ON THE EXISTENCE OF CONDENSER POTENTIALS

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0. Introduction

The existence of condenser potentials was established in the framework of Dirichlet spaces by Beurling and Deny, cf. Deny [5] or Landkof [10], simply by choosing the potential of minimal energy within a certain convex set. This same idea works for non-symmetric Dirichlet spaces, cf. Bliedtner [3].

Let $(\mu_t)_{t>0}$ be a transient convolution semigroup on a locally compact abelian group G and let $\kappa = \int_0^\infty \mu_t dt$ be the potential kernel. The associated negative definite function on $\hat{G}$ is denoted ψ. For an explicit formulation of these concepts see [2]. We may now state that κ satisfies the condenser principle in the special case where κ is associated with an ordinary Dirichlet space, corresponding to ψ being realvalued, cf. [7], or more generally in the case where κ is associated with a non-symmetric Dirichlet space, corresponding to ψ satisfying the inequality $|\operatorname{Im} \psi| \leq A \operatorname{Re} \psi$, cf. [1].

The purpose of this note is to show that every potential kernel κ satisfies the condenser principle.

The condenser potential is constructed as sum of an alternating infinite series. As an application it is proved that the condenser measures are concentrated on the boundaries if and only if $(\mu_t)_{t>0}$ consists of probability measures and is of local type. Results of this type was obtained by Itô [8] for Dirichlet spaces.

A similar approach to condenser potentials but in the context of function kernels has been given by Kishi [9].

In section 3 we finally give an extension of the condenser principle to arbitrary Hunt convolution kernels.

1. The condenser principle

Let G be a locally compact abelian group with a fixed Haar measure ω_G. In this and the next section $(\mu_t)_{t>0}$ denotes a transient convolution semigroup and $\kappa = \int_0^\infty \mu_t \, dt$ is the potential kernel. By $D^+(\kappa)$ we note the set of positive measures μ on G for which $\kappa * \mu$ exists.

For a compact neighbourhood V of 0 we always denote by ε_{CV} the canonical κ-balayaged measure of ε_0 on CV. We put $\eta_V = a_V \kappa * (\varepsilon_0 - \varepsilon_{CV})$ where a_V is the uniquely determined number such that $\eta_V(G) = 1$. Here and in the following we mainly use the terminology from [2].

The existence of condenser potentials is formulated in the following theorem.

THEOREM 1. *Condenser principle.*

Let (Ω_0, Ω_1) be a pair of open subsets of G such that $\bar{\Omega}_0$ is compact and disjoint from $\bar{\Omega}_1$. Then there exist positive measures $\mu_0, \mu_1 \in D^+(\kappa)$ such that $\xi = \kappa * (\mu_0 - \mu_1)$ satisfies

  (i) $0 \leq \xi \leq \omega_G$,

  (ii) $\xi = \omega_G$ *in* Ω_0,

  (iii) $\xi = 0$ *in* Ω_1,

  (iv) supp $(\mu_0) \subseteq \bar{\Omega}_0$, supp $(\mu_1) \subseteq \bar{\Omega}_1$.

Proof. Let λ_0 be the canonical equilibrium distribution for Ω_0. Then $0 \leq \kappa * \lambda_0 \leq \omega_G$ and $\kappa * \lambda_0 = \omega_G$ in Ω_0 (cf. [2] p. 156). We define a sequence $(\lambda_n)_{n \geq 0}$ of measures inductively: λ_{2n+1} is the canonical balayaged measure of λ_{2n} on Ω_1, and λ_{2n+2} is the canonical balayaged measure of λ_{2n+1} on Ω_0, $n \geq 0$. We then have for $n \geq 0$

(1) $\kappa * \lambda_{2n+1} \leq \kappa * \lambda_{2n}$ with equality in Ω_1,

(2) $\kappa * \lambda_{2n+2} \leq \kappa * \lambda_{2n+1}$ with equality in Ω_0,

(3) supp $(\lambda_{2n}) \subseteq \bar{\Omega}_0$, supp $(\lambda_{2n+1}) \subseteq \bar{\Omega}_1$.

The sequence $(\kappa * \lambda_n)$ is decreasing and converges accordingly vaguely to a potential $\kappa * \tau$, and τ is the vague limit of (λ_n) (cf. [2], Th. 16.10). It follows by (3) that τ must be supported by $\bar{\Omega}_0 \cap \bar{\Omega}_1$ and hence zero. Having shown that $(\kappa * \lambda_n)$ decreases to zero, we get that the infinite alternating series

(4) $$\sum_{n=0}^\infty (-1)^n \kappa * \lambda_n$$

is vaguely convergent.

The n'th partial sum and the sum of (4) is denoted respectively S_n and ξ. By (1) we get

$$S_{2n} = \sum_{k=0}^{n-1} (\kappa * \lambda_{2k} - \kappa * \lambda_{2k+1}) \geq 0$$

and $S_{2n} = 0$ in Ω_1, and by (2) we get

$$S_{2n+1} = \kappa * \lambda_0 - \sum_{k=1}^{n} (\kappa * \lambda_{2k-1} - \kappa * \lambda_{2k}) \leq \kappa * \lambda_0 \leq \omega_G$$

and

$$S_{2n+1} = \kappa * \lambda_0 = \omega_G \qquad \text{in } \Omega_0 \, .$$

This shows that $0 \leq \xi \leq \kappa * \lambda_0 \leq \omega_G$ and that $\xi = \omega_G$ in Ω_0, $\xi = 0$ in Ω_1.

We finally have to show that ξ has the form $\kappa * (\mu_0 - \mu_1)$ for measures μ_0 and μ_1 satisfying (iv). In order to obtain this we actually show that the series (4) is absolutely convergent.

We first prove the following formula

$$(5) \qquad \sum_{n=0}^{\infty} (-1)^n \varepsilon_{CV} * \kappa * \lambda_n = \varepsilon_{CV} * \xi \qquad \text{(vaguely)} \, .$$

Since the series (4) is alternating we have for $n \geq 0$

$$0 \leq \xi - S_{2n} \leq \kappa * \lambda_{2n} \, , \qquad 0 \leq S_{2n+1} - \xi \leq \kappa * \lambda_{2n+1} \, ,$$

and hence

$$0 \leq \varepsilon_{CV} * \xi - \varepsilon_{CV} * S_{2n} \leq \varepsilon_{CV} * \kappa * \lambda_{2n} \leq \kappa * \lambda_{2n}$$

and

$$0 \leq \varepsilon_{CV}^* * S_{2n+1} - \varepsilon_{CV} * \xi \leq \varepsilon_{CV} * \kappa * \lambda_{2n+1} \leq \kappa * \lambda_{2n+1} \, ,$$

which show that $(\varepsilon_{CV} * S_n)$ converges vaguely to $\varepsilon_{CV} * \xi$, i.e. that (5) holds.

For every compact neighbourhood V of 0 we finally have

$$(6) \qquad \sum_{n=0}^{\infty} (-1)^n \kappa * (\varepsilon_0 - \varepsilon_{CV}) * \lambda_n = \xi - \varepsilon_{CV} * \xi \, .$$

Let now V be so small that $\overline{\Omega}_0 + V$ is disjoint from $\overline{\Omega}_1 + V$ and let $\phi \in C_c^+(G)$ be chosen so that $\phi = 1$ on $\overline{\Omega}_0 + V$ and $\phi = 0$ on $\overline{\Omega}_1 + V$. By (3) we have

$$\langle \kappa * (\varepsilon_0 - \varepsilon_{CV}) * \lambda_{2n+1}, \phi \rangle = 0 \, ,$$

and

$$\langle \kappa * (\varepsilon_0 - \varepsilon_{CV}) * \lambda_{2n}, \phi \rangle = \frac{1}{a_V} \lambda_{2n}(G) \ ,$$

so by (6)

$$\frac{1}{a_V} \sum_{n=0}^{\infty} \lambda_{2n}(G) = \langle \xi - \varepsilon_{CV} * \xi, \phi \rangle < \infty \ .$$

This implies that the series $\sum_{n=0}^{\infty} \lambda_{2n}$ is vaguely convergent with sum μ_0, which is a positive measure supported by the compact set $\overline{\Omega}_0$. We clearly have

$$\kappa * \mu_0 = \sum_{n=0}^{\infty} \kappa * \lambda_{2n} \ ,$$

so by (4) the sequence

$$\kappa * \left(\sum_{k=0}^{n} \lambda_{2k+1} \right)$$

increases to $\kappa * \mu_0 - \xi$. By Lemma 15.7 of [2] follows that the series $\sum_{k=0}^{\infty} \lambda_{2k+1}$ is vaguely convergent, and the sum μ_1 is a positive measure supported by $\overline{\Omega}_1$. By Lemma 15.3 of [2] we finally get that $\mu_1 \in D^+(\kappa)$ and $\kappa * \mu_1 = \kappa * \mu_0 - \xi$. □

Remarks. 1) It follows from the above proof that μ_0, μ_1 and ξ are canonically constructed and that

$$\xi = \sum_{n=0}^{\infty} (-1)^n \kappa * \lambda_n \ ,$$

where λ_0 is the canonical equilibrium distribution for Ω_0, λ_{2n+1} is the canonical balayaged measure of λ_{2n} on Ω_1 and λ_{2n+2} is the canonical balayaged measure of λ_{2n+1} on Ω_0. Furthermore

$$\mu_0 = \sum_{n=0}^{\infty} \lambda_{2n} \ , \qquad \mu_1 = \sum_{n=0}^{\infty} \lambda_{2n+1} \ .$$

The measure μ_0 (resp. μ_1) is called the *inner* (resp. *outer*) *canonical condenser distribution*, and $\xi = \kappa * (\mu_0 - \mu_1)$ is called the *canonical condenser potential* for the pair (Ω_0, Ω_1).

2) The canonical condenser potential for the pair (Ω_0, Ω_1) is dominated by the canonical equilibrium potential for Ω_0, i.e. $\xi \leq \kappa * \lambda_0$. If

$\Omega_1 = \phi$ they are equal.

3) Since κ satisfies the principle of positivity of mass (cf. [2] Th. 16.25) we have $\mu_1(G) \leq \mu_0(G)$.

Under certain further assumptions we have $\mu_0(G) = \mu_1(G)$, see Corollary 4 below.

2. Applications

We recall that $(\mu_t)_{t>0}$ is called of local type if its infinitesimal generator on $C_0(G)$ is a local operator. This is equivalent with κ satisfying the principle of local unicity of mass: If $\kappa*\sigma_1 = \kappa*\sigma_2$ in an open set U then $\sigma_1 = \sigma_2$ in U. It is also equivalent with supp $(\varepsilon_{CV}) \subseteq \partial V$ for all (or sufficiently small) neighbourhoods V of 0. For these and other equivalent properties see [2] § 18.

We can formulate an equivalent condition in terms of the condenser distributions.

THEOREM 2. *Let κ be a potential kernel for a transient convolution semigroup $(\mu_t)_{t>0}$. Then $(\mu_t)_{t>0}$ is of local type if and only if the following holds:*

For any pair of open sets (Ω_0, Ω_1) for which $\bar{\Omega}_0$ is compact and disjoint from $\bar{\Omega}_1$ the canonical outer condenser distribution μ_1 is supported by $\partial\Omega_1$.

Proof. Suppose first that $(\mu_t)_{t>0}$ is of local type. Since $\kappa*\mu_0 = \kappa*\mu_1$ in Ω_1 we get by the principle of local unicity of mass that $\mu_0 = \mu_1$ in Ω_1, and since supp $(\mu_0) \cap \bar{\Omega}_1 = \phi$ we have $\mu_1(\Omega_1) = \mu_0(\Omega_1) = 0$, hence supp (μ_1) $\subseteq \partial\Omega_1$.

Conversely, suppose that the outer condenser distribution μ_1 always lies on the boundary of Ω_1. The measure $\check{\kappa}$ is potential kernel for the reflected semigroup $(\check{\mu}_t)_{t>0}$, and it suffices to prove that $(\check{\mu}_t)_{t>0}$ is of local type. Let V be a compact neighbourhood of 0 in G and let δ_{CV} denote the canonical $\check{\kappa}$-balayaged measure of ε_0 on CV. Then $\check{\kappa}*\delta_{CV} \leq \check{\kappa}$ with equality in CV. It suffices to prove that $\delta_{CV}(CV) = 0$. Let Ω_0 be an open set such that $\bar{\Omega}_0$ is compact and contained in CV. We choose an open set $\Omega_1 \supseteq V$ such that $\bar{\Omega}_1 \cap \bar{\Omega}_0 = \phi$ and let $\xi = \kappa*(\mu_0 - \mu_1)$ be the canonical condenser potential for the pair (Ω_0, Ω_1).

For an approximate unit $(\phi_i)_{i \in I}$ from $C_c^+(G)$ for which the support of ϕ_i is sufficiently close to 0 for all $i \in I$ we have $(\mu_0 - \mu_1)*\phi_i = 0$ on V,

because supp $(\mu_0) \subseteq \bar{\Omega}_0$ and supp $(\mu_1) \subseteq \partial\Omega_1$. Therefore we have

$$\langle \delta_{CV}, \xi*\phi_i \rangle = \langle \check{\kappa}*\delta_{CV}, (\mu_0 - \mu_1)*\phi_i \rangle = \langle \check{\kappa}, (\mu_0 - \mu_1)*\phi_i \rangle = \xi*\phi_i(0) .$$

Given a compact subset $A \subseteq \Omega_0$ we have $\xi*\phi_i = 1$ on A and $\xi*\phi_i(0) = 0$ if supp (ϕ_i) is sufficiently close to 0, so we have $\delta_{CV}(A) = 0$. By the arbitrariness of A we get $\delta_{CV}(\Omega_0) = 0$ and by the arbitrariness of Ω_0 we get $\delta_{CV}(CV) = 0$. □

It is also possible to give necessary and sufficient conditions that both inner and outer condenser distributions lie on the boundary.

We shall use below that $\mu_t(G) = 1$ for one (or all) $t > 0$ if and only if $\varepsilon_{CV}(G) = 1$ for one (or all) V. (Cf. e.g. [4] Th. 5).

It will be convenient with the following Proposition.

PROPOSITION 3. *Let κ be a potential kernel for a transient convolution semigroup $(\mu_t)_{t>0}$ of probability measures. If $\kappa*\mu = \kappa*\nu$ in the complement of some compact set, then $\mu(G) = \nu(G)$.*

Proof. For an open relatively compact set Ω we denote by λ_Ω the canonical equilibrium distribution for Ω. Then $\kappa*\lambda_\Omega$ tends vaguely to ω_G as Ω increases to G. Since $\kappa*\lambda_\Omega \leq \omega_G$ we get that $\varepsilon_{CV}*\kappa*\lambda_\Omega$ converges vaguely to $\varepsilon_{CV}*\omega_G = \omega_G$ and hence that $\eta_V*\lambda_\Omega$ converges vaguely to 0 as Ω increases to G. This implies that λ_Ω converges vaguely to 0.

Let $\phi \in C_c^+(G)$ be symmetric of integral 1. Then there exists a function $f \in C_c^+(G)$ such that $\kappa*\mu*\phi \leq \kappa*\nu*\phi + f$ and hence

$$\mu(G) = \lim_{\Omega\uparrow G} \kappa*\lambda_\Omega*\mu*\phi(0) \leq \lim_{\Omega\uparrow G} \kappa*\lambda_\Omega*\nu*\phi(0)$$
$$+ \lim_{\Omega\uparrow G} \lambda_\Omega*f(0) = \nu(G) .$$

Similarly we get $\nu(G) \leq \mu(G)$. □

COROLLARY 4. *Let κ be a potential kernel for a transient convolution semigroup of probability measures.*

(i) *For every positive measure $\mu \in D^+(\kappa)$ and every open set Ω for which $C\Omega$ is compact we have $\mu^\Omega(G) = \mu(G)$, where μ^Ω is the canonical balayaged measure of μ on Ω.*

(ii) *For any pair (Ω_0, Ω_1) of open sets for which $\bar{\Omega}_0$ is compact and Ω_1 is the complement of a compact neighbourhood of $\bar{\Omega}_0$, the condenser measures μ_0 and μ_1 for (Ω_0, Ω_1) have same total mass: $\mu_0(G) = \mu_1(G)$.*

Proof. (i) follows since $\kappa * \mu = \kappa * \mu^{\varrho}$ in Ω. (ii). With the terminology of the proof of Theorem 1 we have by (i) that $\lambda_{2n+1}(G) = \lambda_{2n}(G)$ for all $n \geq 0$ and hence $\mu_0(G) = \mu_1(G)$. □

THEOREM 5. *Let κ be a potential kernel for a transient convolution semigroup $(\mu_t)_{t>0}$. Then $(\mu_t)_{t>0}$ is of local type and $\mu_t(G) = 1$ for all $t > 0$ if and only if the following holds:*

For any pair of open sets (Ω_0, Ω_1) for which $\overline{\Omega}_0$ is compact and disjoint from $\overline{\Omega}_1$ the condenser distributions μ_0 and μ_1 are supported by $\partial \Omega_0$ and $\partial \Omega_1$ respectively.

Theorem 5 follows from the following two propositions together with Theorem 2.

PROPOSITION 6. *Let κ be the potential kernel for a transient convolution semigroup $(\mu_t)_{t>0}$ of local type and such that $\mu_t(G) = 1$ for all $t > 0$.*

If $\kappa * \mu = \kappa * \nu + \omega_G$ in an open set Ω then $\mu = \nu$ in Ω.

Proof. Let $f \in C_c^+(\Omega)$ and choose a compact neighbourhood V_0 of 0 such that $\operatorname{supp}(f) - V_0 \subseteq \Omega$. For $V \subseteq V_0$ we then have $\check{\varepsilon}_{CV} * f \in C_c^+(\Omega)$ because $\operatorname{supp}(\varepsilon_{CV}) \subseteq \partial V$, and therefore

$$\langle \kappa * \mu, f * (\varepsilon_0 - \check{\varepsilon}_{CV}) \rangle = \langle \kappa * \nu, f * (\varepsilon_0 - \check{\varepsilon}_{CV}) \rangle + \langle \omega_G, f * (\varepsilon_0 - \check{\varepsilon}_{CV}) \rangle,$$

but the last term vanishes because $\varepsilon_{CV}(G) = 1$. We then have

$$\langle \eta_V * \mu, f \rangle = \langle \eta_V * \nu, f \rangle$$

and for $V \to 0$ we get $\langle \mu, f \rangle = \langle \nu, f \rangle$, hence $\mu = \nu$ in Ω. □

PROPOSITION 7. *Let κ be a potential kernel for a transient convolution semigroup $(\mu_t)_{t>0}$.*

If the canonical equilibrium distribution λ_0 for an arbitrary open relatively compact set Ω_0 is supported by $\partial \Omega_0$, then $\mu_t(G) = 1$ for all $t > 0$.

Proof. It suffices to prove that $\varepsilon_{CV}(G) = 1$ for all sufficiently small compact neighbourhoods V of 0. Let Ω_0 be a non-empty open relatively compact set and let $f \in C_c^+(\Omega_0)$ be such that $\langle \omega_G, f \rangle = 1$. Denoting by λ_0 the canonical equilibrium distribution for Ω_0, we have $\operatorname{supp}(\lambda_0) \subseteq \partial \Omega_0$.

For V so small that $\operatorname{supp}(f) - V \subseteq \Omega_0$ we therefore have

$$0 = \langle \lambda_0, \check{\kappa} * (\varepsilon_0 - \check{\varepsilon}_{CV}) * f \rangle = \langle \kappa * \lambda_0, f - \check{\varepsilon}_{CV} * f \rangle,$$

and hence

$$1 = \langle \kappa*\lambda_0, f \rangle = \langle \kappa*\lambda_0, \check{\varepsilon}_{CV}*f \rangle \le \langle \omega_G, \check{\varepsilon}_{CV}*f \rangle = \varepsilon_{CV}(G) \ .$$

On the other hand we always have $\varepsilon_{CV}(G) \le 1$. □

3. Extension to Hunt convolution kernels

In this section we drop the assumption that $\mu_t(G) \le 1$ for all $t > 0$ and suppose $\kappa = \int_0^\infty \mu_t dt$ is an arbitrary Hunt convolution kernel on G, cf. [6]. For an excessive measure τ on G (i.e. $\mu_t*\tau \le \tau$ for all $t > 0$) and an open set Ω we define the reduced measure R_τ^Ω as the infimum of all excessive measures majorizing τ on Ω. The reduced measure R_τ^Ω is again excessive, and if Ω is relatively compact R_τ^Ω is a potential $R_\tau^\Omega = \kappa*\lambda$ generated by a uniquely determined measure $\lambda \in D^+(\kappa)$, cf. [4].

By the same method of proof as in Theorem 1 we obtain the following generalization of Theorem 1:

THEOREM 8. *Let κ be a Hunt convolution kernel and let τ be an excessive measure. For any pair (Ω_0, Ω_1) of open sets such that $\overline{\Omega}_0$ is compact and disjoint from $\overline{\Omega}_1$ there exist positive measures $\mu_0, \mu_1 \in D^+(\kappa)$ such that $\xi = \kappa*(\mu_0 - \mu_1)$ satisfies*
(i) $0 \le \xi \le \tau$,
(ii) $\xi = \tau$ *in* Ω_0,
(iii) $\xi = 0$ *in* Ω_1,
(iv)   supp $(\mu_0) \subseteq \overline{\Omega}_0$, supp $(\mu_1) \subseteq \overline{\Omega}_1$.
The measure ξ is sum of the infinite alternating series

$$\xi = \sum_{n=0}^{\infty} (-1)^n \kappa*\lambda_n \ ,$$

where λ_0 is the uniquely determined measure such that $R_\tau^{\Omega_0} = \kappa*\lambda_0$, λ_{2n+1} is the canonical balayaged measure of λ_{2n} on Ω_1 and λ_{2n+2} is the canonical balayaged measure of λ_{2n+1} on Ω_0, $n \ge 0$. Furthermore

$$\mu_0 = \sum_{n=0}^{\infty} \lambda_{2n} \quad and \quad \mu_1 = \sum_{n=0}^{\infty} \lambda_{2n+1} \ .$$

Remark. Theorem 1 is obtained for $\tau = \omega_G$ which is excessive if and only if $\mu_t(G) \le 1$ for all $t > 0$, i.e. if and only if κ is the potential kernel for a transient convolution semigroup in the above sense.

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