R)]$, denoted by $J_\xi(t) = \int_0^t \xi(s)dW(s)$, such that*

(a) *J_ξ has continuous sample paths,*

(b) *J_ξ is a martingale,*

(c) $\operatorname{prob}\{\sup_{0\leq t\leq \tau} |J_\xi(t)|_2 > \delta\} \leq \delta^{-2}E\,|J_\xi(\tau)|_2^2$,

(d) *$EJ_\xi(t) = 0$ and $E\,|J_\xi(t)|_2^2 = E\int_0^t |\xi(s)|_2^2\, ds$.*

DEFINITION 2. Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces. A continuous bilinear map S from $\mathscr{H} \times \mathscr{H}$ into $\mathscr{K}$ is said to be of *trace-class-type* if (i) for each $x \in \mathscr{K}$, S_x is a trace class operator of $\mathscr{H}$, where $S_x(\cdot,\cdot) = \langle S(\cdot,\cdot), x\rangle_{\mathscr{K}}$ and (ii) the linear functional $x \to \operatorname{trace}_{\mathscr{H}} S_x$ is continuous.

Notation. The definition implies obviously that there exists a unique element, denoted by TRACE S, of $\mathscr{K}$ such that $\langle \text{TRACE } S, x\rangle_{\mathscr{K}} = \text{trace}_{\mathscr{H}} S_x$ for all $x \in \mathscr{K}$. $\mathscr{S}(\mathscr{H};\mathscr{K})$ will denote the vector space of all trace-class-type bilinear maps from $\mathscr{H}$ into $\mathscr{K}$.

PROPOSITION 1.2. (a) *If $S \in \mathscr{S}(\mathscr{H};\mathscr{K})$ and $\{\phi_k\}$ is an orthonormal basis of $\mathscr{H}$ then $\sum_{k=1}^{\infty} S(\phi_k, \phi_k)$ converges in $\mathscr{K}$ to* TRACE S,

(b) *If $S \in \mathscr{S}(\mathscr{H};\mathscr{K})$ and $T, U \in L(\mathscr{H};\mathscr{H})$, $V \in L(\mathscr{K};\mathscr{K})$ then $S \circ [T \times U]$ and $V \circ S$ belong to $\mathscr{S}(\mathscr{H};\mathscr{K})$ and* TRACE $V \circ S = V(\text{TRACE } S)$,

(c) $L^2(B; L^n(B;R)) \subset \mathscr{S}(H; L^n_{(2)}(H;R))$.

Proof. (a) and (b) appeared in [15] in a similar form. (c) follows from the fact that $L^2(B;R) \approx L(B;B^*) \subset L_1(H;H)$, the Banach space of all trace class operators of H with the trace class norm $|\cdot|_1$. Actually, $|S|_1 \leq \|S\| \int_B \|x\|^2 p_1(dx)$ for all $S \in L^2(B;R)$.

Notation. 1) If $T \in L^n(H;R)$ and $S \in L(L^{n-1}(H;R); L^{n-1}(H;R))$ we define the composition $S \vartriangle T$ of S and T to be an element of $L^n(H;R)$ by $(S \vartriangle T)^{\sim} = S \circ \tilde{T}$. Thus $S \vartriangle T(h_1, h_2, \cdots, h_n) = S(\tilde{T}(h_1))(h_2, \cdots, h_n)$.

2) If $S \in L^n(H;R)$ and $T \in L(L^{n-1}(H;R);R)$ we define $S \vartriangle T$ to be an element of H by: $\langle S \vartriangle T, h\rangle = T(\tilde{S}(h))$, $h \in H$. Of course if $S \in L^n_{(2)}(H;R)$ and $T \in L^{n-1}_{(2)}(H;R)$ then define $\langle S \vartriangle T, h\rangle = \langle T, \tilde{S}(h)\rangle_2$.

Remarks. (1) If $T \in L^n_{(2)}(H;R)$ and $L^{n-1}_{(2)}(H;R)$ is invariant under S then $S \vartriangle T \in L^n_{(2)}(H;R)$.

(2) For the case $n = 2$ in Notation 2, it is easy to see that $S \vartriangle h = S^*h$, $h \in H$.

In [14] we proved an infinite dimensional analogue of well-known Ito's formula [8]. This formula was used in [17] to show the relation between the work of [14] and that of [19]. Later, in [15] we proved another version of Ito's formula and used it to construct diffusion processes in a Riemann-Wiener manifold. We will give three versions of Ito's formula for stochastic processes with state space $L^n(H;R)$, $n \geq 2$. Let $\mathscr{L}[L^n(H;R)]$ consist of all non-anticipating processes $\zeta(t)$ with state space $L^n(H;R)$ such that $\int_0^{\tau} E|\zeta(t)|^2\, dt < \infty$ for each $0 < \tau < \infty$.

THEOREM 1. (Ito's formula). *Let θ be a twice Frechet differentiable map from $L^n(H;R)$ into itself such that for all $S \in L^n(H;R)$* (i) $\theta'(S)(L^n_{(2)}(H;R)) \subset L^n_{(2)}(H;R)$, (ii) $\theta''(S)(L^n_{(2)}(H;R) \times L^n_{(2)}(H;R)) \subset L^n_{(2)}(H;R)$

and (iii) $\theta''(S) \in \mathscr{S}(L^n_{(2)}(H;R); L^n_{(2)}(H;R))$. *If* $\Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds$, *where* $\xi \in \mathscr{L}[L^{n+1}_{(2)}(H;R)]$ *and* $\zeta \in \mathscr{L}[L^n(H;R)]$. *Then*

$$\theta(\Phi(t)) = \theta(\Phi_0) + \int_0^t \theta'(\Phi(s)) \vartriangle \xi(s)dW(s) + \int_0^t \Big\{\theta'(\Phi(s))(\zeta(s)) + \tfrac{1}{2}\,\mathrm{TRACE}\,\theta''(\Phi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]\Big\} ds .$$

Proof. Kunita-Watanabe's method [12; 13] can be employed here. We will sketch the outline only. Let $\varepsilon > 0$ and $\{\sigma_j\}$ be an increasing sequence of stopping time converging to ∞ such that $\sigma_0 = 0$ and for $\sigma_j \leq s$, $t < \sigma_{j+1}$, we have

$$\left|\int_s^t \xi(\tau)dW(\tau)\right|_2 < \varepsilon/2$$

and

$$\left|\int_s^t \zeta(s)ds\right| < \varepsilon/2 .$$

Thus, whenever $\sigma_j \leq s, t < \sigma_{j+1}$

$$|\Phi(t) - \Phi(s)| < \varepsilon .$$

Because θ is twice Frechet differentiable, we have, whenever x and y are near in $L^n(H;R)$,

$$\theta(x) - \theta(y) = \theta'(y)(x-y) + \tfrac{1}{2}\theta''(y)(x-y, x-y) + o(|x-y|^2) .$$

Time parameter will be also subscribed from now on. Let $\tau_j = t \wedge \sigma_j$. Thus

$$\begin{aligned}\theta(\Phi(t)) - \theta(\Phi_0) &= \textstyle\sum_{j=1}^\infty [\theta(\Phi_{\tau_j}) - \theta(\Phi_{\tau_{j-1}})] \\ &= \textstyle\sum_{j=1}^\infty \theta'(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}}) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{j=1}^\infty \theta''(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}}, \Phi_{\tau_j} - \Phi_{\tau_{j-1}}) \\ &\quad + o(|\Phi_{\tau_j} - \Phi_{\tau_{j-1}}|^2) .\end{aligned}$$

Putting $\Phi_{\tau_j} - \Phi_{\tau_{j-1}} = \int_{\tau_{j-1}}^{\tau_j} \xi(s)dW(s) + \int_{\tau_{j-1}}^{\tau_j} \zeta(s)ds$ into the above equation, we see that to finish the proof it is sufficient to show the following two equalities:

$$\theta'(\Phi_s)\left(\int_s^t \xi(\tau)dW(\tau)\right) = \int_s^t \theta'(\Phi_s) \vartriangle \xi(\tau)dW(\tau) \tag{8}$$

$$\theta''(\Phi_s)\left(\int_s^t \xi(\tau)dW(\tau),\ \int_s^t \xi(\tau)dW(\tau)\right) = \int_s^t \mathrm{TRACE}\, \theta''(\Phi_s) \circ [\tilde{\xi}(\tau) \times \tilde{\xi}(\tau)]d\tau \tag{9}$$

(8) is easily checked, while (9) follows from the following observation: If $s \leq u < v$ then

$$E\theta''(\Phi_s)(\tilde{\xi}(u)(W(v) - W(u)), \tilde{\xi}(u)(W(v) - W(u))) = (v - u)E\ \mathrm{TRACE}\, \theta''(\Phi(s)) \circ [\tilde{\xi}(u) \times \tilde{\xi}(u)] .$$

If $s \leq u < v \leq u' < v'$ then

$$E\theta''(\Phi_s)(\tilde{\xi}(u)(W(v) - W(u)), \tilde{\xi}(u')(W(v') - W(u'))) = 0 .$$

THEOREM 2 (Ito's formula). *Let Γ be a twice differentiable map from $L^n_{(2)}(H; R)$ into itself such that $\Gamma''(S) \in \mathscr{S}(L^n_{(2)}(H; R); L^n_{(2)}(H; R))$. If $\Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds$, where $\xi \in \mathscr{L}[L^{n+1}_{(2)}(H; R)]$ and $\zeta \in \mathscr{L}[L^n_{(2)}(H; R)]$. Then*

$$\Gamma(\Phi(t)) = \Gamma(\Phi_0) + \int_0^t \Gamma'(\Phi(s)) \vartriangle \xi(s)dW(s) + \int_0^t \left\{\Gamma'(\Phi(s))(\zeta(s) + \tfrac{1}{2}\,\mathrm{TRACE}\, \Gamma''(\Phi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]\right\} ds .$$

THEOREM 3 (Ito's formula). *Let f be a twice Frechet differentiable function from $L^n(H; R)$ (resp. $L^n_{(2)}(H; R)$) into R. If $\Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds$, where ξ and ζ are same as Theorem 1 (resp. Theorem 2). Then*

$$f(\Phi(t)) = f(\Phi_0) + \int_0^t (\xi(s) \vartriangle f'(\Phi(s)), dW(s)) + \int_0^t \{f'(\Phi(s))(\zeta(s)) + \tfrac{1}{2}\,\mathrm{trace}\, f''(\Phi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]\}ds .$$

Remark. The proof of the above theorems goes in the same way as that of Theorem 1. We point out that $\xi(s) \vartriangle f'(\Phi(s))$ (see Notation 2) following Proposition 1.2) is a non-anticipating process with state space H and the stochastic integral $\int_0^t (\xi(s) \vartriangle f'(\Phi(s)), dW(s))$ was defined in the previous paper [14]. Furthermore, $f''(\Phi(s))\circ[\tilde{\xi}(s)\times\tilde{\xi}(s)]$ is a non-anticipating process with state space $L_1(H;H)$, the Banach space of all trace class operators of H. To see this, note that if $S\in L^2(L^n(H;R);R)$ and $T\in L^{n+1}_{(2)}(H;R)$ then $S\circ[\tilde{T}\times\tilde{T}]$ is a trace class operator of H.

THEOREM 4 (Girsanov-Skorokhod-McKean's formula). *Suppose* $\xi\in\mathscr{L}[L^3_{(2)}(H;R)]$ *and* $\eta\in\mathscr{L}[L^2(H;R)]$ *and with probability* 1, $\{\tilde{\xi}(t)(x),\ \eta(t);\ 0\le t<\infty,\ x\in H\}$ *forms a commutative family of operators of* $H(L^2(H;R)\approx L(H;H))$. *Then the solution of*

$$Y(t)=I+\int_0^t Y(s)\circ_3\xi(s)^*dW(s)+\int_0^t Y(s)\circ\eta(s)ds \tag{10}$$

can be represented by

$$Y(t)=\exp\left\{\int_0^t\xi(s)dW(s)+\int_0^t\left\{\eta(s)-\tfrac{1}{2}\mathrm{TRACE}\,\kappa\circ[\tilde{\xi}(s)\times\tilde{\xi}(s)]\right\}ds\right\}, \tag{11}$$

where κ *is the map from* $L(H;H)\times L(H;H)$ *into* $L(H;H)$ *given by* $\kappa(S,T)=S\circ T$.

Remark. We will discuss stochastic integral equation below. Moreover, in [16] we define, for $S\in L(H;H)$ and $T\in L^3(H;R)$, $S\vartriangle T\in L^3(H;R)$ by $(S\vartriangle T)^{\sim}(x)=S\circ(\tilde{T}(x))$, $x\in H$. It is easy to see that $S\vartriangle T$ is nothing but $T\circ_3 S^*$. Thus equation (10) is the same as the equation in § 4 of [16].

Proof. As in one dimensional case, we can solve (10) directly by using log function. Here, we prove this theorem in the reverse direction. Theorem 5 below implies that (10) has a unique solution. Thus it suffices to check that (11) satisfies (10). Consider the function $\theta(x)=\exp(x)$, $x\in L(H;H)$. θ is a C^∞-function from $L(H;H)$ into itself satisfying the hypothesis of Theorem 1 and, in particular, if x and y commute we have $\theta'(x)y=e^xy$ and $\theta''(x)(y,y)=e^xy^2$. Let

$$\Phi(t)=\int_0^t\xi(s)dW(s)+\int_0^t\{\eta(s)-\tfrac{1}{2}\,\mathrm{TRACE}\,\kappa\circ[\tilde{\xi}(s)\times\tilde{\xi}(s)]\}ds\,.$$

Then $Y(t)=\exp\{\Phi(t)\}$. By stochastic differentiation given in Theorem 1, we have

$$\begin{aligned} dY(t)&=\theta'(\Phi(t))\vartriangle\xi(t)dW(t)+\theta'(\Phi(t))(\eta(t)\\ &\quad-\tfrac{1}{2}\,\mathrm{TRACE}\,\kappa\circ[\tilde{\xi}(t)\times\tilde{\xi}(t)])dt\\ &\quad+\tfrac{1}{2}\,\mathrm{TRACE}\,\theta''(\Phi(t))\circ[\tilde{\xi}(t)\times\tilde{\xi}(t)]dt\,. \end{aligned} \tag{12}$$

Recall the notation 1) following Proposition 1.2. Let $h_1, h_2, h_3 \in H$

$$\begin{aligned}\theta'(\Phi(t)) \vartriangle \xi(t)(h_1, h_2, h_3) &= \theta'(\Phi(t))(\tilde{\xi}(t)h_1)(h_2, h_3)\\ &= e^{\Phi(t)}\tilde{\xi}(t)(h_1)(h_2, h_3) \qquad \text{by commutativity assumption,}\\ &= \langle Y(t)\tilde{\xi}(t)(h_1)h_2, h_3\rangle\\ &= \langle \tilde{\xi}(t)(h_1)h_2, Y(t)^*h_3\rangle\\ &= \xi(t)(h_1, h_2, Y(t)^*h_3)\\ &= \xi(t)\circ_3 Y(t)^*(h_1, h_2, h_3)\ .\end{aligned}$$

Therefore, we have

$$\theta'(\Phi(t)) \vartriangle \xi(t) = \xi(t)\circ_3 Y(t)^*\ . \tag{13}$$

Clearly,

$$\theta'(\Phi(t))(\eta(t)) = Y(t)\circ\eta(t)\ . \tag{14}$$

Moreover, it can be checked easily that

$$\theta'(\Phi(t))(\kappa\circ[\tilde{\xi}(t)\times\tilde{\xi}(t)] = \theta''(\Phi(t))\circ[\tilde{\xi}(t)\times\tilde{\xi}(t)]\ . \tag{15}$$

Putting (13), (14), and (15) into (12), we obtain

$$dY(t) = \xi(t)\circ_3 Y(t)^*dW(t) + Y(t)\circ\eta(t)dt\ ,$$

or

$$Y(t) = I + \int_0^t \xi(s)\circ_3 Y(s)^*dW(s) + \int_0^t Y(s)\circ\eta(s)ds\ .$$

THEOREM 5. *Let f and g be maps from $[t_0, \infty)\times L^n(H;R)\times\Omega$ ($t_0\geq 0, n\geq 2$) into $L^{n+1}_{(2)}(H;R)$ and $L^n(H;R)$, respectively. Assume that f and g satisfy the following conditions:*

(a) *for each $S\in L^n(H;R)$, $f(\cdot, S, \cdot)$ and $g(\cdot, S, \cdot)$ are non-anticipating,*

(b) *there is a constant c such that with probability 1,*

$$|f(t,S) - f(t,T)|_2 + |g(t,S) - g(t,T)| \leq c|S-T|\ ,$$

and

$$|f(t,S)|_2 + |g(t,S)| \leq c(1+|S|)$$

for all $t\in[t_0,\infty)$ and $S, T\in L^n(H;R)$.

Let $\zeta \in \mathscr{L}[L^n(H;R)]$ have continuous sample paths. Then the $L^n(H;R)$-valued stochastic integral equation

$$Y(t) = \zeta(t) + \int_{t_0}^{t} f(s, Y(s))dW(s) + \int_{t_0}^{t} g(s, Y(s))ds$$

has a unique continuous solution $Y \in \mathscr{L}[L^n(H;R)]$. Moreover, $Y(t)$ is a Markov process if $\zeta(t)$ is so.

Proof. We may assume that $t_0 \leq t \leq t_1 < \infty$. Let $\mathfrak{A}$ be the Banach space of all non-anticipating processes $Y(t)$ in $L^n(H;R)$ with norm

$$|||Y||| = \left\{\int_{t_0}^{t_1} E\,|Y(t)|^2\,dt\right\}^{1/2} < \infty\,.$$

Clearly, $\mathscr{L}[L^n(H;R)] \subset \mathfrak{A}$. Define a map Φ in $\mathfrak{A}$ by

$$\Phi(Y)(t) = \zeta(t) + \int_{t_0}^{t} f(s, Y(s))dW(s) + \int_{t_0}^{t} g(s, Y(s))ds\,.$$

It is easy to see that Φ is a map from $\mathfrak{A}$ into itself and $\Phi(Y)$ has continuous sample paths. Furthermore,

$$E\,|\Phi(Y)(t) - \Phi(Z)(t)|^2 \leq \alpha\int_{t_0}^{t} E\,|Y(s) - Z(s)|^2\,ds\,, \tag{16}$$

where α is a constant depending only on c, t_0 and t_1. (16) implies that there exists an N such that whenever $m \geq N$,

$$|||\Phi^m(Y) - \Phi^m(Z)||| \leq \tfrac{1}{2}\,|||Y - Z|||\,.$$

The rest of the proof goes in the same way as Theorem 5.1 of [14].

THEOREM 6. *In the hypothesis of Theorem 5 replace $L^n(H;R)$ by $L^n_{(2)}(H;R)$ and $L^n(H;R)$-norm $|\cdot|$ by $L^n_{(2)}(H;R)$-norm $|\cdot|_2$. Then the $L^n_{(2)}(H;R)$-valued stochastic integral equation*

$$Z(t) = \zeta(t) + \int_{t_0}^{t} f(s, Z(s))dW(s) + \int_{t_0}^{t} g(s, Z(s))ds$$

has a unique continuous solution $Z \in \mathscr{L}[L^n_{(2)})H;R)]$. $Z(t)$ is a Markov process if $\zeta(t)$ is so.

Remark. Theorem 5 with $n = 2$ and Theorem 6 with $n \geq 3$ will be used in the next section. Proof of Theorem 6 is obvious.

2. Regularity Properties

We assume that A and σ satisfy the following conditions:
$(A-1)$ A is of the form $A(t,x) = C + K(t,x)$, where $C \in L(B;B)$ and K is a continuous map from $[0,\infty)\times B$ into $L_{(2)}(H;H)$,
$(A-2)$ There is a constant γ such that for all $t \geq 0$ and $x,y \in B$,

$$|K(t,x) - K(t,y)|_2 \leq \gamma\,\|x-y\| \quad \text{and} \quad |K(t,x)|_2 \leq \gamma(1+\|x\|)\,,$$

$(\sigma-1)$ σ is continuous map from $[0,\infty)\times B$ into B such that for all $t \geq 0$ and $x,y \in B$, $\|\sigma(t,x) - \sigma(t,y)\| \leq \gamma\|x-y\|$ and $\|\sigma(t,x)\| \leq \gamma(1+\|x\|)$.

Although the above conditions are weaker than those in Theorem 5.1 [14], it is easy to see that the proof there goes in the same way to conclude that under $(A-1)$,$(A-2)$ and $(\sigma-1)$ the stochastic integral equation (1) has a unique non-anticipating continuous solution. Moreover, this solution is a Markov process. In the sequel, we denote this solution by $X_x(t)$, where x is the starting point.

DEFINITION 3 [7]. A map f from B into a Banach space D is said to be *Frechet differentiable at x in H-directions* (briefly, H-differentiable at x) if there exists a linear operator $T \in L(H;D)$ such that $\|f(x+h) - f(x) - T(h)\|_D = o(|h|)$, $h \in H$. T is easily checked to be unique and will be denoted by $f'(x)$, called the H-derivative of f at x. f is said to be C_H^1 if $f'(x)$ exists for all $x \in B$ and f' is continuous from B into $L(H;D)$. Inductively, we can define n-th H-differentiability and C_H^n.

Notation. Let D be a Banach space. $\mathscr{L}(D)$ will denote the Banach space of square integrable random variables taking values in D. Note that in Section 1 we used $\mathscr{L}[D]$ to denote the space of all non-anticipating processes ζ such that $\int_0^\tau E\,\|\zeta(t)\|_D^2\,dt < \infty$ for each $0 \leq \tau < \infty$.

DEFINITION 4. A function ξ from B into $\mathscr{L}(D)$ is said to be *mean-square differentiable at x in H-directions* (briefly, MS-H-differentiable at x) if there is $\theta \in \mathscr{L}(L(H;D))$ such that $E\,\|\xi(x+h) - \xi(x) - \theta(h)\|_D^2 = o(|h|^2)$, $h \in H$. θ is unique and will be denoted by $\delta\xi_x$, called the MS-H-derivative of ξ at x. MS-H-differentiability and $MS - C_H^n$ $(n \geq 1)$ are defined in an obvious way.

DEFINITION 5. A transformation Z from B into $\mathscr{L}[D]$ is said to be *MS-H-differentiable* if there is a transformation Y from B into

$\mathscr{L}[L(H;D)]$ such that for each $t \geq 0$, $Y(t)$ is the *MS-H*-derivative of $Z(t)$. Y is unique and will be denoted by δZ. Higher order *MS-H*-derivatives will be denoted by $\delta^n Z$, $n \geq 2$.

EXAMPLE 1. Let $X(t) = x + W(t)$, where W is Wiener process starting at the origin. Then $\delta X_x(t) = I$ for all x and $\delta^n X_x(t) \equiv 0$, $n \geq 2$.

EXAMPLE 2. Consider the Langevin equation $dU(t) = dW(t) - U(t)dt$. Its solution $U(t)$ is called Uhlenbeck-Ornstein process. We have $\delta U_x(t) = e^{-t}I$ for all x, $\delta^n U_x(t) \equiv 0$, $n \geq 2$.

EXAMPLE 3. Let $K \in L_{(2)}(H;H)$, $T \in L(B;H)$ and $x_0 \in H \cap \ker T^*$, where $*$ denotes the adjoint of operators of H. Consider the equation $dX(t) = (I+K)dW(t) + f(|TX(t)|^2)x_0 dt$, where f is a real-valued differentiable function with compact support. We have $\delta X_x(t) = e^{\zeta_x(t)}$, where $\zeta_x \in \mathscr{L}[L(H;H)]$ is given by $\zeta_x(t) = 2\left[\int_0^t f'(|TX_x(s)|^2)\langle T^2 X_x(s), \cdot\rangle ds\right]x_0$.

Remark. Two transformations Z_1 and Z_2 from B into $\mathscr{L}[D]$ have the same *MS-H*-derivative if and only if there exists $\xi \in \mathscr{L}[D]$ such that $Z_1 - Z_2 \equiv \xi$. Moreover, if ξ is an *MS-H*-differentiale function from B into $\mathscr{L}(D)$ then $\xi_{x+h} - \xi_x = \int_0^1 \delta\xi_{x+\tau h}(h)d\tau$, $x \in B$ and $h \in H$.

PROPOSITION 2.1. *Suppose* $\xi \in \mathscr{L}[L_{(2)}^{n+1}(H;R)]$, $n \geq 0$. *Then* $E|J_\xi(t)|_2^4 = 2\int_0^t E[|J_\xi(s)|_2^2 |\xi(s)|_2^2]ds + 4\int_0^t E|\xi(s) \vartriangle J_\xi(s)|^2 ds$.

Remark. $\xi \vartriangle J_\xi \in \mathscr{L}[H]$. See Notation 2) following Proposition 1.2.

Proof. Apply Ito's formula in Theorem 3 to the function $f(x) = |x|_2^4$, $x \in L_{(2)}^n(H;R)$ and to the process $J_\xi(t) = \int_0^t \xi(s)dW(s)$, $\xi \in \mathscr{L}[L_{(2)}^{n+1}(H;R)]$. Note that $f'(x) = 4|x|_2^2\langle x, \cdot\rangle_2$ and $f''(x) = 4|x|_2^2\langle\cdot,\cdot\rangle_2 + 8\langle x,\cdot\rangle_2\langle x,\cdot\rangle_2$. Hence we have

$$f(J_\xi(t)) = \int_0^t (\xi(s) \vartriangle f'(J_\xi(s)), dW(s)) + \frac{1}{2}\int_0^t \operatorname{trace} f''(J_\xi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]ds .$$

After taking expectation, we get

$$E|J_\xi(t)|_2^4 = \frac{1}{2}\int_0^t E \operatorname{trace} f''(J_\xi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]ds . \tag{17}$$

Let $\{e_i\}$ be an orthonormal basis of H, then

$$\begin{aligned}&\operatorname{trace}|J_\xi(s)|_2^2\langle\tilde{\tilde{\xi}}(s),\tilde{\tilde{\xi}}(s)\rangle_2\\ &\quad=\textstyle\sum_j|J_\xi(s)|_2^2\langle\tilde{\tilde{\xi}}(s)e_j,\tilde{\tilde{\xi}}(s)e_j\rangle_2\\ &\quad=|J_\xi(s)|_2^2|\xi(s)|_2^2\,,\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\operatorname{trace}\langle J_\xi(s),\tilde{\tilde{\xi}}(s)\rangle_2\langle J_\xi(s),\tilde{\tilde{\xi}}(s)\rangle_2\\ &\quad=\textstyle\sum_j\langle J_\xi(s),\tilde{\tilde{\xi}}(s)e_j\rangle_2^2\\ &\quad=\textstyle\sum_j\langle\xi(s)\vartriangle J_\xi(s),e_j\rangle^2\\ &\quad=|\xi(s)\vartriangle J_\xi(s)|^2\,.\end{aligned}\tag{19}$$

Combining (17), (18), and (19), we obtain the conclusion.

PROPOSITION 2.2. *Suppose* $\xi\in\mathscr{L}[L_{(2)}^{n+1}(H;R)]$, $n\geq 0$. *Then* $E|J_\xi(t)|_2^4\leq 36t\int_0^t E|\xi(s)|_2^4\,ds$.

Proof. First note that from the previous proposition $E|J_\xi(t)|_2^4$ is an increasing function of t. Hence

$$\begin{aligned}E[|J_\xi(s)|_2^2|\xi(s)|_2^2]&\leq\{E|J_\xi(s)|_2^4\}^{1/2}E|\xi(s)|_2^4\}^{1/2}\\ &\leq\{E|J_\xi(t)|_2^4\}^{1/2}\{E|\xi(s)|_2^4\}^{1/2}\,.\end{aligned}$$

Now, recall that for $S\in L_{(2)}^{n+1}(H;R)$ and $T\in L_{(2)}^{n}(H;R)$, $S\vartriangle T$ is an element in H defined by $\langle S\vartriangle T,h\rangle=\langle T,\tilde{S}(h)\rangle_2$. Thus we have $|S\vartriangle T|\leq|T|_2|S|_2$. So,

$$\begin{aligned}E|\xi(s)\vartriangle\dot{J}_\xi(s)|^2&\leq E[|\xi(s)|_2^2|J_\xi(s)|_2^2]\\ &\leq\{E|J_\xi(t)|_2^4\}^{1/2}\{E|\xi(s)|_2^4\}^{1/2}\,.\end{aligned}$$

Therefore, by the previous proposition

$$\begin{aligned}E|J_\xi(t)|_2^4&\leq 6\{E|J_\xi(t)|_2^4\}^{1/2}\int_0^t\{E|\xi(s)|_2^4\}^{1/2}ds\\ &\leq 6\{E|J_\xi(t)|_2^4\}^{1/2}\left\{t\int_0^t E|(s)|_2^4\,ds\right\}^{1/2}\,,\end{aligned}$$

and

$$E|J_\xi(t)|_2^4\leq 36t\int_0^t E|\xi(s)|_2^4\,ds\,.$$

Notation. 1) Let $S\in L^n(H;R)$. $\hat{S}\in L^{n-1}(H;H)$ is defined by $\langle\hat{S}(h_1,$

$h_2, \cdots, h_{n-1}), h\rangle = S(h, h_1, h_2, \cdots, h_{n-1})$. Note that for $n = 2, \tilde{S} = S$, while $\hat{S} = S^*$.

2) Let $T \in L^3(H; R)$ and $S \in L^n(H; R)$. Define $S: T \in L^{n+1}(H; R)$ by $S: T(h_1, h_2, \cdots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \cdots, h_n), h_{n+1})$. Note that if $T \in L^3_{(2)}(H; R)$ and $S \in L^n_{(2)}(H; R)$ then $S: T \in L^{n+1}_{(2)}(H; R)$ and $|S: T|_2 \leq |S|_2 |T|_2$. But for $n = 2$, $|S: T|_2 \leq |S||T|_2$.

Remark. If $T \in L^3(H; R)$, $S \in L^2(H; R)$ and $\tilde{T}(h)$ commutes with S for all $h \in H$. Then $S: T = T \circ_3 S$.

THEOREM 7. *Assume A and σ satisfy $(A-1)$, $(A-2)$, $(\sigma-1)$ and the following conditions:*
$(A-3)$ $K(t, x)$ is C^2_H in x variable with $K'(t, x) \in L^3_{(2)}(H; R)$ and $K''(t, x) \in L^4_{(2)}(H; R)$. $K'(\cdot, \cdot)$ and $K''(\cdot, \cdot)$ are bounded continuous maps from $[0, \tau) \times B$ into $L^3_{(2)}(H; R)$ and $L^4_{(2)}(H; R)$, respectively, for each τ.
$(A-4)$ for all t and x, $K'(t, x) \in L^3_{(2)}(H; R)$ and $K''(t, x) \in L^4_{(2)}(H; R)$ are symmetric in the first two components,
$(\sigma-2)$ $\sigma(t, x)$ ir C^2_H in x variable with $\sigma'(t, x) \in L(H; H)$ and $\sigma''(t, x) \in L^3_{(2)}(H; R)$. $\sigma'(\cdot, \cdot)$ and $\sigma''(\cdot, \cdot)$ are bounded continuous maps from $[0, \tau) \times B$ into $L(H; H)$ and $L^3_{(2)}(H; R)$ respectively, for each τ. Then the diffusion process given by the solution of the stochastic integral equation

$$X(t) = X(0) + \int_0^t A(s, X(s)) dW(s) + \int_0^t \sigma(s, X(s)) ds \tag{20}$$

is twice MS-H-differentiable. The first derivative at x is given by the solution of the operator-valued stochastic integral equation

$$Y(t) = I + \int_0^t Y(s): K'(s, X_x(s)) dW(s) + \int_0^t \sigma'(s, X_x(s)) \circ Y(s) ds . \tag{21}$$

The second derivative at x is given by the solution of the 3-form-valued stochastic integral equation

$$Z(t) = \phi(t) + \int_0^t Z(s): K'(s, X_x(s)) dW(s) + \int_0^t Z(s) \circ_3 \sigma''(s, X_x(s))^* ds , \tag{22}$$

where

$$\phi(t) = \int_0^t K''(s, X_x(s)) \circ_4 [\delta X_x(s)^2]^* dW(s) + \int_0^t \hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds .$$

Furthermore, $\delta X_x \in \mathscr{L}[L(H; H)]$ and $\delta^2 X_x \in \mathscr{L}[L^3_{(2)}(H; R)]$.

Proof. We need to show that $E\,\|X_{x+h}(t)-X_x(t)-Y(t)h\|^2=o(|h|^2)$, $h\in H$. But it is easy to see that $X_{x+h}(t)-X_x(t)$ is in H. Thus we will show below a stronger statement, namely, $E\,|X_{x+h}(t)-X_x(t)-Y(t)h|^2=o(|h|^2)$, $h\in H$. Assume $0\le t\le\tau$. Let $\psi_h(t)=X_{x+h}(t)-X_x(t)$. Then

$$\psi_h(t)=h+\int_0^t\xi_h(s)(\psi_h(s))dW(s)+\int_0^t\zeta_h(s)(\psi_h(s))ds\,, \tag{23}$$

where $\xi_h(s)$ and $\zeta_h(s)$ are given by

$$\xi_h(s)=\int_0^1 A'(s,X_x(s)+\tau\psi_h(s))d\tau=\int_0^1 K'(s,X_x(s)+\tau\psi_h(s))d\tau\,, \tag{24}$$

$$\zeta_h(s)=\int_0^1\sigma'(s,X_x(s)+\tau\psi_h(s))ds\,. \tag{25}$$

On the other hand,

$$Y(t)h=h+\int_0^t K'(s,X_x(s))(Y(s)h)dW(s)+\int_0^t\sigma'(s,X_x(s))(Y(s)h)ds\,. \tag{26}$$

Here we have used the condition $(A-4)$ to bring h into the integral sign.

Now, it can be shown with some computation that

$$\begin{aligned}E\,|\xi_h(s)(\psi_h(s))-K'(s,X_x(s))(Y(s)h)|_2^2\\ \le c_1E\,|\psi_h(s)-Y(s)h|^2+c_2E(|\psi_h(s)|^2\,|Y(s)h|^2)\,,\end{aligned} \tag{27}$$

and

$$\begin{aligned}E\,|\zeta_h(s)(\psi_h(s))-\sigma'(s,X_x(s))(Y(s)h)|^2\\ \le c_1E\,|\psi_h(s)-Y(s)h|^2+c_2E(|\psi_h(s)|^2\,|Y(s)h|^2)\,,\end{aligned} \tag{28}$$

where c_1 and c_2 are constants independent of s and t. From (23)–(28), we obtain for all $0\le t\le\tau$,

$$E\,|\psi_h(t)-Y(t)h|^2\le c_3\lambda(h)+c_4\int_0^t E\,|\psi_h(s)-Y(s)h|^2\,ds\,,$$

where c_3 and c_4 are constants independent of t, and

$$\lambda(h)=\int_0^\tau E\,|\psi_h(s)|^2\,|Y(s)h|^2\,ds\,.$$

Hence by Gronwall's Lemma,

$$E\,|\psi_h(t)-Y(t)h|^2\le c_3\lambda(h)e^{c_4\tau}\qquad 0\le t\le\tau\,.$$

But $\lambda(h) \leq |h|^2 \int_0^\tau E|\psi_h(s)|^2 \|Y(s)\|_H^2 ds$, hence we are remained to prove that

$$\lim_{|h|\to 0} \int_0^\tau E|\psi_h(s)|^2 \|Y(s)\|_H^2 ds = 0 . \tag{29}$$

By a complicated computation using Proposition 2.2 and Gronwall's Lemma, we have

$$E|\psi_h(t)|^4 \leq \text{constant } |h|^4 , \qquad 0 \leq t \leq \tau ,$$

and

$$E\|Y(t)\|_H^4 \leq \text{constant} , \qquad 0 \leq t \leq \tau .$$

Hence (29) is evident and, in particular, we have also that $Y \in \mathscr{L}[L(H;H)]$.

We should not try to prove the second assertion. But we will show that ϕ given in (22) is in $\mathscr{L}[L^3_{(2)}(H;R)]$. ϕ is clearly non-anticipating.

$$\begin{aligned} |\phi(t)|_2^2 &\leq 2\left|\int_0^t K''(s, X_v(s) \circ_4 [\delta X_x(s)^2]^* dW(s)\right|_2^2 \\ &\quad + 2\left|\int_0^t \hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds\right|_2^2 . \end{aligned} \tag{30}$$

Apply Proposition 1.1 to get

$$\begin{aligned} &E\left|\int_0^t K''(s, X_x(s)) \circ_4 [\delta X_x(s)^2]^* dW(s)\right|_2^2 \\ &\quad = \int_0^t E|K''(s, X_x(s) \circ_4 [\delta X_x(s)^2]^*|_2^2 ds \\ &\quad \leq \int_0^t E\|\delta X_x(s)\|_H^2 |K''(s, X_x(s))|_2^2 ds \\ &\quad \leq \alpha \int_0^t E\|\delta X_x(s)\|_H^2 ds , \end{aligned} \tag{31}$$

where $\alpha = \sup_{0\leq s\leq\tau, x\in B} |K''(s,x)|_2^2 < \infty$.

On the other hand,

$$\begin{aligned} &E\left|\int_0^t \hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds\right|_2^2 \\ &\quad \leq t\int_0^t E|\hat{\sigma}''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)]|_2^2 ds \\ &\quad \leq t\int_0^t E|\sigma''(s, X_x(s))|_2^2 \|\delta X_x(s)\|_H^2 ds \\ &\quad \leq \beta t \int_0^t E\|\delta X_x(s)\|_H^2 ds , \end{aligned} \tag{32}$$

where $\beta = \sup_{0\le s\le\tau, x\in B} |\sigma''(s, X_x(s))|_2^2 < \infty$.

Note that we have used the property that if $S \in L_{(2)}^3(H; R)$ and $T \in L(H; H)$ then $\hat{S}\circ[T\times T] \in L_{(2)}^3(H; R)$ and $|\hat{S}\circ[T\times T]|_2 \le |S|_2 \|T\|_H^2$. This can be seen by observing that $\hat{S}\circ[T\times T] = (S\circ_1 T)\circ_2 T$ and then applying Lemma 1.2. (30), (31) and (32) clearly show that $\int_0^\tau E|\phi(t)|_2^2 dt < \infty$ for each $0\le\tau<\infty$. Hence $\phi \in \mathscr{L}[L_{(2)}^3(H; R)]$.

THEOREM 8. *Assume A and σ satisfy $(A-1)$, $(A-2)$, $(\sigma-1)$ and the following conditions:*

$(A-3)^*$ $K(t,x)$ is C_H^n $(n\ge 2)$ in x variable with $K^{(j)}(t,x) \in L_{(2)}^{j+2}(H; R)$, $j=1,2,\cdots,n$. $K^{(j)}$ is bounded and continuous from $[0,\tau)\times B$ into $L_{(2)}^{j+2}(H; R)$ for each $0\le\tau<\infty$, $j=1,2,\cdots,n$.

$(A-4)^*$ for all t and x, $K^{(j)}(t,x)\in L_{(2)}^{j+2}(H; R)$ is symmetric in the first two components, $j=1,2,\cdots,n$.

$(\sigma-2)^*$ $\sigma(t,x)$ is C_H^n $(n\ge 2)$ in x variable with $\sigma'(t,x)\in L(H; H)$ and $\sigma^{(j)}(t,x)\in L_{(2)}^{j+1}(H; R)$, $j=2,3,\cdots,n$. σ' and $\sigma^{(j)}$ are bounded, continuous from $[0,\tau)\times B$ into $L(H; H)$ and $L_{(2)}^{j+1}(H; R)$, respectively, for each $0\le\tau<\infty$, $j=2,3,\cdots,n$.

Then the diffusion process $X(t)$ given by the solution of the equation

$$X(t) = X(0) + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds$$

is n-th MS-H-differentiable. Furthermore, $\delta X \in \mathscr{L}[L(H; H)]$ and $\delta^j X \in \mathscr{L}[L_{(2)}^{j+1}(H; R)]$, $j=2,3,\cdots,n$.

THEOREM 9. *Suppose A and σ satisfy the conditions $(A-1)$, $(A-2)$, $(A-3)^*$, $(A-4)^*$, $(\sigma-1)$ and $(\sigma-2)^*$. Let $X(t)$ be the diffusion process given by the diffusion coefficients A and σ. If f is a C_H^k-function in B with bounded derivatives, $0\le k\le n$, then the function $\theta(x) = E_x[f(X(t))]$ is also C_H^k. Its first two H-derivatives are*

$$\theta'(x) = E[\delta X_x(t)^*(f'(X_x(t)))]\,, \tag{33}$$

$$\theta''(x) = E\{\delta^2 X_x(t)^\vee(f'(X_x(t))) + f''(X_x(t))\circ[\delta X_x(t)\times\delta X_x(t)]\}\,. \tag{34}$$

Moreover, $\theta''(x)$ is a Hilbert-Schmidt operator of H for all $x\in B$ if f'' is so.

Notation. If $S\in L^n(H; R)$ then $S^\vee \in L(H; L^{n-1}(H; R))$ is defined to be $\check{S}(h) = S(\cdot,\cdot,\cdots,\cdot,h)$. Note that if $S\in L_{(2)}^n(H; R)$ then $\check{S}(h)\in L_{(2)}^{n-1}(H; R)$

and $\check{S}$ is a Hilbert-Schmidt operator from H into $L_{(2)}^{n-1}(H;R)$.

Proof. Let $\psi_h(t) = X_{x+h}(t) - X_x(t)$. Then

$$f(X_{x+h}(t)) - f(X_x(t)) = \int_0^1 \langle f'(X_x(t) + \tau\psi_h(t)), \psi_h(t)\rangle d\tau .$$

Hence

$$\begin{aligned} f(X_{x+h}(t)) - f(X_x(t)) - \langle \delta X_x(t)^*(f'(X_x(t))), h\rangle \\ = \int_0^1 \langle f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t)), \psi_h(t)\rangle d\tau \\ + \langle f'(X_x(t)), \psi_h(t) - \delta X_x(t)h\rangle = \alpha(h) + \beta(h) . \end{aligned}$$

Obviously, $E|\beta(h)| = o(|h|)$ since f' is bounded and $X(t)$ is MS-H-differentiable.

On the other hand,

$$\begin{aligned} E|\alpha(h)| &\le \int_0^1 E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))||\psi_h(t)|\, d\tau \\ &\le \{E|\psi_h(t)|^2\}^{1/2} \int_0^1 \{E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))|^2\}^{1/2} d\tau \\ &\le c|h|\left\{\int_0^1 E|f'(X_x(t) + \tau\psi_h(t)) - f'(X_x(t))|^2\, d\tau\right\}^{1/2} , \end{aligned}$$

where c is a constant independent of h. Apply Lebesgue's dominated convergence theorem to conclude that $E|\alpha(h)| = o(|h|)$. Therefore,

$$E|f(X_{x+h}(t)) - f(X_x(t)) - \langle \delta X_x(t)^*(f'(X_x(t))), h\rangle| = o(|h|) , \qquad h \in H .$$

This proves (33). (34) can be proved in a similar way. Furthermore, $\theta^{(j)}(x)$ $(3 \le j \le k)$ can be expressed by using the first j-th derivatives of f and $X(t)$. Finally $\theta''(x)$ is a Hilbert-Schmidt operator by the remark in Notation above and by the property: if $S \in L_{(2)}^2(H;R)$ and $T \in L(H;H)$ then $S \circ [T \times T] \in L_{(2)}^2(H;R)$. In fact, $S \circ [T \times T] = (S \circ_1 T) \circ_2 T$, hence $|S \circ [T \times T]|_2 \le |S|_2 \|T\|_H^2$ by Lemma 1.2

To finish this paper, we consider the homogeneous case, i.e. A and σ are independent of t. A and σ satisfy $(A-1)$, $(A-2)$ and $(\sigma - 1)$. In this case $X(t)$ generates a semi-group $\{P_t; t \ge 0\}$, $P_t f(x) = E_x[f(X(t))]$. Let C_0 be the Banach space of bounded continuous functions on B vanishing at infinity. C_0 has the sup norm. Assume the B-norm $\|\cdot\|^2$ is C_H^2

such that its second H-derivative has bounded range in $L_1(H; H)$.

THEOREM 10. *The operators P_t, $t \geq 0$, form a strongly continuous contraction semi-group on C_0.*

*Proof**. P_t, $t \geq 0$ are obviously strongly continuous and contractive. We need only to show that $P_t f \in C_0$ whenever $f \in C_0$.

Let $\theta(x) = \log(1 + \|x\|^2)$, $x \in B$. θ is C_H^2 with $|\theta'(x)| \leq C_1\|x\|(1 + \|x\|^2)^{-1}$ and $\|\theta''(x)\|_1 \leq C_2(1 + \|x\|^2)^{-1}$, where C_1 and C_2 are two constants independent of x and $\|\cdot\|_1$ denotes the trace class norm. Apply Ito's formula (Theorem 4.1 [14]) to the function θ and the process $X(t)$,

$$X(t) = x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))ds .$$

$$\theta(X(t)) = \theta(x) + \int_0^t (A^*(X(s))\theta'(X(s)), dW(s))$$
$$+ \int_0^t [(\theta'(X(s)), \sigma(X(s))) + \tfrac{1}{2}\,\mathrm{trace}\, A^*(X(s))\theta''(X(s))A(X(s))]ds .$$

It follows easily that

$$E(\theta(X(t)) - \theta(x))^2 \leq \text{constant} = a ,$$

or

$$E\left[\log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2}\right]^2 \leq a . \tag{35}$$

Now, let $f \in C_0$ and $g = P_t f$. Let $\varepsilon > 0$ be given and N be large enough that

$$|f(x)| < \varepsilon/2 \qquad \text{whenever } \|x\| > N .$$

But

$$g(x) = E_{\{\|X(t)\| > N\}} f(X(t)) + E_{\{\|X(t)\| \leq N\}} f(X(t)) .$$

Hence,

$$|g(x)| \leq \varepsilon/2 + \|f\|_\infty \,\mathrm{prob}\,\{\|X(t)\| \leq N\} ,$$

and

* We learn the proof from Professor K. Ito through a private conversation.

$$
\begin{aligned}
&\operatorname{prob}\{\|X(t)\| \leq N\} \\
&\quad \leq \operatorname{prob}\left\{\log \frac{1+\|X(t)\|^2}{1+\|x\|^2} \leq \log \frac{1+N^2}{1+\|x\|^2}\right\} \\
&\quad = \operatorname{prob}\left\{\log \frac{1+\|X(t)\|^2}{1+\|x\|^2} \leq -\log \frac{1+\|x\|^2}{1+N^2}\right\}, \qquad \|x\| > N, \\
&\quad \leq \operatorname{prob}\left\{\left|\log \frac{1+\|X(t)\|^2}{1+\|x\|^2}\right| \geq \log \frac{1+\|x\|^2}{1+N^2}\right\} \\
&\quad \leq \left(\log \frac{1+\|x\|^2}{1+N^2}\right)^{-2} \times E\left[\log \frac{1+\|X(t)\|^2}{1+\|x\|^2}\right]^2 \\
&\quad \leq a\left(\log \frac{1+\|x\|^2}{1+N^2}\right)^{-2} \leq \frac{\varepsilon}{2\|f\|_\infty} \qquad \text{if } \|x\| \text{ is large}.
\end{aligned}
$$

Thus for large $\|x\|$ we have $|g(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $g \in C_0$.

Appendix

It is a pleasure to thank Professor Loren Pitt for pointing out the fact that P_t, $t \geq 0$, being strongly continuous (Theorem 10) is not obvious. We present a proof as follows.

LEMMA A.1 (Gronwall's inequality). *If h is a non-negative integrable function in $[0, a]$, $a < \infty$, satisfying*

$$h(t) \leq g(t) + \alpha \int_0^t h(s)ds ,$$

where $\alpha > 0$ and g is integrable in $[0, a]$. Then

$$h(t) \leq g(t) + \alpha \int_0^t e^{\alpha(t-s)} g(s)ds .$$

Proof. We can prove inductively that

$$
\begin{aligned}
h(t) \leq g(t) &+ \alpha \int_0^t \left\{\sum_{k=1}^{n-1} [\alpha(t-s)]^k / k!\right\} g(s)ds \\
&+ \alpha \int_0^t h(s)[\alpha(t-s)]^n / n!\, ds .
\end{aligned}
$$

The conclusion then follows from Lebesgue's dominated covergence theorem.

LEMMA A.2. *$E_x \|X(t) - x\|^2 \leq ct(1 + \|x\|^2)$ for all $0 \leq t \leq 1$ and all $x \in B$, where c is a constant independent of t and x.*

Proof. We use the letter c to stand for any constant independent of t and x. Let $0 \leq t \leq 1$.

$$\begin{aligned} X(t) &= x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))ds \\ &= x + CW(t) + \int_0^t K(X(s))dW(s) + \int_0^t \sigma(X(s))ds \,. \end{aligned}$$

Hence

$$\begin{aligned} \|X(t) - x\|^2 &\leq c\left\{\|CW(t)\|^2 + \left\|\int_0^t K(X(s))dW(s)\right\|^2 + \left\|\int_0^t \sigma(X(s))ds\right\|^2\right\} \\ &\leq c\left\{\|CW(t)\|^2 + \left|\int_0^t K(X(s))dW(s)\right|^2 + \int_0^t \|\sigma(X(s))\|^2 ds\right\} \,. \end{aligned} \tag{36}$$

(Recall that $\|\cdot\|$ is dominated by $|\cdot|$).

Thus after taking expectation (36) becomes

$$\begin{aligned} E\|X(t) - x\|^2 &\leq c\left\{t + \int_0^t E|K(X(s))|_2^2 ds + \int_0^t E\|\sigma(X(s))\|^2 ds\right\} \\ &\leq c\left\{t + c\int_0^t E(1 + \|X(s)\|^2)ds + c\int_0^t E(1 + \|X(s)\|^2)ds\right\} \\ &\leq c\left\{t + \int_0^t E\|X(s)\|^2 ds\right\} \\ &\leq c\left\{t + c\int_0^t E[\|X(s) - x\|^2 + \|x\|^2]ds\right\} \\ &\leq c\left\{t(1 + \|x\|^2) + \int_0^t E\|X(s) - x\|^2 ds\right\} \,. \end{aligned}$$

Hence by Lemma A.1 we have

$$\begin{aligned} E\|X(t) - x\|^2 &\leq ct(1 + \|x\|^2) + c\int_0^t e^{c(t-s)} cs(1 + \|x\|^2)ds \\ &\leq ct(1 + \|x\|^2) + c\int_0^t e^{c(t-s)} ct(1 + \|x\|^2)ds \\ &= ct(1 + \|x\|^2)\left[1 + c\int_0^t e^{c(t-s)}ds\right] \\ &\leq ct(1 + \|x\|^2) \,. \end{aligned}$$

Now let $f \in C_0$ be also uniformly continuous. A close examination of the proof of Theorem 10 shows that given $\varepsilon > 0$ there exists N independent of t, $0 \leq t \leq 1$

$$|P_t f(x)| < \varepsilon/2 \qquad \text{whenever } \|x\| > N \,.$$

We may as well assume that

$$|f(x)| < \varepsilon/2 \qquad \text{whenever } \|x\| > N\ .$$

Thus we have for all $0 \leq t \leq 1$

$$(37) \qquad |P_t f(x) - f(x)| < \varepsilon \qquad \text{whenever } \|x\| > N\ .$$

On the other hand, let $\delta > 0$ be such that

$$\|x - y\| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon/2\ .$$

Then for $\|x\| \leq N$,

$$\begin{aligned}|P_t f(x) - f(x)| &\leq E_x |f(X(t)) - f(x)| \\ &= E_{\{\|X(t)-x\|<\delta\}} |f(X(t)) - f(x)| \\ &\quad + E_{\{\|X(t)-x\|\geq\delta\}} |f(X(t)) - f(x)| \\ &\leq \varepsilon/2 + 2\|f\|_\infty \operatorname{prob}\{\|X(t) - x\| \geq \delta\}\ .\end{aligned}$$

But

$$\begin{aligned}\operatorname{prob}\{\|X(t) - x\| \geq \delta\} &\leq \delta^{-2} E\|X(t) - x\|^2 \\ &\leq \delta^{-2} ct(1 + \|x\|^2) \qquad \text{by Lemma A.2} \\ &\leq \delta^{-2} ct(1 + N^2)\ .\end{aligned}$$

Therefore we can choose t_0 small enough such that whenever $t \leq t_0$

$$(38) \qquad |P_t f(x) - f(x)| \leq \varepsilon \qquad \text{for all } \|x\| \leq N\ .$$

Clearly (37) and (38) yield that

$$\|P_t f - f\|_\infty \leq \varepsilon \qquad \text{whenever } t \leq t_0\ .$$

This establishes the strong continuity of P_t, $t \geq 0$.

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