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PROPER MORPHISMS AND EXCELLENT SCHEMES

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Introduction

Let $f: X \to Y$ be a finite type morphism of locally noetherian schemes. It is well known ([3, IV, 7.8.6]) that the excellent property ascends from Y to X. On the other side there are counter-examples where X is excellent and Y is not. First of all it is easy to show that the condition on chains of prime ideals does not descend (see [3, IV, 7.8.4]), even by finite morphisms. Secondly in [2] it is produced an example where X is excellent while Y is not a G-scheme (i.e. it has not the good properties of formal fibers). However in [2] it is also proved that the property concerning the openness of regular loci (the so called "J-2") descends by finite type surjective morphisms. Therefore we are led to the following question: When does the G-scheme property descend? I.e. what conditions do we need on f? A reasonable condition is conjectured (in [2]) as the following: f is proper surjective. The aim of the present paper is precisely to give an answer to such a question. What we really prove is the following. If X is a G-scheme and J-2 (quasi-excellent), then the same is true for Y, provided that f is proper surjective and moreover all the residue fields of Y have characteristic 0. We remark that the result is strongly based on Hironaka's desingularization for quasi-excellent schemes defined over a field of characteristic 0.

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§ 0. Recalls and definitions

All rings are assumed to be commutative noetherian rings with unit and all schemes are assumed to be locally noetherian.

1. We recall that a graded ring is a ring S with a direct decompo-

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sition of the underlying additive group, $S = \oplus_{n=0}^{\infty} S_n$, such that $S_n S_m \subseteq S_{n+m}$ for every $n, m \geqslant 0$.

An element of S_n is called a homogeneous element of degree n.

S_0 is a subring of S and $S_+ = \oplus_{n>0} S_n$ is an ideal of S.

An ideal $\mathfrak{J}$ of S is homogeneous if it has a basis consisting of homogeneous elements.

A homogeneous ideal $\mathfrak{J}$ of S is irrelevant if $\sqrt{\mathfrak{J}} \supseteq S_+$ and otherwise it is relevant.

Since S is noetherian, S is finitely generated as an S_0-algebra.

Convention: Once for all we assume that the graded S_0-algebra $S = \oplus_{n=0}^{\infty} S_n$ is generated over S_0 by $x_0, \cdots, x_n \in S_1$, say $S = S_0[x_0, \cdots, x_n]$.

2. Let $\operatorname{Proj}(S) = \{\mathfrak{P} \in \operatorname{Spec}(S)/\mathfrak{P}$ is a homogeneous relevant ideal$\}$. We can give Proj (S) a structure of scheme. For this construction and for the properties of Proj (S) we refer to [4]. (See also [3, II] where homogeneous prime ideals are defined in a slightly different but equivalent way).

3. The dimension of a scheme X, denoted by $\dim(X)$, is its dimension as a topological space. If $X = \operatorname{Spec}(A)$ for a ring A, then the dimension of X is the same as the Krull dimension of A and we shall write as $\dim(A)$. If $X = \operatorname{Proj}(S)$ then $\dim(X) = d$ means that there exists a chain $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_r$ of relevant homogeneous prime ideals in S, while no such chain of length $r+1$ exists (see [3, II, 2.3.17]).

4. Let X be an integral scheme. We denote by $\phi(X)$ the function field of X. For a ring A we shall write $\phi(A)$ instead of $\phi(\operatorname{Spec}(A))$.

5. A ring A is quasi-excellent (QE for short) iff:

(i) A is a G-ring, i.e. the formal fibers of A are geometrically regular.

(ii) A is J-2; i.e. the regular locus of Spec (A') is Zariski open whenever A' is any A-algebra of finite type.

A ring A is excellent if it is QE and universally catenary (UC for short).

A scheme X is excellent (resp. QE) if there exists a covering of X by open affine subsets $U_i = \operatorname{Spec}(A_i)$ such that A_i is excellent (resp. QE), for each i.

For excellent rings and schemes we refer to [3, IV$_2$] and [6, chap. 13].

6. Let $f: X \to Y$ be a scheme morphism. f is proper if it is separated, of finite type and universally closed. f is projective if it factors into a

closed immersion $i: X \to P_Y^n$ for some n, followed by the projection $P_Y^n \to Y$ (P_Y^n denotes the projective n-space over Y).

EXAMPLE. Let A be a ring, let S be a graded ring with $S_0 = A$, which is finitely generated by the elements of S_1 as an S_0-algebra. Then the natural map $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ is a projective morphism.

7. *Remark.* Let A be a ring. A scheme Y over $\mathrm{Spec}(A)$ is projective if and only if it is isomorphic to $\mathrm{Proj}(S)$ for some graded ring S, where $S_0 = A$, and S is finitely generated by the elements of S_1 as an S_0-algebra ([4, II, 5.18]).

8. Let X, Y be two reduced schemes. A morphism $f: X \to Y$ is birational if for every maximal point $y \in Y$, $f^{-1}\{y\} = \{x\}$ with x maximal point of X and if the homomorphism between the residue fields $k(y) \to k(x)$ deduced by f is a bijection. ([3, IV, 6.15.4]).

If both X and Y are integral schemes, then the generic points of X and Y correspond through f and the fraction fields of X and Y are isomorphic.

9. Let X be a reduced scheme. A scheme Y is a resolution of singularities of X if there is a proper and birational morphism $f: Y \to X$ and Y is regular. If such Y exists, then we say that X is desingularizable.

10. We recall the following results on resolution of singularities due to Hironaka ((a)) and Grothendieck ((b), (c)):

(a) Let X be a reduced noetherian scheme with all the residue fields of characteristic 0. If X is QE then X is desingularizable.

(b) Let X be a locally noetherian scheme. If any integral finite X-scheme is desingularizable, then X is QE.

(c) Let X be a locally noetherian scheme such that all the residue fields of X have characteristic 0. If every closed integral subscheme of X is desingularizable, then X is QE.

For more details see [5], [3, IV, 7.9.5] and also [7, Proposition 3.1., Example 3 and Theorem 3.2 with Remark 1].

§1.

The present section is concerned with some preliminary results on the graded S_0-algebra S and on $\mathrm{Proj}(S)$. Mainly we will see when $\mathrm{Proj}(S)$ and $\mathrm{Spec}(S_0)$ have the same dimension and when ϕ $(\mathrm{Proj}(S))$ is a finite algebraic extension of $\phi(S_0)$.

LEMMA 1.1. *Let S_0 be a domain and $S = S_0[x_0, \cdots, x_n]$ a graded S_0-algebra generated by $x_0, \cdots, x_n \in S_1$. Let f:* $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S_0)$ *be a scheme morphism. Consider the following conditions*:

(a) *For each i $(0 \leqslant i \leqslant n)$ and for each $\mathfrak{P} \in \mathrm{Proj}\,(S)$ with $x_i S \subseteq \mathfrak{P}$, it holds $\mathfrak{P} \cap S_0 \neq (0)$.*

(b) *For each i, S_+ is a minimal prime ideal of $x_i S$.*

(c) *There exists i such that S_+ is a minimal prime ideal of $x_i S$.*

Then we have: (a) $\to$ (b) $\to$ (c).

Proof. (a) $\to$ (b). By (a) it follows that there is an irrelevant ideal $\mathfrak{Q} \in \mathrm{Ass}\,(x_i S)$ such that $\mathfrak{Q} \cap S_0 = (0)$. In fact assume the contrary and consider $\sqrt{x_i S} = \bigcap_{j=1}^{k} \mathfrak{Q}_j$ where $\mathfrak{Q}_j \in \mathrm{Ass}\,(x_i S)$ for $1 \leqslant j \leqslant k$. Then $\mathfrak{Q}_j \cap S_0 \neq (0)$ for each j and $\sqrt{x_i S} \cap S_0 \neq (0)$. But this means that there are $t \in S_0$, $t \neq 0$ and $r \in N$ such that $t^r \in x_i S$, and this is absurd because the degree of t^r is zero if $t \in S_0$ while the elements of $x_i S$ have positive degree.

So there exists an irrelevant minimal prime ideal $\mathfrak{Q}$ of $x_i S$ with $\mathfrak{Q} \cap S_0 = (0)$. But $\mathfrak{Q} \supseteq S_+$ because it is irrelevant and $\mathfrak{Q} \subseteq S_+$ because $\mathfrak{Q} \cap S_0 = (0)$. Therefore $\mathfrak{Q} = S_+$.

(b) $\to$ (c). Obvious.

LEMMA 1.2. *Let S be a graded ring, with S_0 domain, and assume that f:* $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S_0)$ *is a surjective morphism. Then there exists a homogeneous relevant prime ideal $\mathfrak{P}_0$ of S_0 such that the induced morphism f':* $\mathrm{Proj}\,(S/P_0) \to \mathrm{Spec}\,(S_0)$ *is again surjective.*

Proof. Since $(0) \in \mathrm{Spec}\,(S_0)$ and f is surjective, there exists $\mathfrak{P}_0 \in \mathrm{Proj}\,(S)$ such that $\mathfrak{P}_0 \cap S_0 = (0)$. Now consider the following diagram

$$\begin{tikzcd} \mathrm{Proj}\,(S) \arrow[rr, "f"] & & \mathrm{Spec}\,(S_0) \\ & \mathrm{Proj}\,(S/\mathfrak{P}_0) \arrow[ul, dashed, "g"] \arrow[ur, "f'"'] & \end{tikzcd}$$

where g is the closed immersion determined by the surjective homomorphism of graded rings $S \to S/\mathfrak{P}_0$ and $f' = f \circ g$. Then f' is surjective because it is proper, hence closed and $(0) \in \mathrm{Im}\,(f')$.

PROPOSITION 1.3. *Let S_0 be a domain and let $S = S_0\,[x_0, \cdots, x_n]$ be a graded domain generated by $x_0, \cdots, x_n \in S_1$ over S_0. Let f:* $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S_0)$ *be a surjective morphism. Consider the following conditions*:

(d) $\dim (S_0) = \dim (\mathrm{Proj}\,(S))$.

(e) $\phi(\mathrm{Proj}\,(S))$ *is a finite algebraic extension of* $\phi(S_0)$. *Then condition* (c) *of* 1.1 *implies* (d) *and* (d) *implies* (e).

Proof. (c) $\to$ (d). The morphism f: $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S_0)$ is closed and surjective so $\dim(\mathrm{Proj}\,(S)) \geqslant \dim(S_0)$ ([3, IV, 5.4.1 (ii)]). Now we distinguish two cases:

(i) $\dim(S_0) = +\infty$. Then, by the foregoing inequality, $\dim(\mathrm{Proj}\,(S)) = +\infty$, that is (d) holds.

(ii) $\dim(S_0) \lneqq +\infty$. Then it is enough to verify the inequality $\dim(\mathrm{Proj}\,(S)) \leqslant \dim(S_0)$. It is clear that (ii) implies $\dim(S) \lneqq +\infty$.

Let $\mathfrak{Q} = (x_0, \cdots, x_n)$, then $\mathrm{ht}(\mathfrak{Q}) \leqslant 1$ by (c). But $\mathfrak{Q} \neq (0)$ implies $\mathrm{ht}(\mathfrak{Q}) = 1$ by the hypothesis that S is a domain.

Now we prove that $\dim(S) = \dim(S_0) + 1$. We have $\dim(S) - \dim(S/\mathfrak{Q}) \geqslant \mathrm{ht}(\mathfrak{Q})$ ([6, Sec. 12. A]), that is, $\dim(S) - \dim(S_0) \geqslant \mathrm{ht}(\mathfrak{Q}) = 1$. On the other hand, we compute the dimension of the fiber of the natural morphism φ: $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(S_0)$ over the generic point $(0) \in \mathrm{Spec}\,(S_0)$, i.e. $\dim(\phi(S_0)[x_0, \cdots, x_n]) = \dim(\phi(S_0)[x_0, \cdots, x_n]/(x_0, \cdots, x_n)) + \mathrm{ht}\,(x_0, \cdots, x_n) = 1$ ([6, Sec. 14. H]). Since we have $\dim(S) - \dim(S_0) \leqslant \dim(\phi(S_0)[x_0, \cdots, x_n])$ ([4, II, Example 3.22]) we get $\dim(S) \leqslant \dim(S_0) + 1$, hence $\dim(S) = \dim(S_0) + 1$.

On the other hand, since $\mathrm{Proj}\,(S)$ is a topological subspace of $\mathrm{Spec}\,(S)$, it is true that $\dim(\mathrm{Proj}\,(S)) \leqslant \dim(S)$. If we show that $\dim(S) \gneqq \dim(\mathrm{Proj}\,(S))$, then by the foregoing inequality, we may deduce $\dim(\mathrm{Proj}\,(S)) = \dim(S_0)$.

Now, let $\mathfrak{q}_0 \subsetneqq \cdots \subsetneqq \mathfrak{q}_r$ be a maximal chain of homogeneous primes of $\mathrm{Proj}\,(S)$ such that $\dim(\mathrm{Proj}\,(S)) = r$. Consider the ideal $\mathfrak{q}'$ of S generated by $\mathfrak{q}_r$ and $x_0, \cdots, x_n$. Then $\mathfrak{q}'$ is proper and different from $\mathfrak{q}_r$, because otherwise $x_0 = \cdots = x_n = 0$, but in this case $\mathrm{Proj}\,(S) = \emptyset$ and so $\dim(\mathrm{Proj}\,(S)) \leqslant \dim(S_0)$. Let $\mathfrak{P}$ be a minimal prime ideal of $\mathfrak{q}'$. Then $\mathfrak{q}_0 \subsetneqq \cdots \subsetneqq \mathfrak{q}_r \subsetneqq \mathfrak{P}$ is a chain of $\mathrm{Spec}\,(S)$, that is $\dim(S) \geqslant r + 1 > \dim(\mathrm{Proj}\,(S))$.

(d) $\to$ (e). Since f: $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S_0)$ is a surjective morphism of integral schemes of the same dimension by (d), the fiber over the generic point of $\mathrm{Spec}\,(S_0)$ has dimension 0 and hence it is finite. By [4, II, Example 3.7], it follows that $\phi(\mathrm{Proj}\,(S))$ is a finite field extension of $\phi(S_0)$.

PROPOSITION 1.4. *Let* S_0 *be a domain and let* $S = S_0[x_0, \cdots, x_n]$ *be a graded domain generated by* $x_0, \cdots, x_n \in S_1$ *over* S_0. *Let* f: $\mathrm{Proj}\,(S) \to$

$\mathrm{Spec}\,(S_0)$ *be a surjective morphism. If the condition* (e) *of* 1.3 *holds then there exist a finite extension* S'_0 *of* S_0 *and a proper birational morphism* g: $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S'_0)$ *such that the following diagram*

$$\begin{tikzcd} \mathrm{Proj}\,(S) \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathrm{Spec}\,(S_0) \\ & \mathrm{Spec}\,(S'_0) \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative.

Proof. Observe that, if we define a finite extension S'_0 of S_0 such that there exists a morphism g: $\mathrm{Proj}\,(S) \to \mathrm{Spec}\,(S'_0)$ which makes the diagram commutative, then we deduce that g is proper. In fact, since $f\ (=h \circ g)$ and h are proper (in addition h is finite), g is also such ([4, II, 4.8. (e)]).

Now we have to define S'_0 such that g is birational too. Consider the integral closure X'' of $\mathrm{Spec}\,(S_0)$ in $\mathrm{Proj}\,(S)$ ([3, II, 6.3]). Then X'' is an affine scheme $\mathrm{Spec}\,(S''_0)$, because the morphism $h'\colon X'' \to \mathrm{Spec}\,(S_0)$ is integral. Moreover there is a natural morphism g': $\mathrm{Proj}\,(S) \to X''$. S'_0 is a suitable subring of S''_0. In fact, let $L = \phi(\mathrm{Proj}\,(S))$ and $K = \phi(S_0)$, then by (e) it follows that $L = K[t_1, \cdots, t_m]$, where t_i is algebraic over K for $i = 1, \cdots, m$. Let $f_i(X)$ be the minimal polynomial of t_i over K ($1 \leqslant i \leqslant m$), then it holds $f_i(t_i) = t_i^{s_i} + (a_{i1}/b_{i1})t_i^{s_i-1} + \cdots + (a_{is_i}/b_{is_i}) = 0$ where a_{ij}, $b_{ij} \in S_0$ for $1 \leqslant j \leqslant s_i$. Multiplying this equation by $(b_{i1} \cdots b_{is_i})^{s_i} = b_i^{s_i}$, it becomes an equation of integral dependence for $b_i t_i$ over S_0. Put $S'_0 = S_0[b_1 t_1, \cdots, b_m t_m]$. Then S'_0 is finite as an S_0-module and clearly $\phi(S'_0) = \phi(\mathrm{Proj}\,(S))$. Moreover there is a morphism g'': $\mathrm{Spec}\,(S''_0) \to \mathrm{Spec}\,(S'_0)$. If we put $g = g'' \circ g'$, then g is a proper and birational morphism.

§ 2.

Here we prove our main theorem on descent of excellent property by proper surjective morphisms.

THEOREM 2.1. *Let* Y *be a locally noetherian scheme defined over a field of characteristic* 0. *Suppose* $f\colon X \to Y$ *is a proper surjective scheme morphism, then* X *is QE if and only if* Y *is QE.*

Proof. The "if" part is clear by definition (see [6 chap. 13]). Conversely, in our hypothesis we may apply 2.3 of [2] and we deduce that Y is J-2. So it is enough to show that Y is a G-scheme.

We verify that we may assume:

1) Y is affine, say $Y = \operatorname{Spec}(S_0)$.

2) S_0 in 1) is a domain.

3) S_0 is local.

1) In fact, if $\{V_i\}$ is an open affine covering of Y and $U_i = f^{-1}(V_i)$, then $f_{|U_i}\colon U_i \to V_i$ is proper ([4, II, 4.8]) and surjective (since f is surjective it follows $f(f^{-1}(U_i)) = V_i$). V_i satisfies the hypotheses and U_i is QE for any i. Hence it suffices to prove that V_i is QE, but this means that we may assume $Y = \operatorname{Spec}(S_0)$.

2) It is known that $Y = \operatorname{Spec}(S_0)$ is a G-scheme if and only if $\operatorname{Spec}(S_0/\mathfrak{P})$ is a G-scheme for every $\mathfrak{P} \in \operatorname{Spec}(S_0)$ with $\operatorname{ht}(\mathfrak{P}) = 0$.

Let $\mathfrak{P} \in \operatorname{Spec}(S_0)$ with $\operatorname{ht}(\mathfrak{P}) = 0$. For proper and surjective morphism $f\colon X \to \operatorname{Spec}(S_0)$, let f' be the morphism obtained from f by the base extension $h\colon \operatorname{Spec}(S_0/\mathfrak{P}) \to \operatorname{Spec}(S_0)$, where h is finite. Now consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \operatorname{Spec}(S_0) \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & \operatorname{Spec}(S_0/\mathfrak{P}) \end{array}$$

where $X' = X \otimes_{\operatorname{Spec}(S_0)} \operatorname{Spec}(S_0/\mathfrak{P})$. Then f' is proper and surjective (such properties are stable under base extension by [3, II, 5.4.2 and I, 3.5.2]) and X' is QE because h', obtained from h by the base extension f, is finite and X is QE by hypothesis. Hence it follows that we may assume S_0 is a domain.

3) Proceeding similarly to point 2)—that is using the fact that our properties are stable under base extension—we show that S_0 may be taken local.

It is known that S_0 is a G-ring if and only if $(S_0)_\mathfrak{m}$ is a G-ring for every $\mathfrak{m} \in \operatorname{Max}(S_0)$. For $f\colon X \to Y = \operatorname{Spec}(S_0)$, let f' be the morphism obtained from f by the base extension $h\colon \operatorname{Spec}((S_0)_\mathfrak{m}) \to \operatorname{Spec}(S_0)$, where h is a morphism essentially of finite type. We have

$$\begin{array}{ccc} X & \xrightarrow{f} & \operatorname{Spec}(S_0) \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & \operatorname{Spec}((S_0)_\mathfrak{m}) \end{array}$$

where $X' = X \otimes_{\operatorname{Spec}(S_0)} \operatorname{Spec}((S_0)_\mathfrak{m})$. Then f' is proper and surjective and X' is QE because h' is essentially of finite type. So we may assume that S_0 is local.

Summarizing, we have a proper and surjective morphism $f\colon X \to Y$ where X is QE and $Y = \operatorname{Spec}(S_0)$ with S_0 a local domain. In this case we may apply 5.6.2 of [3, II] and we find a projective scheme X' over $\operatorname{Spec}(S_0)$ and a morphism $g\colon X' \to X$ projective and surjective. The scheme X' is isomorphic to $\operatorname{Proj}(S)$ for some graded ring S (Remark 7). Then $\operatorname{Proj}(S)$ is QE because this property ascends by g, and $h = f \circ g\colon X' \to Y$ is surjective because it is the composition of two surjective morphisms.

By 1.2 we may replace S by $S/\mathfrak{P}_0$ for a suitable $\mathfrak{P}_0 \in \operatorname{Proj}(S)$ and assume that S is a domain. Now it is enough to show the theorem with $X = \operatorname{Proj}(S)$ and $Y = \operatorname{Spec}(S_0)$, where S_0 is a local domain and $S = S_0[x_0, \cdots, x_n]$ is a domain.

We proceed by double induction with respect to (n, d) where n is the number of minimal generators of S over S_0 and $d = \dim(S_0)$. Note that assuming S_0 local it holds $\dim(S_0) \lneqq +\infty$ and so the proof by induction covers all the cases.

For $(0, d)$ it holds $S = S_0[x_0]$. If we prove that x_0 is transcendental over S_0, then we see that S is isomorphic to $S_0[X_0]$ with X_0 indeterminate and $\operatorname{Spec}(S_0)$ is isomorphic to $\operatorname{Proj}(S)$.

We show by absurdity that x_0 is transcendental over S_0. So suppose that we have an equation $a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_mx_0^m = 0$ of algebraic dependence for x_0 of minimal degree m with $a_i \in S_0$. Then $a_0 \in S_0 \cap (x_0) = (0)$, i.e. $a_0 = 0$. But this means that x_0 is a zero-divisor, and this is impossible because S is a domain.

For $(n, 0)$, it follows immediately that S_0 is QE. In fact it is a field.

Assuming that the theorem is true for $(n-1, d)$ and $(n, d-1)$, we prove it for (n, d). We distinguish two cases:

Case 1. There exist i $(0 \leqslant i \leqslant n)$ and $\mathfrak{P} \in \operatorname{Proj}(S)$ with $x_iS \subseteq \mathfrak{P}$ such that $\mathfrak{P} \cap S_0 = (0)$. Take such a $\mathfrak{P} \in \operatorname{Proj}(S)$ and consider the quotient $S/\mathfrak{P}$. The surjective homomorphism of graded rings $S \to S/\mathfrak{P}$ gives rise to a closed immersion $g\colon \operatorname{Proj}(S/\mathfrak{P}) \to \operatorname{Proj}(S)$ which, in particular, is of finite type, hence the QE property ascends to $\operatorname{Proj}(S/\mathfrak{P}))$ from $\operatorname{Proj}(S)$. Consider the following commutative diagram

$$\begin{tikzcd} \operatorname{Proj}(S) \arrow[rr, "f"] & & \operatorname{Spec}(S_0) \\ & \operatorname{Proj}(S/\mathfrak{P}) \arrow[ul, "g"] \arrow[ur, "h"'] & \end{tikzcd}$$

Obviously h is projective ([4, II, Example 4.8.1]). Moreover, in our case, the prime ideal (0) of S_0 belongs to $\operatorname{Im}(h)$ and, since h is closed, we have $(\bar{0}) \subseteq \operatorname{Im}(h)$, that is h is surjective.

Applying now the inductive hypothesis we get that S_0 is QE. (In fact $x_i \in \mathfrak{P}$ hence the number of generators of $S/\mathfrak{P}$ over S_0 is strictly less than n.)

Case 2. For each i $(0 \leqslant i \leqslant n)$ and for each $\mathfrak{P} \in \operatorname{Proj}(S)$ with $x_i S \subseteq \mathfrak{P}$, it holds that $\mathfrak{P} \cap S_0 \neq (0)$. In that case condition (a) of 1.1 holds. Then applying 1.1, 1.3, and 1.4, we have a ring S_0' finite over S_0 and a proper birational morphism $g\colon \operatorname{Proj}(S) \to \operatorname{Spec}(S_0')$. Moreover the morphism h: $\operatorname{Spec}(S_0') \to \operatorname{Spec}(S_0)$ defined in 1.4 is finite and surjective. Hence by [2, 1.3], it suffices to verify that S_0' is a G-ring. We recall that S_0 and S_0' have the residue fields of characteristic 0. So, in order to see that S_0' is a G-ring, it is sufficient to verify that every closed integral subscheme of $\operatorname{Spec}(S_0')$ is desingularizable. (See 0.10 (c)).

First prove that S_0' is desingularizable. In fact $\operatorname{Proj}(S)$ satisfies the hypothesis of Hironaka's theorem (0.10 (a)) and it is desingularizable. Let Z be a resolution of $\operatorname{Proj}(S)$, then, through the morphism $g\colon \operatorname{Proj}(S) \to \operatorname{Spec}(S_0')$, Z resolves also $\operatorname{Spec}(S_0')$.

Now we see that every integral quotient $S_0'/\mathfrak{P}$ is desingularizable. For $\mathfrak{P} \in \operatorname{Spec}(S_0')$, put $\mathfrak{p} = \mathfrak{P} \cap S_0$. Then $\mathfrak{P} \neq (0)$ implies $\mathfrak{p} = \mathfrak{P} \cap S_0 \neq (0)$ because S_0' is integral over S_0. For the morphism $f\colon \operatorname{Proj}(S) \to \operatorname{Spec}(S_0)$ take the base extension $\varphi\colon \operatorname{Spec}(S_0/\mathfrak{p}) \to \operatorname{Spec}(S_0)$ and consider the following diagram

$$\begin{tikzcd}
\operatorname{Proj}(S) \arrow[r, "g"] \arrow[rr, bend left, "f"] & \operatorname{Spec}(S_0') \arrow[r, "h"] & \operatorname{Spec}(S_0) \\
\operatorname{Proj}(S \otimes_{S_0} S_0/\mathfrak{p}) \arrow[u] \arrow[r, "g'"] \arrow[rr, bend right, "f'"'] & \operatorname{Spec}(S_0'/\mathfrak{p}S_0') \arrow[u] \arrow[r, "h'"] & \operatorname{Spec}(S_0/\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

where the morphism f' and h' are obtained by φ respectively from f and h. Then f' is a surjective and projective morphism by [3, II, 5.6.5. and I, 3.5.2.] and $\operatorname{Proj}(S \otimes_{S_0} S_0/\mathfrak{p})$ is clearly QE. Since $\mathfrak{p} \neq (0)$, we have $\dim(S_0/\mathfrak{p}) \lneqq \dim(S_0)$ and applying the inductive hypothesis, we deduce that $S_0/\mathfrak{p}$ is QE. Moreover since h' is finite ([3, II, 6.1.5]) the QE property passes from $S_0/\mathfrak{p}$ to $S_0'/\mathfrak{p}S_0'$ and from $S_0'/\mathfrak{p}S_0'$ to the quotient $S_0'/\mathfrak{P}$ $(\mathfrak{P} \supseteq \mathfrak{p}S_0')$. Therefore, by Hironaka's theorem, $S_0'/\mathfrak{P}$ is desingularizable. This concludes our proof: We have seen that S_0' is a G-ring, so also S_0 is a G-ring, hence QE.

Remark 2.2. We need in our proof of the fact that Proj (S) is desingularizable. Therefore we use both the G-scheme and the J-2 properties. We are not able to make the G-scheme property descend separately.

Remark 2.3. The UC property does not descend by proper surjective morphisms. See [3, IV, 7.8.4].

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