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ON EIGENVALUES IN THE CONTINUUM OF 2-BODY OR MANY-BODY SCHRÖDINGER OPERATORS

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Introduction

Let us consider the following two problems.

(A) Does either

$$\liminf_{R\to\infty} R^{\alpha}\int_{R_0\le|x|\le R}|u(x)|^2\,dx>0 \tag{1}$$

or

$$\liminf_{R\to\infty}(\log R)^{-1}\int_{R_0\le|x|\le R}|u(x)|^2\,dx>0 \tag{2}$$

hold for the not identically vanishing solution $u(x)\in H^2_{\mathrm{loc}}(\Omega)$ of the equation

$$-\Delta u(x)+V(x)u(x)=\lambda u(x) \tag{3}$$

for $x\in\Omega\subset\boldsymbol{R}^n$ $(n\geq 3)$, where λ is a constant satisfying $\lambda>E_0$ and $V(x)$ is a 2-body or many-body potential?

(B) Can the selfadjoint realization of $-\Delta+V(x)$ in $L^2(\Omega)$ have eigenvalues in (E_0,∞)?

In (A) we would like to take α satisfying (1) and E_0 as small as possible. If (1) with $\alpha<0$ or (2) is satisfied, (B) is solved negatively.

In our previous papers (Mochizuki [7] and Uchiyama [10]) it was shown that (1) with $\alpha<0$ or (2) holds under some conditions on $V(x)$. The results are an extension of Weidmann [11] and are summarized in Proposition 1. The problem (B) is solved negatively by some papers (e.g., Weidmann [11], [12], Agmon [1], [2], Albeverio [3], Müller-Pfeiffer [8], Kalf [6], Simon [9], and Jansen-Kalf [5]).

In this paper we give a slight modification of our previous results. Theorem 1 can be easily reduced from Proposition 1. Theorem 2 which

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asserts the non-existence in (E_0, ∞) of eigenvalues of $-\Delta + V(x)$ is a corollary of Theorem 1. Jansen-Kalf [5] gives similar results to Theorem 1 in the 2-body case. On the other hand, our theorem can apply to many-body problem.

In § 2, we give some examples. Especially, example **I** shows that our results in many-body case are pure extension of Weidmann [12], Agmon [1] and Albeverio [3].

1. Theorems

We shall consider solutions of the equation

$$-\Delta u + V(x)u - \lambda u = 0 \tag{1.1}$$

in an exterior domain $\Omega \subset \boldsymbol{R}^n$ $(n \geq 3)$ of some compact set, where Δ is the n-dimensional Laplacian, λ is a real number and $V(x)$ is assumed to satisfy the following conditions:

(I) $V(x)$ is a real-valued function which belongs to the *Stummel class* Q_μ^{ext}. Namely, for some constant $\mu > 0$ and $R_0 > 0$ such that $\{x ; |x| > R_0\} \subset \Omega$, we have

$$\begin{cases} \sup\limits_{x;|x|>R_0} \displaystyle\int_{|x-y|<1} |V(y)|^2 |x-y|^{-n+4-\mu}\, dy < \infty & (\text{if } n \geq 4) \\ \sup\limits_{x;|x|>R_0} \displaystyle\int_{|x-y|<1} |V(y)|^2\, dy < \infty & (\text{if } n = 3)\,. \end{cases}$$

(II) Let $V(x) = V(r\omega) = V(r, \omega)$, where $r = |x|$ and $\omega = x/|x|$. Then there exists a null set $e \subset S^{n-1} = \{x ; |x| = 1\}$ such that $V(r, \omega)$ is differentiable in $r > R_0$ for any $\omega \in S^{n-1} \backslash e$. $r\dfrac{\partial V}{\partial r} \in Q_\mu^{\text{ext}}$. Further, there exists at least one $\gamma \in (0, 2]$ such that

$$\sum (r, \gamma) = \sup_{\omega \in S^{n-1}\backslash e} \left\{ r\frac{\partial V(r, \omega)}{\partial r} + \gamma V(r, \omega) \right\} < \infty$$

for $r > R_0$ and

$$E(\gamma) = \frac{1}{\gamma} \limsup_{r \to \infty} \sum (r, \gamma) < \infty\,.$$

(III) The unique continuation property holds.

In the following, by a solution u of equation (1.1) is meant an H^2_{loc}-function which satisfies (1.1) in the distribution sense in Ω. Here $H^j(\Omega)$

denotes the class of L^2-functions in Ω such that all distribution derivatives up to j belong to $L^2(\Omega)$ and H^j_{loc} denotes the class of locally H^j-functions in Ω.

LEMMA 1. *Let u be a solution of* (1.1). *Suppose that there exists a real C^1-function $\zeta(t)$ of $t>0$ such that $\zeta(|x|)u$ and $\zeta'(|x|)u$ are in $L^2(\Omega)$, where $\zeta' = d\zeta/dt$. Then we have for any $R_2 > R_0$*

$$\int_{|x|>R_2+1} \zeta^2\{|\nabla u|^2 + |V(x)|\,|u|^2\}dx \le C \int_{|x|>R_2} (\zeta^2 + \zeta'^2)\,|u|^2\,dx\,, \tag{1.2}$$

where C is a positive constant independent of $\zeta(t)$ and R_2.

Proof. Let $\phi_s(t)$ $(s > R_2 + 2)$ be a C^1-function of $t>0$ satisfying the following conditions: $0 \le \phi_s(t) \le 1$ and $|\phi_s'(t)| \le C_1$, where C_1 is independent of s; $\phi_s(t) = 1$ for $R_2 + 1 < t < s-1$, and $\phi_s(t) = 0$ for $t < R_2$ and $t > s$. Multiply (1.1) by $2\phi_s(|x|)^2\zeta(|x|)^2\bar{u}$ and integrate over Ω. Integration by parts gives

$$\begin{aligned} 2\int_\Omega \phi_s^2\zeta^2\,|\nabla u|^2\,dx = &-\int_\Omega 4\phi_s\zeta\frac{\partial u}{\partial r}\,(\phi_s'\zeta + \phi_s\zeta')\bar{u}dx \\ &- 2\int_\Omega \phi_s^2\zeta^2(V(x) - \lambda)\,|u|^2\,dx\,. \end{aligned}$$

Hence we have

$$\begin{aligned} &\int_{R_2<|x|<s} \phi_s^2\zeta^2(|\nabla u|^2 + |V(x)|\,|u|^2)dx \\ &\quad\le \int_{R_2<|x|<s} \{4(\phi_s'\zeta + \phi_s\zeta')^2 + \phi_s^2\zeta^2(2\,|\lambda| + 3\,|V(x)|)\}|u|^2\,dx\,. \end{aligned} \tag{1.3}$$

On the other hand, for the potential $V(x)$ satisfying condition (I) Ikebe-Kato [4] proves the fact that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\int_{|x|>R_2} |V(x)|\,|f(x)|^2\,dx \le \int_{|x|>R_2} \{\delta\,|\nabla f(x)|^2 + C_\delta\,|f(x)|^2\}dx$$

for any $f(x) \in H^1(\Omega)$ with support in $\{x\,;|x| \ge R_2\}$. Put $f = \phi_s\zeta u$. Then we have

$$\begin{aligned} &\int_{R_2<|x|<s} \phi_s^2\zeta^2\,|V(x)|\,|u|^2\,dx \\ &\quad\le \int_{R_2<|x|<s} \{\delta\,|\nabla(\phi_s\zeta u)|^2 + C_\delta\phi_s^2\zeta^2\,|u|^2\}dx \end{aligned}$$

$$\leq \int_{R_2<|x|<s} \{2\delta |\nabla u|^2 + [2\delta(\phi_s'\zeta + \phi_s\zeta')^2 + C_\delta \phi_s^2 \zeta^2] |u|^2\} dx .$$

This and (1.3) show that

$$\begin{aligned}(1-6\delta)&\int_{R_2+1<|x|<s-1} \zeta^2(|\nabla u|^2 + |V(x)||u|^2)dx \\ &\leq \int_{R_2<|x|<s} \{(4+6\delta)(\phi_s'\zeta + \phi_s\zeta')^2 + \phi_s^2\zeta^2(2|\lambda| + 3C_\delta)\}|u|^2 dx \\ &\leq \tilde{C}\int_{R_2<|x|<s} (\zeta^2 + \zeta'^2)|u|^2 dx .\end{aligned}$$

Hence, choosing $6\delta < 1$ and letting $s \to \infty$, we have (1.2). q.e.d.

LEMMA 2. *Suppose that* $V(x)$ *satisfies* (I) *and* (II). *Let*

$$\Gamma = \{\gamma \in (0,2]; E(\gamma) < \infty\} \quad \textit{and} \quad E_0 = \inf_{\gamma \in \Gamma} E(\gamma) . \tag{1.4}$$

Then we have $E_0 > -\infty$.

Proof. We assume the contrary. Then for any positive integer p there exists $\gamma_p \in \Gamma$ and $r_p > R_0$ satisfying

$$\frac{\partial}{\partial r}(r^{\gamma_p}V(r,\omega)) = r^{\gamma_p - 1}\left(r\frac{\partial V}{\partial r} + \gamma_p V\right) \leq -\gamma_p p r^{\gamma_p - 1}$$

for any $r \geq r_p$ and $\omega \in S^{n-1}\backslash e$. Integrating both sides with respect to r from ρ to $t\rho$, where $\rho > r_p$ and $t \geq 1$, we have for any $\omega \in S^{n-1}\backslash e$

$$(t\rho)^{\gamma_p}V(t\rho,\omega) - \rho^{\gamma_p}V(\rho,\omega) \leq -p\{(t\rho)^{\gamma_p} - \rho^{\gamma_p}\} .$$

Put $y = \rho\omega$. Then it follows that

$$p(t^{\gamma_p} - 1) \leq -t^{\gamma_p}V(ty) + V(y) . \tag{1.5}$$

By (I), there exists a constant $K > 0$ such that

$$\int_{|x-y|<1} |V(y)|^2 dy \leq K \qquad \text{for any } x \text{ satisfying } |x| > R_0 .$$

Thus, integrating (1.5) over $\{y; |x-y| < 1\}$ with respect to y, we have for any x such that $|x| \geq r_p + 1$

$$\begin{aligned} M_n p^2(t^{\gamma_p} - 1)^2 &\leq 2\left\{t^{2\gamma_p}\int_{|x-y|<1} |V(ty)|^2 dy + K\right\} \\ &\leq 2\{t^{2\gamma_p}t^{-n}(2\sqrt{n}t)^n + 1\}K ,\end{aligned} \tag{1.6}$$

where M_n is the volume of the unit ball of $\boldsymbol{R}^n$. Put $t = 2^{1/\gamma p}$ in (1.6). Then we have for any positive integer p

$$M_n p^2 \leq 2\{4(2\sqrt{n})^n + 1\}K .$$

This is a contradiction and the lemma is proved. q.e.d.

Now our results for the problem (A) can be stated in the following

THEOREM 1. *Suppose that $V(x)$ satisfies conditions* (I), (II) *and* (III). *Let Γ and E_0 be as in the above lemma, and let u be a not identically vanishing solution of* (1.1) *with $\lambda > E_0$. Then we have for any $\gamma \in \Gamma$ satisfying $E_0 \leq E(\gamma) < \lambda$*

$$\liminf_{R\to\infty} R^{-1+(\gamma/2)} \int_{R_0<|x|<R} |u(x)|^2\,dx > 0 \qquad \text{if } 0 < \gamma < 2 \tag{1.7}$$

and

$$\liminf_{R\to\infty} (\log R)^{-1} \int_{R_0<|x|<R} |u(x)|^2\,dx > 0 \qquad \text{if } \gamma = 2\,. \tag{1.8}$$

As a corollary of this theorem, we have the following theorem which solves the problem (B) negatively.

THEOREM 2. *Let E_0 be as in Lemma* 2. *Then any selfadjoint realization of $-\Delta + V(x)$ in $L^2(\Omega)$ has no eigenvalues in (E_0, ∞).*

In order to prove Theorem 1, we use the following proposition which is obtained in Mochizuki [7; Theorem 1.3] (cf., also Uchiyama [10; Lemma 3.15] where is proved the case $\frac{2}{3} < \gamma \leq 2$ by a different method).

PROPOSITION 1. *Let u be a solution in Ω of the equation*

$$-\Delta u - q(x)u = 0\,, \tag{1.9}$$

where $q(x)$ satisfies conditions (I), (III) *and*

(II)′ *there exist constants $0 < \gamma \leq 2, \sigma > 0$ and $R_1 > 0$ such that $\{x\,;|x| > R_1\} \subset \Omega$, $r(\partial q/\partial r) \in Q_\mu^{\text{ext}}$ and*

$$r\frac{\partial q}{\partial r} + \gamma q \geq \sigma \qquad \text{for } r \geq R_1 \text{ and } \omega \in S^{n-1}\backslash e\,,$$

where $r = |x|, \omega = x/|x|$ and $e \subset S^{n-1}$ is the null set. If u satisfies the condition

$$\liminf_{R\to\infty} R^{\gamma/2} \int_{|x|=R} \left\{\left|\frac{\partial u}{\partial r}\right|^2 + (1 + |q(x)|)\,|u|^2\right\} dS = 0\,, \tag{1.10}$$

then u must identically vanish in Ω.

For the sake of self-containedness of this paper, we shall give a brief proof of this proposition in Appendix.

Proof of Theorem 1. We fix a $\gamma \in \Gamma$ satisfying $E_0 \leq E(\gamma) < \lambda$. $E(\gamma) > -\infty$ by Lemma 2. Put $q(x) = \lambda - V(x)$. Then by (II) we see that for any $\delta > 0$ there exists an $R(\delta) > R_0$ such that

$$r\frac{\partial q}{\partial r} + \gamma q \geq \gamma(\lambda - E(\gamma) - \delta) \qquad \text{for } r > R(\delta) \text{ and } \omega \in S^{n-1} \backslash e\,.$$

We choose $\delta = (\lambda - E(\gamma))/2$ and put $R_1 = R(\delta)$ and $\sigma = \gamma\delta > 0$. Then $q(x) = \lambda - V(x)$ satisfies (I), (III) and (II)$'$ with these γ, σ and R_1. Let u be a non-trivial solution of (1.1). Then by Proposition 1, we see that there exist some $C_1 > 0$ and $R_2 > R_1$ such that

$$\int_{|x|=s}\left\{\left|\frac{\partial u}{\partial r}\right|^2 + (1 + |q(x)|)\,|u|^2\right\}dS \geq C_1 s^{-\gamma/2} \qquad \text{for } s \geq R_2\,. \tag{1.11}$$

Let $\zeta_R(t)$ be a C^1-function of $t > 0$ satisfying the following conditions: $0 \leq \zeta_R(t) \leq 1$ for $t > 0$, $\zeta_R(t) = 1$ for $0 < t < R - 1$, where $R > R_2 + 2$, $\zeta_R(t) = 0$ for $t > R$ and $|\zeta_R'(t)| \leq C_2$ for $t > 0$, where C_2 is a positive constant independent of R. Multiply (1.11) by $\zeta_R(s)^2$ and integrate over $(R_2 + 1, \infty)$. Then we have

$$\int_{|x|>R_2+1} \zeta_R^2\left\{\left|\frac{\partial u}{\partial r}\right|^2 + (1 + |q(x)|)|u|^2\right\}dx$$
$$\geq \begin{cases} \dfrac{C_1}{1-(\gamma/2)}\{R^{1-(\gamma/2)} - (R_2+1)^{1-(\gamma/2)}\} & \text{if } 0 < \gamma < 2 \\ C_1\{\log R - \log(R_2+1)\} & \text{if } \gamma = 2\,. \end{cases}$$

Combining this and Lemma 1 with $\zeta(t) = \zeta_R(t)$, we obtain (1.7) and (1.8).

q.e.d.

Remark 1. If $\Omega = \boldsymbol{R}^n$ and $R_0 = 0$ in conditions (I) and (II), then condition (III) is not required to obtain the above theorems. In fact, in this case, Proposition 1 is proved in Uchiyama [10; Lemma 4.3] without (III).

Remark 2. If the interval $(0, 2]$ appearing in condition (II) and Lemma 2 is replaced by $(0, 2-\delta]$ for any $\delta > 0$, then similar results can be obtained for a more general elliptic operators by use of Theorem

1.1 of Mochizuki [7]. For example, the above Theorem 2 holds true for the operator $-\Delta + V(x) + \tilde{V}(x)$, where $V(x)$ satisfies (I), (II) with $\gamma \in (0, 2-\delta]$ and (III), and $\tilde{V}(x)$ is a short range potential:

$$|x|\,\tilde{V}(x) \to 0 \qquad \text{as } |x| \to \infty$$

satisfying also (I) and (III).

2. Examples and remarks

I. Let the potential $V(x), x = (x_1, \cdots, x_{3N}) \in \boldsymbol{R}^{3N}$, have the form

$$V(x) = \sum_{j=1}^{N} V_j(\boldsymbol{r}_j) + \sum_{1 \le j < k \le N} V_{jk}(\boldsymbol{r}_{jk}) \,, \tag{2.1}$$

where $\boldsymbol{r}_j = (x_{3j-2}, x_{3j-1}, x_{3j})$ and $\boldsymbol{r}_{jk} = \boldsymbol{r}_j - \boldsymbol{r}_k$. We use the notation

$$r_j = |\boldsymbol{r}_j|,\ r_{jk} = |\boldsymbol{r}_{jk}| \quad \text{and} \quad r = \left(\sum_{j=1}^{N} r_j^2\right)^{1/2} = |x| \,.$$

Then we have

$$r\frac{\partial V_j}{\partial r} = r_j\frac{\partial V_j}{\partial r_j} \quad \text{and} \quad r\frac{\partial V_{jk}}{\partial r} = r_{jk}\frac{\partial V_{jk}}{\partial r_{jk}} \,. \tag{2.2}$$

PROPOSITION 2.1. *Suppose that $V(x)$ satisfies conditions* (I), (III) *and*

$$\begin{cases} \dfrac{1}{\gamma}\left(r_j\dfrac{\partial V_j}{\partial r_j} + \gamma V_j\right) \le E_j^\gamma & (r_j > 0) \\ \dfrac{1}{\gamma}\left(r_{jk}\dfrac{\partial V_{jk}}{\partial r_{jk}} + \gamma V_{jk}\right) \le E_{jk}^\gamma & (r_{jk} > 0) \end{cases} \tag{2.3}$$

for some constants $E_j^\gamma, E_{jk}^\gamma$ and $0 < \gamma \le 2$. Then there exists an E_0 such that

$$E_0 \le \sum_{j=1}^{N} E_j^\gamma + \sum_{1 \le j < k \le N} E_{jk}^\gamma \tag{2.4}$$

and $-\Delta + V(x)$ has no eigenvalues in (E_0, ∞).

Proof. It follows from (2.2) and (2.3) that

$$\frac{1}{r}\left(r\frac{\partial V}{\partial r} + \gamma V\right) \le \sum_{j=1}^{N} E_j^\gamma + \sum_{1 \le j < k \le N} E_{jk}^\gamma$$

for $r > 0$ and $\omega = x/r \in S^{n-1}\backslash e$. Hence, $V(x)$ satisfies condition (II) with

$E(\gamma) \leq \sum_{j=1}^{N} E_j^\gamma + \sum_{1\leq j<k\leq N} E_{jk}^\gamma$ and Theorem 2 leads the assertion.

q.e.d.

This results can be applied to generalized Coulomb-Yukawa potentials:

$$V_j = -\frac{c_j}{r_j^{\beta_j}} e^{-\alpha_j r_j}, \qquad V_{jk} = \frac{c_{jk}}{r_{jk}^{\beta_{jk}}} e^{-\alpha_{jk} r_{jk}}, \tag{2.5}$$

where $c_j, c_{jk}, \alpha_j, \alpha_{jk}, \beta_j$ and β_{jk} are all non-negative constants. We assume

$$0 < \beta_j < 3/2, \qquad 0 < \beta_{jk} < 3/2, \tag{2.6}$$

$$\max_{1\leq j\leq N} \{\beta_j\} \leq \min_{1\leq j<k\leq N} \{\beta_{jk}\}. \tag{2.7}$$

By (2.6) we see that the potential $V(x) = \sum_{j=1}^{N} V_j + \sum_{1\leq j<k\leq N} V_{jk}$ satisfies condition (I). Condition (III) easily follows from (2.5). If we choose

$$\gamma \leq \min_{1\leq j<k\leq N} \{\beta_{jk}\}, \tag{2.8}$$

then we have

$$\begin{aligned} &\frac{1}{\gamma}\left(r_{jk}\frac{\partial V_{jk}}{\partial r_{jk}} + \gamma V_{jk}\right) \\ &\quad \leq \frac{1}{\gamma} c_{jk} \sup_{0<r_{jk}<\infty} e^{-\alpha_{jk} r_{jk}} \{(\gamma - \beta_{jk}) r_{jk}^{-\beta_{jk}} - \alpha_{jk} r_{jk}^{1-\beta_{jk}}\} = 0 . \end{aligned} \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\gamma}\left(r_j\frac{\partial V_j}{\partial r_j} + \gamma V_j\right) \\ &\quad \leq \frac{1}{\gamma} c_j \sup_{0<r_j<\infty} e^{-\alpha_j r_j} \{(\beta_j - \gamma) r_j^{-\beta_j} + \alpha_j r_r^{1-\beta_j}\} = E_j^\gamma . \end{aligned} \tag{2.10}$$

Thus, for the potential $V(x)$ given by (2.1), (2.5), (2.6) and (2.7), there exists an E_0 such that

$$E_0 \leq \sum_{j=1}^{N} E_j^\gamma \tag{2.11}$$

and $-\Delta + V(x)$ has no eigenvalues in (E_0, ∞).

Note that in (2.10) each $E_j^\gamma < \infty$ if α_j and β_j satisfy one of the following three conditions.

$$\alpha_j = 0 \quad \text{and} \quad \beta_j \leq \gamma \qquad \text{(Coulomb type)}, \tag{2.12}$$

(2.13) $\alpha_j > 0$ and $\beta_j \leq \min\{\gamma, 1\}$ (Yukawa type) ,

(2.14) $\alpha_j > 0$ and $\beta_j < \gamma$ (Yukawa type) .

If α_j and β_j satisfy (2.12), then we have $E_j^\gamma = 0$ $(j = 1, \cdots, N)$. Thus, for the Coulomb type potential $V(x)$, $-\Delta + V(x)$ has no positive eigenvalues. The concrete Yukawa potential is given by

$$(2.15) \qquad V(x) = -\sum_{j=1}^{N} \frac{c_j}{r_j} e^{-\alpha_j r_j} + \sum_{1 \leq j < k \leq N} \frac{c_{jk}}{r_{jk}} e^{-\alpha_{jk} r_{jk}} .$$

In this case we have $E_j^1 = c_j \alpha_j$ since $\beta_j = \gamma = 1$ in (2.10), and hence $E_0 \leq \sum_{j=1}^{N} c_j \alpha_j$.

The generalized Coulomb-Yukawa potentials (2.5) have been studied by Weidmann [12], Agmon [1] and Albeverio [3]. Their results can be applied to show that the Schrödinger operator with the potential (2.15) has no eigenvalues in $(\sum_{j=1}^{N} c_j \alpha_j, \infty)$. However, we can show the

PROPOSITION 2.2. *Let $V(x)$ be the Yukawa potential given by* (2.15). *Then we have*

$$(2.16) \qquad E_0 \leq \limsup_{r \to \infty} \sum_{j=1}^{N} c_j \alpha_j e^{-\alpha_j r_j} = \sum_{j=1}^{N} c_j \alpha_j - \min_{1 \leq j \leq N} \{c_j \alpha_j\} .$$

Hence, $-\Delta + V(x)$ has no eigenvalues in $(\sum_{j=1}^{N} c_j \alpha_j - \min_{1 \leq j \leq N} \{c_j \alpha_j\}, \infty)$.

Proof. We have

$$\begin{aligned} E_0 = E(1) &= \limsup_{r \to \infty} \left(r \frac{\partial V}{\partial r} + V \right) \\ &= \limsup_{r \to \infty} \left(\sum_{j=1}^{N} c_j \alpha_j e^{-\alpha_j r_j} - \sum_{1 \leq j < k \leq N} c_{jk} \alpha_{jk} e^{-\alpha_{jk} r_{jk}} \right) \\ &\leq \limsup_{r \to \infty} \sum_{j=1}^{N} c_j \alpha_j e^{-\alpha_j r_j} . \end{aligned}$$

There exists a sequence $r(p) = \sqrt{r_1(p)^2 + \cdots + r_N(p)^2} \to \infty$ (as $p \to \infty$) such that

$$\limsup_{r \to \infty} \sum_{j=1}^{N} c_j \alpha_j e^{-\alpha_j r_j} = \lim_{p \to \infty} \sum_{j=1}^{N} c_j \alpha_j e^{-\alpha_j r_j(p)} .$$

Since $r(p) \to \infty$ as $p \to \infty$, there exists at least one $k, 1 \leq k \leq N$, such that

$$\lim_{p\to\infty}\sum_{j=1}^{N} c_j\alpha_j e^{-\alpha_j r_j(p)} \le \sum_{j=1}^{N} c_j\alpha_j - c_k\alpha_k$$
$$\le \sum_{k=1}^{N} c_j\alpha_j - \min_{1\le j\le N}\{c_j\alpha_j\} .$$

On the other hand, let $c_k\alpha_k = \min_{1\le j\le N}\{c_j\alpha_j\}$ and choose $r_j(p) = 0$ $(j \neq k)$ and $r_k(p) \to \infty$ as $p \to \infty$. Then

$$\sum_{j=1}^{N} c_j\alpha_j - c_k\alpha_k = \lim_{p\to\infty}\sum_{j=1}^{N} c_j\alpha_j e^{-\alpha_j r_j(p)}$$
$$\le \limsup_{r\to\infty}\sum_{j=1}^{N} c_j\alpha_j e^{-\alpha_j r_j} .$$

Summing up these results, we have (2.16). q.e.d.

Remark. If $N = 2$, then we have

$$E_0 = c_1\alpha_1 + c_2\alpha_2 - \min\{c_1\alpha_1, c_2\alpha_2\} . \tag{2.17}$$

In fact, assuming $c_1\alpha_1 \ge c_2\alpha_2$ and choosing $r_1(p) = 0$ and $r_2(p) = p$, we have

$$E_0 = \limsup_{r\to\infty}(c_1\alpha_1 e^{-\alpha_1 r_1} + c_2\alpha_2 e^{-\alpha_2 r_2} - c_{12}\alpha_{12}e^{-\alpha_{12}r_{12}})$$
$$\ge \lim_{p\to\infty}(c_1\alpha_1 + c_2\alpha_2 e^{-\alpha_2 p} - c_{12}\alpha_{12}e^{-\alpha_{12}p}) = c_1\alpha_1 .$$

II. Let us consider in $\boldsymbol{R}^6$ the operator

$$L = -\Delta_1 - \Delta_2 - \frac{2}{r_1} - \frac{2}{r_2} , \tag{2.18}$$

where $\Delta_j = \sum_{k=0}^{2}\partial^2/\partial x_{3j-k}^2$ and $r_j = (\sum_{k=0}^{2}|x_{3j-k}|^2)^{1/2}$ $(j = 1, 2)$. The negative eigenvalues of each $-\Delta_j - \dfrac{2}{r_j}$ form the set $\left\{-\dfrac{1}{n^2}\right\}_{n=1,2,\dots}$. Thus, we see that $\left\{-\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right\}_{n_1,n_2=1,2,\dots}$ are eigenvalues of L and the essential spectrum of L is $[-1, \infty)$. This shows that $(-\delta, \infty)$ is never the continuous spectrum of L for any $\delta > 0$, though $(0, \infty)$ is continuous as is seen in **I.**

III. The potential

$$V(x) = \frac{-32\sin r[g(r)^3\cos r - 3g(r)^2\sin^3 r + g(r)\cos r + \sin^3 r]}{[1 + g(r)^2]^2} \tag{2.19}$$

in $\boldsymbol{R}^3$, where $g(r) = 2r - \sin 2r$, is given by von Neumann and Wigner as an example which has the eigenvalue $+1$ with eigenfunction

$$u(x) = \frac{\sin r}{r(1 + g(r)^2)} .$$

Simon [9] proved that $-\Delta + V(x)$ with the above potential $V(x)$ has no eigenvalues in $(16, \infty)$ using the equality

$$\limsup_{r\to\infty} \left(r\frac{\partial V}{\partial r} + V\right) = 16$$

which follows from the following property of $V(x)$:

$$V(x) = -\frac{8 \sin 2r}{r} + \tilde{V}(x) ,$$

where $\tilde{V}(x)$ and $\partial\tilde{V}(x)/\partial r$ behave like $O(r^{-2})$ as $r \to \infty$. This property shows that

$$\gamma E(\gamma) = \limsup_{r\to\infty} \left(r\frac{\partial V}{\partial r} + \gamma V\right) = 16 \qquad \text{for any } \gamma .$$

Thus, choosing $\gamma = 2$, we can apply Theorem 2 to see that $-\Delta + V(x)$ has no eigenvalues in $(8, \infty)$.

IV. The potential

$$V(x) = \frac{-32k^2\alpha^2 \sin kr[(kr + 1/2\alpha) \cos kr - \sin kr]}{[1 + \alpha h(r)]^2} \tag{2.20}$$

in $\boldsymbol{R}^3$, where k, α are non-zero real constants and $h(r) = 2kr - \sin 2kr$, is given by Moses and Tuan (cf. Albeverio [3]) as an example which has the eigenvalue $+k^2$ with eigenfunction

$$u(x) = \frac{\sin kr}{r(1 + \alpha h(r))} .$$

The above $V(x)$ has the following property:

$$V(x) = -\frac{4k \sin 2kr}{r} + \tilde{V}(x) ,$$

where $\tilde{V}(x)$ and $\partial\tilde{V}(x)/\partial r$ behave like $O(r^{-2})$ as $r \to \infty$. Thus, we have

$$\gamma E(\gamma) = \limsup_{r\to\infty}\left(r\frac{\partial V}{\partial r} + \gamma V\right) = 8k^2 \qquad \text{for any } \gamma .$$

Choosing $\gamma = 2$, we see that $-\Delta + V(x)$ has no eigenvalues in $(4k^2, \infty)$. Note that in Kalf [6] is studied the potential $V_1(x) = V(x) - \frac{(n-1)(n-3)}{4r^2}$ in $\boldsymbol{R}^n$ $(n \geq 3)$, where $V(x)$ is given above. Using his virial theorem, Kalf proved that $-\Delta + V_1(x)$ has no eigenvalues in $(\Lambda/2, \infty)$, where

$$\Lambda = \sup_{x\in\boldsymbol{R}^n}\left(r\frac{\partial V}{\partial r} + 2V\right) \geq 8k^2 .$$

In this case we have also $E_0 = 4k^2$.

V. Kalf [6] also proved that the potential

$$V_1(x) = \frac{\beta}{r^2} + \sin(\log r) \qquad \text{in } \boldsymbol{R}^n \ (n \geq 3) ,$$

where $\beta > -[(n-2)/2]^2$, does not have eigenvalues in $(\sqrt{5}/2, \infty)$ (this can also proved by use of a theorem due to Agmon [1]). We consider here the following potential

$$V(x) = \frac{\beta}{r^2} + \psi(x)\sin(\log r) \qquad \text{in } \boldsymbol{R}^n \ (n \geq 3) , \tag{2.21}$$

where $\psi(x)$ satisfies (I), (III) and

$$\lim_{r\to\infty}\left(r\left|\frac{\partial\psi}{\partial r}\right| + |\psi - 1|\right) = 0 . \tag{2.22}$$

Then we have

$$\gamma E(\gamma) = \limsup_{r\to\infty}\left(r\,\frac{\partial V}{\partial r} + \gamma V\right) = \sqrt{1+\gamma^2} .$$

Thus, it follows that $-\Delta + V(x)$ does not have eigenvalues in $(\sqrt{5}/2, \infty)$. Note that in this case Kalf's virial theorem shows the non-existence of eigenvalues in $(\Lambda/2, \infty)$, where

$$\Lambda = \sup_{x\in\boldsymbol{R}^n}\left(r\frac{\partial V}{\partial r} + 2V\right) \geq \sqrt{5} .$$

If $\psi(x) = 1$, we have $\Lambda = \sqrt{5}$.

Remark. Applying Jansen-Kalf [5] to **III**, **IV** and **V**, we can have the same results as mentioned above.

Appendix (Proof of Proposition 1)

We use the notation: $B(R,t)=\{x; R<|x|<t\}$ for $0<R<t$, $B(R) = \{x; |x|>R\}$ for $R>0$ and $S(R)=\{x; |x|=R\}$ for $R>0$.

Let u be a solution of (1.9) satisfying also condition (1.10). Obviously, we may take u to be a real-valued function. Let $\rho(t)$ be a real-valued, C^3-function of $t>0$ and put

$$v(x)=e^{\rho(r)}u(x) \qquad (r=|x|)\,. \tag{3.1}$$

Then v satisfies the equation

$$-\Delta v+2\rho'\frac{\partial v}{\partial r}-\tilde{q}v=0 \qquad \text{in } x\in B(R_1)\,, \tag{3.2}$$

where

$$\tilde{q}=q+\left(\rho'^2-\rho''-\frac{n-1}{r}\rho'\right). \tag{3.3}$$

We multiply (3.2) by $r^\beta v$ and integrate over $B(R,t)$, where $R_1\le R<t$. Integrating by parts, we have

$$\begin{aligned}&\int_{B(R,t)} r^\beta(|\nabla v|^2-\tilde{q}\,|v|^2)dx\\ &\quad=\left[\int_{S(t)}-\int_{S(R)}\right]r^\beta\frac{\partial v}{\partial r}vdS-\int_{B(R,t)} r^\beta\left(\frac{\beta}{r}+2\rho'\right)\frac{\partial v}{\partial r}vdx\,.\end{aligned} \tag{3.4}$$

Similarly, integration by parts of (3.2) multiplied by $2r^\alpha\dfrac{\partial v}{\partial r}$ gives

$$\begin{aligned}&\left[\int_{S(t)}-\int_{S(R)}\right]r^\alpha\left(2\left|\frac{\partial v}{\partial r}\right|^2-|\nabla v|^2+\tilde{q}\,|v|^2+\frac{n-1-\gamma+\alpha}{r}\frac{\partial v}{\partial r}v\right)dS\\ &\quad=\int_{B(R,t)} r^{\alpha-1}\left\{(2-\gamma)\left(|\nabla v|^2-\left|\frac{\partial v}{\partial r}\right|^2\right)+(4\rho' r-\gamma+2\alpha)\left|\frac{\partial v}{\partial r}\right|^2\right.\\ &\qquad\left.+\left(r\frac{\partial\tilde{q}}{\partial r}+\gamma\tilde{q}\right)|v|^2+\frac{n-1-\gamma+\alpha}{r}(2\rho' r+\alpha-1)\frac{\partial v}{\partial r}v\right\}dx\,.\end{aligned} \tag{3.5}$$

First we put $\rho(r)=0$ and $\alpha=\gamma/2$ in (3.5). Then $v=u$ and $\tilde{q}=q$. Noting the condition (II)$'$, the equality

$$\int_{B(R,t)} r^{\alpha-2} \frac{\partial v}{\partial r} v dx$$
$$= \frac{1}{2}\left[\int_{S(t)} - \int_{S(R)}\right] r^{\alpha-2} |v|^2 dS - \frac{n-3+\alpha}{2} \int_{B(R,t)} r^{\alpha-3} |v|^2 dx$$

and the inequality $(n-1-\gamma+\alpha)(\alpha-1)(n-3+\alpha) \leq 0$, we have

$$\left[\int_{S(t)} - \int_{S(R)}\right] r^{\alpha}\left\{2\left|\frac{\partial u}{\partial r}\right|^2 - |\nabla u|^2 + q|u|^2 + \frac{n-1-\alpha}{r}\left(\frac{\partial u}{\partial r}u - \frac{\alpha-1}{2r}|u|^2\right)\right\} dS \geq \sigma \int_{B(R,t)} r^{\alpha-1} |u|^2 dx .$$

By (1.10), we can let $t \to \infty$ to obtain

$$\sigma \int_{B(R)} r^{\alpha-1} |u|^2 dx \leq \int_{S(R)} r^{\alpha}\left\{|\nabla u|^2 - q|u|^2 + \frac{(n-1-\alpha)^2}{4r^2}|u|^2\right\} dS < \infty. \tag{3.6}$$

This and Lemma 1 with $\zeta(r)^2 = r^{\alpha-1}$ imply that

$$\int_{B(R_1)} r^{\alpha-1}(|\nabla u|^2 + |u|^2) dx < \infty .$$

Integrating (3.6) with respect to R from s to t, where $R_1 \leq s < t$, using (3.4) with $\beta = \alpha$ and $\rho(r) = 0$, and letting $t \to \infty$, we obtain

$$\sigma \int_{B(s)} (r-s) r^{\alpha-1} |u|^2 dx$$
$$\leq \frac{1}{2} \int_{S(s)} r^{\alpha}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2\right) dS + C_1 \int_{B(s)} r^{\alpha-1}\left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2\right) dx < \infty ,$$

where $C_1 = \dfrac{\alpha}{2} + \dfrac{(n-1-\alpha)^2}{4R_1}$. In consideration of Lemma 1, we can repeat the integration with respect to s. Then we finally have

$$\int_{B(R_1)} r^{m}(|\nabla u|^2 + |u|^2) dx < \infty \tag{3.7}$$

for arbitrary $m > 0$.

Next we prove that for any $k > 0$ and $0 < \nu < 1$

$$\int_{B(R_1)} e^{kr^{\nu}}(|\nabla u|^2 + |u|^2) dx < \infty . \tag{3.8}$$

For this purpose we put $\rho(r) = m \log r$, where $m \geq n$, and $\alpha = -n+1+\gamma$ in (3.5). Then noting (3.7) and $4\rho' r - \gamma + 2\alpha > 0$, we have for $R \geq R_1$

$$\int_{S(R)} r^{\alpha}(|\nabla v|^2 - \tilde{q}\,|v|^2)dS \geq \int_{B(R)} r^{\alpha-1}\left(r\frac{\partial\tilde{q}}{\partial r} + \gamma\tilde{q}\right)|v|^2\,dx\,. \tag{3.9}$$

Here, by (3.3) and (II)′,

$$r\frac{\partial\tilde{q}}{\partial r} + \gamma\tilde{q} \geq \sigma - (2-\gamma)\frac{m(m-n+2)}{r^2} \qquad \text{for } r \geq R_1 \text{ and } \omega \in S^{n-1}\backslash e\,.$$

Multiply (3.9) by $R^{-2m-\alpha}$ and integrate over (s, ∞), where $s \geq R_1$. Then we have

$$\int_{B(s)} r^{-2m}(|\nabla v|^2 - \tilde{q}\,|v|^2 dx \geq \int_s^{\infty}\left\{\sigma - (2-\gamma)\left(\frac{m}{R}\right)^2\right\}dR\int_{B(R)} r^{-2m-1}|v|^2\,dx\,.$$

If we put $\beta = -2m$ in (3.4) and let $t \to \infty$, then

$$\int_{B(s)} r^{-2m}(|\nabla v|^2 - \tilde{q}\,|v|^2)dx = -\int_{S(s)} r^{-2m}\frac{\partial v}{\partial r}v\,dS$$
$$= -\frac{1}{2}\frac{d}{ds}\int_{S(s)} r^{-2m}|v|^2\,dS - \frac{2m-n+1}{2}\int_{S(s)} r^{-2m-1}|v|^2\,dS\,.$$

Thus, noting $r^{-2m}|v|^2 = |u|^2$, we have

$$-\frac{1}{2}\left[\frac{d}{ds}\int_{S(s)}|u|^2\,dS + \frac{m}{s}\int_{S(s)}|u|^2\,dS\right] \geq \int_s^{\infty}\left\{\sigma - (2-\gamma)\left(\frac{m}{R}\right)^2\right\}dR\int_{B(R)} r^{-1}|u|^2\,dx\,. \tag{3.10}$$

We fix arbitrary $k > 0$ and $0 < \nu < 1$, let $m = \tilde{k}\nu s^{\nu}$ $(\tilde{k} = k+1)$ and choose $R_2 = R_2(\tilde{k}, \nu) \geq R_1$ so large that

$$\sigma - (2-\gamma)\left(\frac{\tilde{k}\nu}{R_2^{1-\nu}}\right)^2 \geq 0\,.$$

Then it follows from (3.10) that

$$\frac{d}{ds}\int_{S(s)}|u|^2\,dS + \tilde{k}\nu s^{\nu-1}\int_{S(s)}|u|^2\,dS \leq 0 \qquad \text{for } s \geq R_2\,.$$

Therefore, for any $k > 0$ and $0 < \nu < 1$,

$$\int_{S(s)} |u|^2 \, dS \le C_2 e^{-\tilde{k} s^\nu} \qquad (\tilde{k} = k + 1) \, ,$$

where C_2 is independent of s. This and Lemma 1 with $\zeta(r)^2 = e^{kr^\nu}$ prove (3.8).

Now we can prove $u = 0$. We return once more to (3.5) with $\rho(r) = kr^\nu$ and $\alpha = -n + 1 + \gamma$. Since

$$\tilde{q} = q + \left\{\left(\frac{k\nu}{r^{1-\nu}}\right)^2 - \frac{k\nu(n-2+\nu)}{r^{2-\nu}}\right\} ,$$

we have for $R \ge R_1$

$$\begin{aligned} &-R^\alpha \int_{S(R)} \left(2\left|\frac{\partial v}{\partial r}\right|^2 - |\nabla v|^2 + \tilde{q}\,|v|^2\right) dS \\ &\ge \int_{B(R)} r^{\alpha-1}(4k\nu r^\nu - \gamma + 2\alpha)\left|\frac{\partial v}{\partial r}\right|^2 dx \\ &\quad + \int_{B(R)} r^{\alpha-1}\left\{\sigma + (\gamma - 2 + 2\nu)\left(\frac{k\nu}{r^{1-\nu}}\right)^2 + (-\gamma + 2 - \nu)\frac{k\nu(n-2+\nu)}{r^{2-\nu}}\right\} \\ &\qquad \times |v|^2 \, dx \, . \end{aligned} \tag{3.11}$$

If we choose ν such that $\dfrac{2-\gamma}{2} < \nu < 1$ and $R_3 = R_3(\nu) \ge R_1$ sufficiently large, then for any $k \ge 1$ and $r \ge R_3$

$$\begin{cases} 4k\nu r^\nu - \gamma + 2\alpha \ge 0 \, , \\ (\gamma - 2 + 2\nu)\left(\dfrac{k\nu}{r^{1-\nu}}\right)^2 + (-\gamma + 2 - \nu)\dfrac{k\nu(n-2+\nu)}{r^{2-\nu}} \ge 0 \, . \end{cases}$$

Therefore, by (3.11),

$$\int_{S(R)} \left(2\left|\frac{\partial v}{\partial r}\right|^2 - |\nabla v|^2 + \tilde{q}\,|v|^2\right) dS \le 0 \qquad \text{for } R \ge R_3 \, . \tag{3.12}$$

Since $v = e^{kr^\nu} u$, we can write the left side of (3.12) in the form

$$e^{2kR^\nu}\{k^2 M_1(R) + kM_2(R) + M_3(R)\} ,$$

where

$$M_1(R) = \frac{2\nu^2}{R^{2-2\nu}} \int_{S(R)} |u|^2 \, dS$$

and $M_2(R)$ and $M_3(R)$ are independent of k. Suppose that $M_1(R) > 0$ for some $R \ge R_3$. Then k can be chosen so large that (3.12) is no longer valid. Hence $u = 0$ in $B(R_3)$. By the unique continuation property (III), we have $u=0$ in Ω and Proposition 1 is proved.

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