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KIRILLOV MODELS FOR DISTINGUISHED REPRESENTATIONS

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§ 0. Introduction

In the theory of automorphic forms on covering groups of the general linear group, a central role is played by certain local representations which have unique Whittaker models. A representation with this property is called distinguished. In the case of the 2-sheeted cover of GL_2, these representations arise as the the local components of generalizations of the classical θ-function. They have been studied thoroughly in [GPS]. The Weil representation provides these representations with a very nice realization, and the local factors attached to these representations can be computed using this realization. It has been shown [KP] that only in the case of a certain 3-sheeted cover do we find other principal series of covering groups of GL_2 which have a unique Whittaker model. It is natural to ask if these distinguished representations also have a realization analgous to the Weil representation.

In this paper we investigate this question by constructing explicit Kirillov models for the distinguished principal series of the 2-sheeted and 3-sheeted covers of GL_2 over a p-adic field. In the case of the 2-sheeted cover, we find that the action of the Weyl element is given by a very simple formula. We use this formula to give a new computation of local L and ε-factors, the previous method in [GPS] relying heavily on the Weil representation. For the 3-sheeted cover, we calculate the action of the Weyl element, but it appears that the formula does not simplify, leading us to believe that there is no local analogue of the Weil representation in this case.

One ingredient of these calculations is the evaluation of Salie's sum over quotient rings of the ring of integers in a p-adic field. In this paper we carry this out by a direct calculation. It can also be shown that the

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classical evaluation of this sum is a simple consequence of the equivalence between the Kirillov model and the Weil model of the distinguished principal series representation of the 2-sheeted cover of GL_2.

§ 1. Preliminaries

1.1. Notations

Let k be a p-adic field containing the n-th roots of unity μ_n. Let $\mathcal{O}$ denote the ring of integers, $\mathcal{P}$ the prime ideal, and $\mathcal{U}$ the units. τ will denote a local uniformizing parameter. Let $\bar{k} = \mathcal{O}/\mathcal{P}$, let q be the cardinality of $\bar{k}$, and let v be the normalized valuation on k. Let χ be an additive character of k of conductor $\mathcal{O}$. Let $\mathcal{S}(k)$ denote the space of locally constant, compactly supported functions on k. $(\cdot, \cdot)$ will denote the n-th order Hilbert symbol on k.

1.2. The metaplectic groups

Let $G = GL(2, k)$. Define

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c, & \text{if } c \neq 0 \\ d, & \text{if } c = 0\,, \end{cases}$$

For $0 \leq c < n$, define a cocycle on G by

$$\beta^{(c)}(g_1, g_2) = \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)}\right)\left(\det g_1, \frac{x(g_1g_2)}{x(g_1)}\right)(\det g_1, \det g_2)^c\,.$$

These define extensions $\tilde{G}^{(c)}$ of G. They are realized as the set of pairs (g, ζ), with $g \in G$ and $\zeta \in \mu_n$, with the multiplication defined by $(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\beta^{(c)}(g, g'))$. We will usually write $\tilde{G}$ and β for $\tilde{G}^{(c)}$ and $\beta^{(c)}$. Further properties of the cocycle and the group $\tilde{G}$ can be found in [KP]. Let $\tilde{A} = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right) \middle| a, b \in k^\times, \zeta \in \mu_n\right\}$, $\tilde{N} = \left\{\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \zeta\right) \middle| x \in k, \zeta \in \mu_n\right\}$, and $\tilde{B} = \tilde{A}\tilde{N}$. Let $Z = \left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in k^\times\right\}$, and $\tilde{Z} = \{(z, \zeta) \,|\, z \in Z, \zeta \in \mu_n\}$. Note that β splits over $\tilde{N}$. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $\tilde{h} = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right) \in \tilde{A}$, let $\delta(\tilde{h}) = |ab^{-1}|^{1/2}$. For $g \in G$, we will sometimes denote $(g, 1)$ by g.

1.3. The principal series

Let $\tilde{\mu}$ be a genuine d-dimensional representation of $\tilde{A}$. Extend $\tilde{\mu}$ to $\tilde{B}$ by $\tilde{\mu}(\tilde{a}\tilde{n}) = \tilde{\mu}(\tilde{a})$. The principal series representation $H_{\tilde{\mu}} = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\tilde{\mu}$ is the space of locally constant functions $f\colon \tilde{G} \to C^d$ such that $f(\tilde{n}\tilde{h}\tilde{g}) = \delta(\tilde{h})\tilde{\mu}(\tilde{h})f(\tilde{g})$, where $\tilde{h} \in \tilde{A}$, $\tilde{n} \in \tilde{N}$. $\tilde{G}$ acts on $H_{\tilde{\mu}}$ by right translations. Since functions in this space are determined by their restrictions to the group

$$\left\{(\omega^{-1}, 1)\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, 1\right)\middle| x \in k^x\right\},$$

we define $\phi_f(x) = f\left((w^{-1}, 1)\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$. For $f \in H_{\tilde{\mu}}$, $|x|\tilde{\mu}\left(\left(\begin{pmatrix}x & 0\\ 0 & x^{-1}\end{pmatrix}, 1\right)\right)\phi_f(x)$ is constant for large $|x|$. Let $k_{\tilde{\mu}}$ be the space of all locally constant functions ϕ on k such that $|x|\tilde{\mu}\left(\begin{pmatrix}x & 0\\ 0 & x^{-1}\end{pmatrix}, 1\right)\phi(x)$ is constant for large $|x|$. For $\phi \in k_{\tilde{\mu}}$, define the Fourier transform of ϕ by

$$\hat{\phi}(x) = \sum_{n \in Z} \int_{\nu(y)=n} \phi(y)\bar{\chi}(xy)dy\,.$$

The mapping $\phi \to \hat{\phi}$ is injective for all $\tilde{\mu}$ if $n > 1$, since $\hat{\phi} = 0$ implies ϕ is constant, which is possible only if $\tilde{\mu}\left(\begin{pmatrix}x & 0\\ 0 & x^{-1}\end{pmatrix}, 1\right) = |x|^{-1}I$ for $|x|$ large, which never happens if $n > 1$. We thus obtain a representation of $\tilde{G}$ in the space $\hat{k}_{\tilde{\mu}} = \{\hat{\phi} \mid \phi \in k_{\tilde{\mu}}\}$. The action $\hat{T}_{\tilde{\mu}}$ of $\tilde{G}$ on $\hat{k}_{\tilde{\mu}}$ is given by

$$\hat{T}_{\tilde{\mu}}\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right)\phi(x) = \chi(bx)\phi(x)\,, \tag{1.3.1}$$

$$\hat{T}_{\tilde{\mu}}\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, 1\right)\phi(x) = |ab^{-1}|^{1/2}\tilde{\mu}\left(\begin{pmatrix}b & 0\\ 0 & a\end{pmatrix}, (a, ab)\right)\phi(xab^{-1})\,, \tag{1.3.2}$$

$$\hat{T}_{\tilde{\mu}}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, 1\right)\phi(x) = \int J_{\tilde{\mu}}(z, x)\phi(z)dz\,, \tag{1.3.3}$$

where

$$J_{\tilde{\mu}}(x, z) = \int \tilde{\mu}\left(\begin{pmatrix}y & 0\\ 0 & y^{-1}\end{pmatrix}, 1\right)\chi\left(xy + \frac{z}{y}\right)\frac{dy}{|y|}\,.$$

§ 2. The two-sheeted cover

We now proceed to examine the two-sheeted cover of G. To be consistent with [GPS], we take the cocycle corresponding to $c = 0$ and denote it simply by β. We note that $\beta\left(\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \begin{pmatrix}a' & 0\\ 0 & d'\end{pmatrix}\right) = (a, d')$.

2.1. Representations of $\tilde{A}$

Let

$$\tilde{A}_0 = \left\{\left(\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \zeta\right)\middle| \nu(a), \nu(d) \equiv 0(2)\right\}.$$

We choose the ordered set of coset representatives of $\tilde{A}_0$ in $\tilde{A}$ to be

$$\left\{I, \begin{pmatrix}\tau & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & \tau\end{pmatrix}, \begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix}\right\}.$$

β is trivial on $\tilde{A}_0 \times \tilde{A}_0$, and so $\tilde{A}_0$ is abelian and its genuine characters are of the form $\tilde{\mu}_0\left(\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \gamma\right) = \gamma\mu_1(a)\mu_2(d)$, where μ_1 and μ_2 are characters of $k^\times$. Now form $\tilde{\mu} = \mathrm{Ind}_{\tilde{A}_0}^{\tilde{A}}\tilde{\mu}_0$ with respect to the above set of coset representatives. μ will denote the character $\mu_1\mu_2^{-1}$, and we will sometimes write $\tilde{\mu}(\mu_1, \mu_2)$ for $\tilde{\mu}$ to denote the dependence on μ_1 and μ_2.

2.2. Intertwining operators

Given μ_1 and μ_2, define a representation of $\tilde{A}$ by $\tilde{\mu}(\mu_1, \mu_2)^w(\tilde{a}) = \tilde{\mu}(\mu_1, \mu_2)(w\tilde{a}w^{-1})$. The representations $T_{\tilde{\mu}}$ and $T_{\tilde{\mu}^w}$ are equivalent, the equivalence $I\colon H_{\tilde{\mu}} \to H_{\tilde{\mu}^w}$ being given by

$$(I\phi)(\tilde{g}) = \int_N \phi(wn\tilde{g})dn\,,$$

the integral converging if μ is ramified or if $\mu(x) = |x|^\alpha$, with $\mathrm{Re}(\alpha) > 0$. The representations $\tilde{\mu}(\mu_2, \mu)$ and $\tilde{\mu}(\mu_1, \mu_2)^w$ are also equivalent. We have $X\tilde{\mu}(\mu_2, \mu_1) = \tilde{\mu}(\mu_1, \mu_2)^w X$, where

$$X = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & (\tau, \tau) & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

With respect to the form $\langle v, w\rangle = \sum_{i=1}^4 v_i w_i$ on C^4, the contragradient of $\tilde{\mu}(\mu_1, \mu_2)$ is $\tilde{\mu}(\mu_1^{-1}, \mu_2^{-1})$. If we let $(v, w) = \langle v, X^{-1}w\rangle$, then $(\tilde{\mu}(\tilde{h})v, \tilde{\mu}^w(\tilde{h})w) = (v, w)$ and $\tilde{\mu}$ and $\tilde{\mu}^w$ are contragredient with respect to the form (v, w). Now form

$$(\phi, \phi') = \int_{\tilde{B}\backslash\tilde{G}} (\phi(g), \phi'(g))dg$$

on $H_{\tilde{\mu}} \times H_{\tilde{\mu}^w}$. We have the intertwining operator $I\colon H_{\tilde{\mu}} \to H_{\tilde{\mu}^w}$, so we may form, for $\phi_1, \phi_2 \in H_{\tilde{\mu}}$,

$$(\phi_1, I\phi_2) = \int_{\tilde{B}\backslash\tilde{G}} (\phi_1(g), I\phi_2(g))dg\,.$$

Switching to functions on k, this becomes

$$\int \left\langle \phi_1\left(w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right), X^{-1}I\phi_2\left(w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\right\rangle dx. \tag{2.2.1}$$

We have

$$I\phi_2\left(w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \int \tilde{\mu}\left(\begin{pmatrix}y & 0\\ 0 & y^{-1}\end{pmatrix}, 1\right) f_2(x+y)\frac{dy}{|y|}, \tag{2.2.2}$$

where $f_i(x) = \phi_i\left(w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$. But (2.2.2) equals

$$\int_{\nu(y)\equiv 0(2)} \mu(y)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & (\tau, y) & 0 & 0\\ 0 & 0 & (\tau, y) & 0\\ 0 & 0 & 0 & 1\end{bmatrix} f_2(x+y)\frac{dy}{|y|} \tag{2.2.3}$$

$$+\int_{\nu(y)\equiv 1(2)} \mu(y)\begin{bmatrix}0 & 0 & 0 & \mu_1\mu_2(\tau)(y, \tau)\\ 0 & 0 & \mu^{-1}(\tau)(\tau, \tau) & 0\\ 0 & \mu(\tau) & 0 & 0\\ \mu_1\mu_2(\tau^{-1})(\tau, \tau y) & 0 & 0 & 0\end{bmatrix} f_2(x+y)\frac{dy}{|y|}.$$

Our goal is to find the explicit intertwining operator in the Fourier transform realization. We need to find an operator J such that (2.2.1) equals $\langle \hat{f}_1, X^{-1}J\hat{f}_2\rangle$. Plugging (2.2.2) into (2.2.1), using (2.2.3), and assuming that $\mu_1\mu_2^{-1}(x) = |x|^r$, we find an operator J on $k^\times$ so that if $\nu(y) \equiv 0(2)$,

$$J(y) = |y|^{-r}\begin{bmatrix}\dfrac{1-q^{-1}}{1-q^{-2r}} & 0 & 0 & \mu_1\mu_2(\tau)(\tau, y)c(\tau)q^{r-1/2}\\ 0 & 0 & (\tau, \tau)\Gamma(2r) & 0\\ 0 & q^{-2r}\Gamma(2r) & 0 & 0\\ \mu_1\mu_2(\tau^{-1})(\tau, \tau y)c(\tau)q^{r-1/2} & 0 & 0 & \dfrac{1-q^{-1}}{1-q^{-2r}}\end{bmatrix}.$$

For $\nu(y) \equiv 1(2)$,

$$J(y) = |y|^{-r}\begin{bmatrix}q^{-r}\Gamma(2r) & 0 & 0 & 0\\ 0 & (\tau, y)c(\tau)q^{r-1/2} & q^{r}(\tau, \tau)\dfrac{1-q^{-1}}{1-q^{-2r}} & 0\\ 0 & q^{-r}\dfrac{1-q^{-1}}{1-q^{-2r}} & (\tau, y)c(\tau)q^{r-1/2} & 0\\ 0 & 0 & 0 & q^{-r}\Gamma(2r)\end{bmatrix}.$$

Recall that for a character π of $k^\times$, we have the gamma function $\Gamma(\pi)$

defined as in [T], p. 48. Any such π can be decomposed uniquely as $\pi(x) = \pi^*(x)|x|^\alpha$, where π^* is the character of $k^\times$ given by $\pi^*(x) = \pi(u)$ if $x = u\tau^n$, $u \in U$. If π is ramified of conductor $h \geq 1$, then $\Gamma(\pi) = c_{\pi^*}q^{h(\alpha-1/2)}$, where $|c_{\pi^*}| = 1$. For $\alpha \in C$, $\Gamma(\alpha)$ will denote $\Gamma(|\cdot|^\alpha)$, and $c(\tau)$ will denote c_{π^*}, for $\pi(x) = (\tau, x)$. Note that $c(\tau)^2 = (\tau, -1) = (\tau, \tau)$. If π is unramified, $\pi(x) = |x|^\alpha$, $\alpha \neq 0$, then $\Gamma(\pi) = \dfrac{1 - q^{\alpha-1}}{1 - q^{-\alpha}}$.

Now suppose $\gamma = 1/2$, the case of interest. If $\nu(y) \equiv 0(2)$,

$$J(y) = |y|^{-1/2}\begin{bmatrix} 1 & 0 & 0 & \mu_1\mu_2(\tau)(\tau, y)c(\tau) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu_1\mu_2(\tau^{-1})(\tau, \tau y)c(\tau) & 0 & 0 & 1 \end{bmatrix},$$

and if $\nu(y) \equiv 1(2)$,

$$J(y) = |y|^{-1/2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (\tau, y)c(\tau) & (\tau, \tau)q^{1/2} & 0 \\ 0 & q^{-1/2} & (\tau, y)c(\tau) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2.3. The image when $\gamma = 1/2$

We can realize $H_{\tilde{\mu}}$ as the space $k_{\tilde{\mu}}$ of all locally constant functions ϕ on k such that $|x|\mu\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)\phi(x)$ is constant for $|x|$ large. For $1 \leq l \leq 4$, define

$$g_l(x) = \begin{cases} |x|^{-1}\tilde{\mu}\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & x\end{pmatrix}(x, x)\right)e_l, & |x| \geq 1 \\ 0, & |x| < 1. \end{cases}$$

Here e_l is the l-th standard basis vector of C^4. Suppose that for $|x|$ large, $|x|\tilde{\mu}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}, 1\right)\phi(x) = v \in C^4$. Write $v = \sum_{l=1}^4 v_l e_l$, and let $\psi(x) = \phi(x) - \sum_{l=1}^4 v_l g_l(x)$. But for $|x|$ large,

$$\begin{aligned}\phi(x) &= |x|^{-1}\tilde{\mu}\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & x\end{pmatrix}, (x, x)\right)v \\ &= \sum_{l=1}^4 v_l|x|^{-1}\tilde{\mu}\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & x\end{pmatrix}, (x, x)\right)e_l = \sum_{l=1}^4 v_l g_l(x),\end{aligned}$$

so for $|x|$ large, $\psi(x) = 0$. Clearly, $\psi(x) \in \mathscr{S}(k)$. Any $\phi \in k_{\tilde{\mu}}$ is thus of the form $\phi(x) = \psi(x) + \sum_{l=1}^4 v_l g_l(x)$ for some $\psi \in \mathscr{S}(k)$. Taking Fourier transforms, $\hat{\phi}(x) = \hat{\psi}(x) + \sum_{l=1}^{4} v_l \hat{g}_l(x)$.

A calculation shows that for $\nu(y) \equiv 0(2)$ and $\gamma = 1/2$, we have

$$\hat{g}_1(y) = \begin{bmatrix} 1 - q^{-(\nu(y)+2))/2} \\ 0 \\ 0 \\ \mu_1\mu_2(\tau^{-1})q^{-(\nu(y)+1)/2}(\tau, y)(\tau, \tau)c(\tau)q^{-1/2} \end{bmatrix},$$

and for $\nu(y) \equiv 1(2)$ and $\gamma = 1/2$,

$$\hat{g}_1(y) = \begin{bmatrix} \frac{1}{1-q^{-1}}(1 - q^{-1} - q^{-(\nu(y)+1)/2} + q^{-1}q^{-(\nu(y)+3)/2}) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $h_1(y) = \hat{g}_1(y) - e_1$, then $J(y)h_1(y) = 0$ for all $y \in k^\times$.

Similar calculations yield functions h_l on $k^\times$, $2 \leq l \leq 4$, so that $\hat{g}_l(y) = h_l(y) + a_l(y)$, where $a_l(y) = a_{l,i}$ if $\nu(y) \equiv i(2)$, $i = 1, 2$, and $a_{l,i}$ are constants. For each l, $J(y)h_l(y) = 0$ for all $y \in k^\times$.

Choosing $\phi \in k_{\bar{\mu}}$, writing $\hat{\phi} = \hat{\psi} + \sum_{l=1}^{4} v_l \hat{g}_l$, and applying the operator J, we obtain $J\hat{\phi} = J\hat{\psi} + J\sum_{l=1}^{4} v_l a_l$. For $\nu(y) \equiv i(2)$, write

$$a(y) = \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix}. \tag{2.3.3}$$

Then for $\nu(y) \equiv 0(2)$,

$$J(y)a(y) = |y|^{-1/2} \begin{bmatrix} \alpha_0 + \mu_1\mu_2(\tau)(\tau, y)c(\tau)\delta_0 \\ 0 \\ 0 \\ \mu_1\mu_2(\tau^{-1})(\tau, \tau y)c(\tau)\alpha_0 + \delta_0 \end{bmatrix},$$

and for $\nu(y) \equiv 1(2)$,

$$J(y)a(y) = |y|^{-1/2} \begin{bmatrix} 0 \\ (\tau, y)c(\tau)\beta_1 + (\tau, \tau)q^{1/2}\gamma_1 \\ q^{-1/2}\beta_1 + (\tau, y)c(\tau)\gamma_1 \\ 0 \end{bmatrix}.$$

2.4. The Kirillov model

In this section we assume $\mu(x) = \mu_1\mu_2^{-1}(x) = |x|^{1/2}$, and we undertake a preliminary study of the Kirillov model. We already have an explicit realization of the image of $\hat{k}_{\tilde{\mu}}$ when $\gamma = 1/2$ under the intertwining operator J. We denote this image by $r_{\tilde{\mu}}$.

Define an operator $\mathscr{E}$ on $r_{\tilde{\mu}}$ by $\mathscr{E}f(y) = |y|^{1/2}\tilde{\mu}^w\left(\begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix}, (y, y)\right)f(y)$. Let $a(y)$ be as in Section 2.3. Then a calculation shows that if $\nu(y) \equiv 0(2)$,

$$(\mathscr{E}Ja)(y) = \mu_1(y)\begin{bmatrix} \alpha_0 + \mu_1\mu_2(\tau)(\tau, y)c(\tau)\delta_0 \\ 0 \\ 0 \\ \mu_1\mu_2(\tau^{-1})(\tau, \tau)c(\tau)\alpha_0 + \delta_0(\tau, y)\end{bmatrix}. \tag{2.4.1}$$

Notice that row (4) $= X_0 \cdot$ row (1), where $X_0 = \mu_1\mu_2(\tau^{-1})(\tau, \tau)c(\tau)$. If $\nu(y) \equiv 1(2)$, we have

$$(\mathscr{E}Ja)(y) = \mu_1(y)\begin{bmatrix} \mu_2(\tau)q^{-1/2}\beta_1 + \mu_2(\tau)(\tau, y)c(\tau)\gamma_1 \\ 0 \\ 0 \\ (\tau, \tau)c(\tau)\beta_1\mu_2(\tau^{-1}) + (\tau, y)\mu_2(\tau^{-1})q^{1/2}\gamma_1\end{bmatrix}. \tag{2.4.2}$$

In this case, row (4) $= X_1 \cdot$ row (1), where $X_1 = \mu_2(\tau^{-2})q^{1/2}(\tau, \tau)c(\tau)$. Since $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$, we have $X_0 = X_1$. We will simply denote this quantity by X from now on.

Define an operator $\mathscr{C}$ on $\mathscr{E}J\hat{k}_{\tilde{\mu}}$ by $(\mathscr{C}h)(x) = h_1(x)$, the first component of $h(x)$ in C^4. By the above calculations, $\mathscr{C}^{-1}$ exists, and for a function $s\colon k^\times \to C$,

$$(\mathscr{C}^{-1}s)(x) = \begin{bmatrix} s(x) \\ 0 \\ 0 \\ \mu_1\mu_2(\tau^{-1})(\tau, \tau)c(\tau)s(x)\end{bmatrix}.$$

We let $K(r_{\tilde{\mu}}) = \mathscr{C}\mathscr{E}J\hat{k}_{\tilde{\mu}}$ and define an action $\tilde{\pi}$ of $\tilde{G}$ on $K(r_{\tilde{\mu}})$ via the map $\mathscr{C}\mathscr{E}$; i.e., $\tilde{\pi}(\tilde{g}) = \mathscr{C}\mathscr{E}r_{\tilde{\mu}}(\tilde{g})\mathscr{E}^{-1}\mathscr{C}^{-1}$.

PROPOSITION 2.4.1. *For* $\phi \in K(r_{\tilde{\mu}})$, $\tilde{\pi}\left(\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}, 1\right)\phi(x) = \chi(bx)\phi(ax)$.

Proof. Let $f = \mathscr{E}^{-1}\mathscr{C}^{-1}\phi \in J\hat{k}_{\tilde{\mu}}$. Then

$$\tilde{\pi}\left(\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\phi(x) = \left(\left(\mathscr{C}\mathscr{E}r_{\tilde{\mu}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\mathscr{E}^{-1}\mathscr{C}^{-1}\phi\right)(x)\right.$$

$$= \left(\mathscr{E}r_{\tilde{\mu}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\mathscr{E}^{-1}\mathscr{C}^{-1}\phi\right)_1(x).$$

But

$$\begin{aligned}
&\left(\mathscr{E}r_{\tilde{\mu}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\mathscr{E}^{-1}\mathscr{C}^{-1}\phi\right)(x) \\
&= |x|^{1/2}\tilde{\mu}^{w}\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, (x, x)\right)\left(r_{\tilde{\mu}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\mathscr{E}^{-1}\mathscr{C}^{-1}\phi\right)(x) \\
&= |x|^{1/2}\tilde{\mu}^{w}\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, (x, x)\right)|a|^{1/2}\tilde{\mu}^{w}\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, (a, a)\right)\mathscr{E}^{-1}\mathscr{C}^{-1}\phi(ax) \text{ by (1.3.2)} \\
&= |ax|^{1/2}\tilde{\mu}^{w}\left(\begin{pmatrix} 1 & 0 \\ 0 & ax \end{pmatrix}, (ax, ax)\right)|ax|^{-1/2}\tilde{\mu}^{w}\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & ax \end{pmatrix}\right), (ax, ax)\right)^{-1}(\mathscr{C}^{-1}\phi)(ax) \\
&= (\mathscr{C}^{-1}\phi)(ax) \\
&= \begin{bmatrix} \phi(ax) \\ 0 \\ 0 \\ \mu_1\mu_2(\tau^{-1})(\tau, \tau)c(\tau)\phi(ax) \end{bmatrix}.
\end{aligned}$$

The first component of this is just $\phi(ax)$, which completes the proof if $b = 0$. A similar argument, using (1.3.1), does the case $a = 1$.

The following proposition is Proposition 3.3.4 in [GPS].

PROPOSITION 2.4.2. *Near* 0, *any* $\xi \in K(r_{\tilde{\mu}})$ *is a linear combination of characters* $(\alpha, y)\mu_1(y)$, *where* $\alpha \in k^\times/k^{\times 2}$. $\mathscr{S}(k^\times)$ *thus has codimension* $[k^\times : k^{\times 2}]$ *in* $K(r_{\tilde{\mu}})$.

Proof. We apply $\mathscr{C}\mathscr{E}J$ to functions of the form (2.3.3). For $\psi \in \mathscr{S}(k)$, $\mathscr{C}\mathscr{E}J\hat{\psi}$ vanishes near zero. (2.4.1) and (2.4.2) give formulas for $(\mathscr{E}Ja)(y)$. Combining these into a single formula and applying $\mathscr{C}$, we have

$$\begin{aligned}
\Big((\mathscr{C}\mathscr{E}Ja)(y) &= \frac{1}{2}(1 + (\varepsilon, y))\mu_1(y)(\alpha_0 + \mu_1\mu_2(\tau)(\tau, y)c(\tau)\delta_0) \\
&\quad + \frac{1}{2}(1 - (\varepsilon, y))\mu_1(y)(\mu_2(\tau)q^{-1/2}\beta_1 + \mu_2(\tau)(\tau, y)\mu_2(\tau^{-1})q^{1/2}\gamma_1),
\end{aligned}$$

which is the desired linear combination.

2.5. The action of in $K(r_{\tilde{\mu}})$

Recall that for any $\tilde{g} \in \tilde{G}$, $f \in K(r_{\tilde{\mu}})$, we have

$$(\tilde{\pi}(\tilde{g})f)(t) = (\mathscr{C}\mathscr{E}\hat{T}(\tilde{g})\mathscr{E}^{-1}\mathscr{C}^{-1}f)(t)\,.$$

Using (1.3.3), we see that

$$\begin{aligned}(2.5.1)\qquad (\hat{T}(\omega, 1)\mathscr{E}^{-1}\mathscr{C}^{-1}f)(t) &= \int J_{\tilde{\mu}^w}(u, t)(\mathscr{E}^{-1}\mathscr{C}^{-1}f)(u)du \\ &= \int J_{\tilde{\mu}^w}(u, t)|u|^{-1/2}\tilde{\mu}^w\left(\left(\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}, (u, u)\right)^{-1}\right)\begin{bmatrix} f(u) \\ 0 \\ 0 \\ Xf(u)\end{bmatrix} du\,.\end{aligned}$$

(Recall that $X = \mu_1\mu_2(\tau^{-1})(\tau, \tau)c(\tau)$.)

In order to simplify (2.5.1), we need some explicit expressions for $J_{\tilde{\mu}^w}(u, t)$. Recall that for $\mu \in k^\times$,

$$J_\mu(u, t) = \int \mu(t)\bar{\chi}\left(ux + \frac{t}{x}\right)\frac{dx}{|x|}\,.$$

If we denote the matrix $J_{\tilde{\mu}^w}(u, t)$ by $J_{ij}(u, t)$, a calculation shows that (2.5.1) equals

$$\frac{1}{2}\int (1 + (\varepsilon, u))|u|^{-1/2}\mu_1(u^{-1})\begin{bmatrix} J_{11}(u, t) + J_{14}(u, t)(u, \tau)X \\ 0 \\ 0 \\ J_{41}(u, t) + J_{44}(u, t)(u, \tau)X\end{bmatrix} f(u)du$$

$$+ \frac{1}{2}\int (1 - (\varepsilon, u))|u|^{-1/2}\mu_1(u^{-1})\begin{bmatrix} 0 \\ J_{22}(u, t)\mu_1(\tau^{-1}) + J_{23}(u, t)\mu_1(\tau)(u, \tau)X \\ J_{32}(u, t)\mu(\tau^{-1}) + J_{33}(u, t)\mu_1(\tau)(u, \tau)X \\ 0\end{bmatrix} f(u)du\,.$$

Applying $\mathscr{C}\mathscr{E}$, we see that if $\nu(t) \equiv 0(2)$, $\tilde{\pi}(\omega, 1)f(t)$ equals

$$(2.5.2)\qquad \frac{1}{2}|t|^{1/2}\mu_1(t)\int (1 + (\varepsilon, u))|u|^{-1/2}\mu_1(u^{-1})(J_{11}(u, t) + X(u, \tau)J_{14}(u, t))f(u)du\,.$$

If $\nu(t) \equiv 1(2)$, it equals

$$\begin{aligned}(2.5.2)\qquad \frac{1}{2}|t|^{1/2}\mu_1(t\tau)\int (1 - (\varepsilon, u))|u|^{-1/2}\mu_1(u^{-1})(J_{22}(u, t)\mu_1(\tau^{-1}) \\ + J_{23}(u, t)\mu_1(\tau)(u, \tau)X)f(u)du\,.\end{aligned}$$

In the next sections we will further simplify these expressions.

2.2. Salié's sum over $\mathcal{O}/\mathscr{P}^m$

In this section we evaluate some sums we need to simplify the formulas for $\tilde{\pi}(w, 1)$ in the Kirillov model.

Let $\mathscr{R} = \mathcal{O}/\mathscr{P}^m$, $m \geq 1$. Let μ be a character of $\mathscr{R}^*$, the units of $\mathscr{R}$. Let n be the conductor of μ. We will denote a coset $x + \mathscr{P}^m$ simply by x. Let ψ be an additive character of $\mathscr{R}$ of conductor m, i.e., ψ is non-trivial on $\mathscr{P}^{m-1}/\mathscr{P}^m$.

For $x \in \mathscr{R}^*$, let $S_\mu(x) = \sum_{u \in \mathscr{R}^*} \mu(u)\psi(x(u + 1/u))$.

PROPOSITION 2.6.1.

(1) *If m is even, $m \geq 2n$, then* $S_\mu(x) = q^{m/2}[\psi(2x) + \mu(-1)\psi(-2x)]$.

(2) *If m is odd, $m \geq 3$, and $m \geq 2n - 1$, then*

$$S_\mu(x) = q^{(m-1)/2}G(\phi)\phi(x)[\psi(2x) + \phi(-1)\mu(-1)\psi(-2x)],$$

where ϕ is the quadratic character on $F_q^\times$ and $G(\phi)$ is the Gauss sum on $F_q^\times$ of ϕ.

Proof. Assume m is even and $m \geq 2n$. Write $\mathscr{U}/\mathscr{U}_n = \{u\}$, where $\mathscr{U}_n = 1 + \mathscr{P}^n$, and write $\mathscr{U}_n/\mathscr{U}_m = \{v\}$. Then

$$S_\mu(x) = \sum_{u,v} \mu(uv)\psi(x(uv + u^{-1}v^{-1})).$$

Writing $v = 1 + y$, with $y \in \mathscr{P}^n/\mathscr{P}^m$, it becomes

$$\sum_u \mu(u) \sum_v \psi(x(u(1 + y) + u^{-1}(1 + y)^{-1})).$$

We have $1/(1 + y) = \sum_{l=0}^{m-1} (-1)^l y^l$, giving

$$\sum_u \mu(u)\psi(x(u + u^{-1})) \sum_v \psi\left(x\left(uy + u^{-1}\left(\sum_{l=1}^{m-1} (-1)^l y^l\right)\right)\right).$$

Writing $\mathscr{P}^n/\mathscr{P}^{m/2} = \{w\}$ and $\mathscr{P}^{m/2}/\mathscr{P}^m = \{z\}$, we get $\{y\} = \{w + z\}$. Since $z \in \mathscr{P}^{m/2}$, $z^l \in \mathscr{P}^m$ for $l \geq 2$, so that $\psi(xu^{-1}z^l) = 1$ in these cases, giving

$$\sum_u \mu(u)\psi(x(u + u^{-1})) \sum_w \psi(xw(u - u^{-1}))\psi(xu^{-1}(w^2 - w^3 + \cdots + (-1)^{m-1}w^{m-1}))$$
$$\times \sum_z \psi(x(uz - u^{-1}z(1 + 2w + \cdots + (-1)^{m-1}(m - 1)w^{m-2}))).$$

The inner sum is $\sum_z \psi(xu^{-1}z(u^2 - 1 + 2w - 3w^2 + \cdots + (-1)^m(m - 1)w^{m-2}))$. This is a sum over $\mathscr{P}^{m/2}/\mathscr{P}^m$, and it is non-zero if and only if $xu^{-1}(u^2 - 1 + 2w - \cdots + (-1)^m(m - 1)w^{m-2}) \in \mathscr{P}^m$, which is true only when $w = 0$ and $u = \pm 1$. This gives the result $q^{m/2}[\psi(2x) + \mu(-1)\psi(-2x)]$.

Now assume m is odd and $(m + 1)/2 \geq n$. Again let $\mathscr{U}/\mathscr{U}_n = \{u\}$, $\mathscr{U}_n/\mathscr{U}_m = \{v\}$. We write $\mathscr{P}^n/\mathscr{P}^m = \{w + z \mid w \in \mathscr{P}^n/\mathscr{P}^{(m+1)/2},\ z \in \mathscr{P}^{(m+1)/2}/\mathscr{P}^m\}$.

Proceeding as before, we find that

$$S_\mu(x) = \sum_u \mu(u)\psi(x(u + u^{-1})) \sum_w \psi(x(u - u^{-1})w)\psi(xu^{-1}(w^2 - w^3 + \cdots + w^{m-1})) \times \sum_z \psi(xz(u + u^{-1}(-1 + 2w - 3w^2 + \cdots + (m-1)w^{m-2}))) .$$

But the inner sum over z equals zero unless $u^2 + 2w - \cdots + (m-1)w^{m-2} \in \mathscr{U}_{m-2}$, which is true if and only if $u^2 = 1$ and $2w - 3w^2 + \cdots + (m-1)w^{m-2}) \in \mathscr{P}^{m-1/2}$. Letting $\{v\} = \mathscr{P}^{(m-1)/2}/\mathscr{P}^{(m+1)/2}$, we have

$$S_\mu(x) = q^{(m-1)/2}[\psi(2x) \sum_v \psi(x(v^2 - v^3 + \cdots + v^{m-1})) + \mu(-1)\psi(-2x) \sum_v \psi(-x(v^2 - v^3 + \cdots + v^{m-1}))] .$$

But $v \in \mathscr{P}^{(m-1)/2}$, so $v^3, v^4, \cdots, v^{m-1}$ are all in $\mathscr{P}^m$, leaving

$$q^{(m-1)/2}[\psi(2x) \sum_v \psi(xv^2) + \mu(-1)\psi(-2x) \sum_a \psi(-xv^2)] .$$

It is easy to see that $\sum_v \psi(xv^2) = G(\phi)\phi(x)$, where ϕ is the unique non-trivial character of order 2 on F_q^x and $G(\phi)$ is the associated Gauss sum. Similarly, $\sum_v \psi(-xv^2) = \phi(-1)\phi(x)G(\phi)$. Therefore,

$$S_\mu(x) = q^{(m-1)/2}G(\phi)\phi(x)[\psi(2x) + \phi(-1)\mu(-1)\psi(-2x)] .$$

This result, in the special case $\mathscr{R} = Z/p^mZ$ and $\mu(x) = \phi(x)$, was proved recently in [I]. We also quote, without proof, the analogous result in the case of a finite field. It is well known and appears, for example, in [E] as Theorem 2.6.

PROPOSITION 2.6.2. *Let $\mu \in \hat{F}_q^\times$, $x \in F_q$, and ψ a non-trivial character of F_q. Then unless $\mu = 1$ and $x = 0$,*

$$S_\mu(x) = \frac{G(\mu\phi)}{\mu(x)G(\phi)} \sum_{w \in F_q} \mu\phi(w^2 - 1)\psi(2wx) .$$

If we take $\mu = \phi$, there are other proofs of Proposition 2.6.1 which include the result for F_q as a special case, but they are more cumbersome than the one we have given.

With the notation of Proposition 2.6.1, we also have

PROPOSITION 2.6.3. *For all $x \in \mathscr{R}^*$, $\sum_{u \in R^*} \mu(u)\psi(x(u + \varepsilon/u)) = 0$, where ε is non-square unit and $\deg(\mu) = n < m$.*

Proof. Write $\mathscr{U}/\mathscr{U}_m = \{vw \mid v \in \mathscr{U}/\mathscr{U}_n,\ w \in \mathscr{U}_n/\mathscr{U}_m\}$, Then

$$\sum_{u\in R^*}\mu(u)\psi\left(x\left(u+\frac{\varepsilon}{u}\right)\right)=\sum_{v,w}\mu(vw)\psi\left(x\left(vw+\frac{\varepsilon}{vw}\right)\right)$$
$$=\sum_{v}\mu(v)\sum_{w}\psi\left(x\left(vw+\frac{\varepsilon}{vw}\right)\right).$$

Writing $\{w\}=\{1+y\,|\,y\in\mathscr{P}^n/\mathscr{P}^m\}$, we get

$$\sum_{v}\mu(v)\sum_{y}\psi\left(x\left(v(1+y)+\frac{\varepsilon}{v(1+y)}\right)\right).$$

Taking $1/(1+y)=\sum_{l=0}^{m-1}(-1)^l y^l$ gives

$$\sum_{v}\psi\left(x\left(v+\frac{\varepsilon}{v}\right)\right)\sum_{y}\psi\left(x\left(vy+\varepsilon v^{-1}\sum_{l=1}^{m-1}(-1)^l y^l\right)\right).$$

But the sum over y is zero unless $v+\varepsilon v^{-1}(-1+y+\cdots+(-1)^{m-1}y^{m-2})\in\mathscr{P}^{m-n}$, which is false in all cases.

2.7. The Kirillov model (cont.)

In this section we use the results of Section 2.6 to simplify the matrix $J_{\tilde{\mu}w}(u,t)$ of Section 2.5.

Recall that for $\mu\in\hat{k}^x$, $u,t\in k^x$,

$$J_\mu(u,t)=\int\mu(x)\bar{\chi}\left(ux+\frac{t}{x}\right)\frac{dx}{|x|}.$$

For k a positive integer, let

$$F_\mu(k,t)=\int_{|x|=q^k}\mu(x)\bar{\chi}\left(x+\frac{t}{x}\right)\frac{dx}{|x|}.$$

The following result appears in [T, p. 69].

LEMMA 2.7.1.

(1) *If μ is unramified, $\mu\neq 1$, then*

$$J_\mu(u,t)=\begin{cases}\mu(t)\Gamma(\mu^{-1})+\mu^{-1}(u)\Gamma(\mu), & |ut|\le q\\ \mu^{-1}(u)F_\mu\left(\dfrac{m}{2},ut\right), & |ut|=q^m,\ m>1,\ m\ \textit{even}\\ 0, & |ut|=q^m,\ m>1,\ m\ \textit{odd}.\end{cases}$$

(2) *If μ is ramified,* $\deg(\mu)=h\ge 1$, *then*

$$J_\mu(u,t)=\begin{cases}\mu(t)\Gamma(\mu^{-1})+\mu^{-1}\Gamma(\mu), & |ut|\le q\\ \mu^{-1}(u)F_\mu\left(\dfrac{m}{2},ut\right). & |ut|=q^m,\ m\ge 2h,\ m\ \textit{even}\\ 0, & |ut|=q^m,\ m>2h,\ m\ \textit{odd}.\end{cases}$$

If $\nu(t) \equiv 0(2)$, we need only calculate

$$J_{11}(u, t) + X(u, \tau)J_{14}(u, t)$$

when $\nu(u) = 0(2)$. Assuming $\nu(u)$, $\nu(t)$ even and $|ut| \leq q$, we have

$$J_{11}(u, t) = (|t|^{-1/2} - q^{-1}|u|^{1/2})$$

and

$$J_{14}(u, t) = \mu_1\mu_2(\tau)c(\tau)(|t|^{-1/2}(\tau, t) + |u|^{1/2}(\tau, u)q^{-1}).$$

In this case, therefore, we see that

$$J_{11}(u, t) + X(u, \tau)J_{14}(u, t) = |t|^{-1/2}(1 + (\tau, ut)). \tag{2.7.1}$$

Now suppose $\nu(u)$, $\nu(t)$ even and $|ut| > q$. Then

$$J_{11}(u, t) = \frac{1}{2}\sum_{k=0}^{1} J_{\varepsilon^k\mu^{-1}}(u, t) = \frac{1}{2}\sum_{k=0}^{1}(\varepsilon^k, \mu)\mu(u)F_{\varepsilon^k\mu^{-1}}\left(\frac{m}{2}, ut\right),$$

where $m = -\nu(ut)$. This equals

$$\begin{aligned} &\frac{1}{2}\mu(u)\sum_{k=0}^{1}\int_{|y|=q^{m/2}}\bar{\chi}\left(y + \frac{ut}{y}\right)(\varepsilon^k, y)\mu^{-1}(y)\frac{dy}{|y|} \\ &= \frac{1}{2}|u|^{1/2}q^{-m/4}\int_{|y|=q^{m/4}}\bar{\chi}\left(y + \frac{ut}{y}\right)(1 + (\varepsilon, y))\frac{dy}{|y|}. \end{aligned} \tag{2.7.2}$$

Also,

$$\begin{aligned} J_{14}(u, t) &= \frac{1}{2}\mu_1\mu_2(\tau)\sum_{k=1}^{1}\zeta^{-k}J_{\varepsilon^k\tau\mu^{-1}}(u, t) \\ &= \frac{1}{2}\mu_1\mu_2(\tau)(\tau, u)|u|^{1/2}q^{-m/4}\int_{|y|=q^{m/2}}\bar{\chi}\left(y + \frac{ut}{y}\right)(\tau, y)(1 - (\varepsilon, y))\frac{dy}{|y|}. \end{aligned} \tag{2.7.3}$$

If $m/2$ is even, we therefore need to calculate

$$\int_{|y|=q^{m/2}}\bar{\chi}\left(y + \frac{ut}{y}\right)\frac{dy}{|y|}.$$

If $ut = x^2$, a change of variables shows this equals

$$q^{-m/4}\sum_{u}\bar{\chi}\left(x\left(u + \frac{1}{u}\right)\right),$$

where u is a set of coset representatives for $\mathscr{U}/\mathscr{U}_{m/2}$. By Proposition 2.6.1, this equals $q^{-m/4}[\bar{\chi}(2x) + \bar{\chi}(-2x)]$. If $ut = \varepsilon x^2$, the integral equals

$$q^{-m/4}\sum_{u}\bar{\chi}\left(x\left(u + \frac{\varepsilon}{u}\right)\right),$$

which is zero by Proposition 2.6.3. If $m/2$ is odd, we need to calculate

$$\int_{|y|=q^{m/2}} \bar{\chi}\left(y + \frac{ut}{y}\right)(\tau, y)\frac{dy}{y}.$$

Arguing as above, this is zero if $ut = \varepsilon x^2$. If $ut = x^2$, this equals

$$q^{-1/2}q^{-m/4}G(\phi)\phi(x)[\bar{\chi}(2x) + \bar{\chi}(-2x)].$$

Substituting into (2.7.2) and (2.7.3), we find that for $ut = x^2$ and $m/2$ even,

$$J_{11}(u, t) + X(u, \tau)J_{14}(u, t) = |u|^{1/2}q^{-m/2}[\bar{\chi}(2x) + \bar{\chi}(-2x)].$$

If $ut = x^2$ and $m/2$ is odd, it equals

$$(\tau, \tau)c(\tau)|u|^{1/2}q^{-m/2}q^{-1/2}G(\phi)\phi(x)[\bar{\chi}(2x) + \bar{\chi}(-2x)].$$

But $G(\phi) = q^{1/2}c(\tau)$ and $c(\tau)^2 = (\tau, \tau)$, so this becomes $|t|^{-1/2}[\bar{\chi}(2x) + \bar{\chi}(-2x)]$.

Noting that the above formulas for the case $|ut| > q$ give (2.7.1) if $x \in \mathcal{O}$, we obtain

LEMMA 2.7.2. *Suppose $\nu(t)$ and $\nu(u)$ are even. Then*

$$J_{11}(u, t) + X(u, \tau)J_{14}(u, t) = \begin{cases} 0, & \text{if } ut = \varepsilon x^2 \\ |t|^{-1/2}[\bar{\chi}(2x) + \bar{\chi}(-2x)], & \text{if } ut = x^2. \end{cases}$$

Now consider the case when $\nu(t)$ and $\nu(u)$ are odd. We need to calculate $J_{22}(u, t)$ and $J_{23}(u, t)$. We simply state the results.

LEMMA 2.7.3. *Suppose $\nu(t)$ and $\nu(u)$ are odd. If $|ut| \leq q$, then* (1) *and* (2) *hold. If $|ut| > q$, $|ut| = q^m$, then* (3), (4), *and* (5) *hold.*

(1) $J_{22}(u, t) = c(\tau)[|t|^{-1/2}(\tau, t) + |u|^{1/2}(\tau, u)q^{-1}]$.

(2) $J_{23}(u, t) = \mu^{-1}(\tau)(\tau, \tau)[|t|^{-1/2} - q^{-1}|u|^{1/2}]$.

(3) *If $ut = \varepsilon x^2$, $J_{22}(u, t) = J_{23}(u, t) = 0$.*

(4) *If $ut = x^2$,*

$$J_{22}(u, t) = \begin{cases} 0, & \text{if } m/2 \text{ even} \\ (\tau, u)|u|^{1/2}q^{-m/2}c(\tau)[\bar{\chi}(2x) + \bar{\chi}(-2x)], & \text{if } m/2 \text{ odd}. \end{cases}$$

(5) *If $ut = x^2$,*

$$J_{23}(u, t) = \begin{cases} 0, & \text{if } m/2 \text{ odd} \\ |u|^{1/2}q^{-m/2}\mu^{-1}(\tau)(\tau, \tau)[\bar{\chi}(2x) + \bar{\chi}(-2x)], & \text{if } m/2 \text{ even}. \end{cases}$$

LEMMA 2.7.4. *Let $A = \mu_1(\tau^{-1})J_{22}(u, t) + \mu_1(\tau)X(u, \tau)J_{23}(u, t)$. Suppose $\nu(t)$ and $\nu(u)$ are odd. Then*

(1) *If* $|ut| \leq q$, $A = c(\tau)\mu_1(\tau^{-1})|t|^{-1/2}[(\tau, t) + (\tau, u)]$.
(2) *If* $|ut| > q$ *and* $ut = \varepsilon x^2$, *then* $A = 0$.
(3) *If* $|ut| > q$ *and* $ut = x^2$, *then* $A = \mu_1(\tau^{-1})c(\tau)(u, \tau)|t|^{-1/2}[\bar{\chi}(2x) + \bar{\chi}(-2x)]$.

Now we substitute the results of Lemmas (2.7.2) and (2.7.4) in formulas (2.5.2) and (2.5.3). For $\nu(t)$ even, $\tilde{\pi}(w, 1)\xi(t)$ equals

$$\begin{aligned}(2.7.5)\quad & |t|^{1/2}\mu_1(t)\int_{\nu(u)\equiv 0(2)} |u|^{-1/2}\mu_1(u^{-1})[J_{11}(u, t) + X(u, \tau)J_{14}(u, t)]\xi(u)du \\ &= \mu_1(t)\int_{u\in t(k^\times)^2} |u|^{-1/2}\mu_1(u^{-1})[\bar{\chi}(2\sqrt{ut}) + \bar{\chi}(-2\sqrt{ut})]\xi(u)du\,.\end{aligned}$$

For $\nu(t)$ odd, $\tilde{\pi}(w, 1)\xi(t)$ equals

$$\begin{aligned}(2.7.6)\quad & |t|^{1/2}\mu_1(t)\mu_1(\tau)\int_{\nu(u)\equiv 1(2)} |u|^{-1/2}\mu_1(u^{-1})[\mu_1(\tau^{-1})J_{22}(u, t) \\ &\qquad + X\mu_1(\tau)(u, \tau)J_{23}(u, t)]\xi(u)du \\ &= \mu_1(t)c(\tau)\int_{v\in t(k^\times)^2} |u|^{-1/2}\mu_1(u^{-1})(\tau, u)[\bar{\chi}(2\sqrt{ut}) + \bar{\chi}(-2\sqrt{ut})]\xi(u)du\,.\end{aligned}$$

Formulas (2.7.5) and (2.7.6) can combined into a single formula.

2.8. The functional equation and local L factors

We first recall some notation from [GPS]. Let (π, V) be an irreducible admissible genuine representation of $\tilde{G}$. The central character ω_π of π is defined by

$$\pi\begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix} = \omega_\pi(z^2)I\,,$$

$z \in k^\times$. Let $\Omega(\omega_\pi)$ denote the set of genuine characters of $\tilde{Z}$ whose restriction to Z^2 equals ω_π. If ψ is an additive character of k and $\rho \in \Omega(\omega_\pi)$, a (ψ, ρ)-Whittaker functional for π is a functional l on V such that

$$l\left(\pi\left(\tilde{z}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}v\right)\right) = \rho(\tilde{z})\psi(x)l(v)\,,$$

for $\tilde{z} \in \tilde{Z}$, $x \in k$, $v \in V$.

The dimension of the space $W_{\psi,\rho}$ of (ψ, p)-Whittaker functionals is at most one. Given ψ, there exists at least one $\rho \in \Omega(\omega_\pi)$ such that $\dim W_{\psi,\rho} > 0$. Let $\Omega(\pi, \psi) = \{\mu \in \Omega(\omega_\pi) \,|\, \dim W_{\psi,\rho} > 0\}$. If $\Omega(\pi, \psi)$ has precisely one element, π is called distinguished. It is known, at least in odd residual characteristic, that any distinguished representation is one of the Weil representations r_χ, as defined in [GPS].

Suppose that π is distinguished representation and $\Omega(\pi, \psi) = \{\rho\}$. Let $W(\pi, \psi)$ be the space of all complex-valued functions on $\tilde{G}$ of the form $W(g) = l(\pi(g)\phi)$, for ϕ in the space of π. For $W \in W(\pi, \psi)$, α a quasi-character of $k^\times$, and $s \in C$, let

$$L_W(g, \alpha, s) = \int_{k^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \alpha(x) |x|^{s-1/2} \frac{dx}{|x|}.$$

The following result appears in [GPS].

PROPOSITION 2.8.1.

(1) *The integral defining $L_W(g, \alpha, s)$ converges for* Re s *sufficiently large and continues analytically to a meromorphic function on* $\boldsymbol{C}$.

(2) *There is a rational function $\gamma_\pi(s, \alpha, \psi)$, independent of W and g, such that*

$$L_W(wg, \mu^{-1}\alpha^{-1}, 1-s) = \gamma_\pi(s, \alpha, \psi) L_W(g, \alpha, s).$$

Here, μ is a certain projective character arising from the Weil representation.

(3) $\gamma_{\pi\otimes\alpha}(s, 1) = \gamma_\pi(s, \alpha)$.

The local L factor $L_\pi(s, \alpha)$ is the g.c.d. of the functions

$$L_\xi(s, \alpha) = \int \xi(x) \alpha(x) |x|^{s-1/2} \frac{dx}{|x|},$$

with $\xi \in K(\pi)$. It is specified uniquely by writing $L_\pi(s, \alpha) = P(q^{-s})^{-1}$, P a polynomial with constant term one. For μ a quasi-character of $k^\times$, let

$$L(s, \mu) = \begin{cases} (1 - \mu(\varpi) q^{-s})^{-1}, & \text{if } \mu \text{ is unramified} \\ 1, & \text{otherwise.} \end{cases}$$

Using Proposition 2.5.2, we easily get the following result, which already appears in [GPS].

PROPOSITION 2.8.2. *If $\mu(x) = |x|^{1/2}$, then $L_{r_{\tilde{\mu}}}(s, \alpha) = L(2s - 1/2, \mu_1^2 \alpha^2 |\cdot|^{-1/2})$.*

Proof. Since Proposition 2.4.2 describes the behavior near zero of the functions in the Kirillov space, the proof is the same as that of Proposition 3.6.2 of [GPS].

2.9. Local γ-factors

In this section we compute the local γ-factors $\gamma_{r_{\tilde{\mu}}}(s, \alpha)$, where α is a quasicharacter of $k^\times$. From the functional equation, we have

$$\gamma_{r_{\tilde{\mu}}}(s, \alpha) = \frac{L_W(w, \rho^{-1}, 1-s)}{L_W(I, 1, s)}\,.$$

For

$$W_N\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \xi_N(x) = \begin{cases} q^N, & x \in \mathscr{U}_N \\ 0, & x \in \mathscr{U}_N\,, \end{cases}$$

we have $L_W(I, \alpha, s) = 1$, so

$$\gamma_{r_{\tilde{\mu}}}(s, 1) = \int W\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix} \rho^{-1}(x)|x|^{-s+1/2}\frac{dx}{|x|} = \int (\pi(w)\xi)(x)\rho^{-1}(x)|x|^{-s+1/2}\frac{dx}{|x|}\,.$$

We will begin by calculating $(\pi(w)\xi)(x)$, for $\xi = \xi_N$. We choose N large enough so that μ_1 is trivial on $\mathscr{U}_N$. Using (2.5.2), we see that

$$(\pi(w)\xi)(x) = \begin{cases} q^N|x|^{1/2}\mu_1(x)\int_{\mathscr{U}_N} K(u, x)du, & \text{if } \nu(x) \equiv 0(2) \\ 0, & \text{if } \nu(x) \equiv 1(2)\,, \end{cases}$$

where $K(u, x) = J_{11}(u, x) + X(u, \tau)J_{14}(u, x)$. If $\nu(x) \equiv 0(2)$, Lemma 2.7.2 says that

$$K(u, x) = \begin{cases} 0, & \text{if } ux \in \varepsilon k^{\times 2} \\ |x|^{-1/2}[\bar{\chi}(2y) + \bar{\chi}(-2y)], & \text{if } ux = y^2\,. \end{cases}$$

We have $u \in \mathscr{U}_N \subset k^{\times 2}$, so $ux \in k^{\times 2} \Leftrightarrow x \in k^{\times 2}$, so $x \notin k^{\times 2}$ implies $(\pi(w)\xi)(x) = 0$.

Assume that $x \in k^{\times 2}$. If $x \in \mathcal{O}$, $K(u, x) = |x|^{-1/2}$, so

$$(\pi(w)\xi)(x) = q^N\mu_1(x)\int_{\mathscr{U}_N} 2du = 2\mu_1(x)\,.$$

If $x \notin \mathcal{O}$ and $\sqrt{x} \in \mathscr{P}^{-N}$, we write $x = s^2$, $u = v^2$, $v = 1 + z$, $z \in \mathscr{P}^N$. Then $\bar{\chi}(2\sqrt{ux}) = \bar{\chi}(2sv) = \bar{\chi}(2s(1+z)) = \bar{\chi}(2s)\bar{\chi}(2sz)$. But $2sz \in \mathcal{O}$, so $\bar{\chi}(2sz) = 1$ and $K(u, x) = |x|^{-1/2}\mu_1(x)[\bar{\chi}(2s) + \bar{\chi}(-2s)]$. Therefore, $(\pi(w)\xi)(x) = \mu_1(x)[\bar{\chi}(2\sqrt{x}) + \bar{\chi}(-2\sqrt{x})]$. If $\sqrt{x} \notin \mathscr{P}^{-N}$, write $x = s^2$, with $\nu(x) = -M < -N, M > 0$. Write $\mathscr{U}_N/\mathscr{U}_M = \{u_i\}$. Then

$$\int_{\mathscr{U}_N} \bar{\chi}(2\sqrt{x}\sqrt{u})du = \sum_i \int_{u_i\mathscr{U}_M} \bar{\chi}(2\sqrt{x}\sqrt{u})du = \sum_i \int_{\mathscr{U}_M} \bar{\chi}(2\sqrt{x}\sqrt{u_i}\sqrt{u})du\,.$$

For $u \in \mathscr{U}_M$, $u = v^2$, write $v = 1 + z$, $z \in \mathscr{P}^M$. Then $\nu(2\sqrt{x}\sqrt{u_i}z) = \nu(\sqrt{x}z) \geq 0$, so $\bar{\chi}(2\sqrt{x}\sqrt{u_i}z) = 1$, and the sum becomes

$$\sum_i \int_{\mathscr{U}_M} \bar{\chi}(2\sqrt{x}\sqrt{u_i})du = q^{-M}\sum_i \bar{\chi}(2\sqrt{x}\sqrt{u_i})\,.$$

But $\{u_i\}$ are representatives for $\mathscr{U}_N/\mathscr{U}_M$, so the sum equals zero. We summarize this as:

LEMMA 2.9.1.

$$(\pi(w)\xi)(x) = \begin{cases} 0, & \textit{if } x \notin k^{\times 2} \\ \mu_1(x)[\bar{\chi}(2\sqrt{x}) + \bar{\chi}(-2\sqrt{x})], & \textit{if } \sqrt{x} \in \mathscr{P}^{-N} \\ 0, & \textit{if } \sqrt{x} \notin \mathscr{P}^{-N}. \end{cases}$$

Now we calculate

$$\gamma_{r_{\tilde{\mu}}}(s, 1) = \int (\pi(w)\xi)(x)\rho^{-1}(x)|x|^{-s+1/2}\frac{dx}{|x|}.$$

Let $\alpha(x) = \mu_1(x)\mu_2(x)$. It can be easily be shown that $\rho^2 = \alpha^2$. By Lemma 2.9.1,

$$\begin{aligned}
\gamma_{r_{\tilde{\mu}}}(s, 1) &= \int_{\mathscr{P}^{-N}\cap k^{\times 2}} \mu_1(x)[\bar{\chi}(2\sqrt{x}) + \bar{\chi}(-2\sqrt{x})]\rho^{-1}(x)|x|^{-s+1/2}\frac{dx}{|x|} \\
&= \sum_{n=0}^{\infty}\int_{\mathscr{U}^2} \alpha(\sqrt{\tau^{-2N+2n}})|\sqrt{\tau^{-2N+2n}}|^{1/2}[\bar{\chi}(2\sqrt{\tau^{-2N+2n}x}) + \bar{\chi}(-2\sqrt{\tau^{-2N+2n}x})] \\
&\quad \times \rho^{-1}(\tau^{-2N+2n}x)|\tau^{-2N+2n}x|^{-s+1/2}\frac{dx}{|x|} \\
&= \sum_{n=0}^{\infty} \alpha(\tau^{n-N})q^{(N-n)(-2s+3/2)}\rho^{-1}(\tau^{2n-2N}) \\
&\quad \times \int_{\mathscr{U}^2} \alpha(\sqrt{x})[\bar{\chi}(2\tau^{n-N}\sqrt{x}) + \bar{\chi}(-2\tau^{n-N}\sqrt{x})]\rho^{-1}(x)dx \\
&= \sum_{n=0}^{\infty} \alpha^{-1}(\tau^{n-N})q^{(N-n)(-2s+3/2)}\int_{\mathscr{U}} \alpha^{-1}(x)[\bar{\chi}(2\tau^{n-N}x) + \bar{\chi}(-2\tau^{n-N}x)]dx
\end{aligned}$$

But the integral is zero if $n \geq N$, so the sum $\sum_{n=0}^{\infty}$ can be replaced by $\sum_{n=0}^{N-1}$. Letting $m = n - N$, the sum becomes

$$\sum_{m=-N}^{-1} \alpha^{-1}(\tau^m)q^{-m(-2s+3/2)}\int_{\mathscr{U}} \alpha^{-1}(x)[\bar{\chi}(2\tau^m x) + \bar{\chi}(-2\tau^m x)]dx. \tag{2.9.1}$$

On the other hand, consulting p. 170 of [GPS], we see that

$$\begin{aligned}
\gamma_{r_{\tilde{\mu}}}(s, 1) &= \int_{k^\times} \alpha(x)|x|^{1/2}[\bar{\chi}(2x) + \bar{\chi}(-2x)]\alpha(x)^{-2}|x|^{1-2s}\frac{dx}{|x|} \\
&= \sum_{m=-\infty}^{\infty} \alpha^{-1}(\tau^m)q^{-m(-2s+3/2)}\int_U \alpha^{-1}(x)[\bar{\chi}(2\tau^m x) + \bar{\chi}(-2\tau^m x)]dx.
\end{aligned}$$

If $m \geq 0$ or $m < -N$, the integral equals zero, so we have

$$\sum_{m=-N}^{-1} \alpha^{-1}(\tau^m) q^{-m(-2s+3/2)} \int_{\mathfrak{U}} \alpha^{-1}(x)[\bar{\chi}(2\tau^m x) + \bar{\chi}(-2\tau^m x)]dx\,,$$

which equals (2.9.1), showing that our result agrees with that of [GPS].

§ 3, The three-sheeted cover

We will study the Kirillov model of the distinguished non-supercuspidal genuine representations of a 3-sheeted cover of GL_2. By [KP], such representations exist only if $c = 2$, so we place ourselves in this case from now on. In this chapter, $(\cdot, \cdot)$ will denote the cubic residue symbol.

3.1. Representations of $\tilde{A}$

Let

$$\tilde{A}_0 = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right) \middle| \nu(a), \nu(b) \equiv 0(3)\right\}.$$

$\tilde{A}_0$ is abelian. Given characters μ_1, μ_2 of $k^\times$, we define a genuine character of $\tilde{A}_0$ by $\tilde{\mu}_0\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right) = \theta(\zeta)\mu_1(a)\mu_2(b)$, where $\theta\colon \mu_3 \to \boldsymbol{C}$ is a non-trivial homomorphism defined on the third roots of unity. $\tilde{\mu}_0$ can be extended to

$$N(\tilde{\mu}_0) = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right) \middle| a \in k^\times,\ b = az \text{ for } z \in k^\times,\ \nu(z) \equiv 0(3)\right\}.$$

The extensions are given by:

$$\tilde{\mu}_0\left(s_i\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \zeta\right)\right) = \theta(\zeta)\delta^i \mu_2\mu_2(\tau^i)\mu_1(a)\mu_2(b), \qquad \text{where } \delta^3 = 1\,.$$

With respect to an ordered set of representatives of $N(\tilde{\mu}_0)$ in $\tilde{A}$, we form $\tilde{\mu} = \mathrm{Ind}_{N(\tilde{\mu}_0)}^{\tilde{A}}$. We will write $\tilde{\mu}(\mu_1, \mu_2, \theta)$ for $\tilde{\mu}$.

The representations $\tilde{\mu}(\mu_2, \mu_1, \theta)$ and $\tilde{\mu}(\mu_1, \mu_2, \theta)^w$ are equivalent.

3.2. The Kirillov model when $\gamma = 1/3$

We assume $\mu(x) = \mu_1\mu_2^{-1}(x) = |x|^\gamma$ and proceed as in section 2.2 to find an explicit intertwining operator J in the Fourier transform realization of the principal series. For each $y \in k^\times$, $J(y)$ is a 3×3 complex matrix.

Let $r_{\tilde{\mu}}$ be the image of $\hat{k}_{\tilde{\mu}}$ under the intertwining operator J when $\gamma = 1/3$. We define operators $\mathscr{C}$ and $\mathscr{E}$ as in the case $n = 2$: $(\mathscr{C}h)(x) = h_1(x)$, and $(\mathscr{E}f)(y) = |y|^{1/2}\tilde{\mu}^w\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}, 1\right)f(y)$. Let $K(\gamma_{\tilde{\mu}}) = \mathscr{C}\mathscr{E}J\hat{k}_{\tilde{\mu}}$. From calculations as in the proof of Proposition 2.4.1, we get

PROPOSITION 3.2.1. *For* $\phi \in K(r_{\tilde{\mu}})$, $\tilde{\pi}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, 1\right)\phi(x) = (a, x)\chi(bx)\phi(ax)$.

We also have the analogue of Proposition 2.4.2.

PROPOSITION 3.2.2. *Any* $\xi \in K(r_{\tilde{\mu}})$ *is a linear combination of characters* $(\alpha, y)\mu_1(y)$, *where* $\alpha \in k^\times/k^{\times 3}$. $\mathscr{S}(k^\times)$ *thus has codimension* $[k^\times : k^{\times 3}]$ *in* $K(r_{\tilde{\mu}})$.

3.3 The action of w in $K(r_{\tilde{\mu}})$

For $f \in K(r_{\tilde{\mu}})$, $(\hat{T}(w, 1)\mathscr{E}^{-1}\mathscr{C}^{-1}f)(t)$ equals

$$\int J_{\tilde{\mu}^w}(u, t)|u|^{-1/2}\tilde{\mu}^w\left(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}, 1\right)^{-1}\begin{pmatrix} f(u) \\ m'f(u) \\ 0 \end{pmatrix} du\,, \tag{3.5.1}$$

where $m' = \delta q^{-1/6}c(\tau^2)\mu_1^{-1}(\tau)$, $\delta^3 = 1$, and

$$J_{\tilde{\mu}^w}(u, t) = \int \tilde{\mu}^w\left(\begin{pmatrix} -x^{-1} & 0 \\ 0 & -x \end{pmatrix}, 1\right)\bar{\chi}\left(\frac{t}{x} + ux\right)\frac{dx}{|x|}\,.$$

The resulting matrix will be denoted $(J_{ij}(u, t))$.

A calculation shows that

$$J_{\tilde{\mu}^w}(u, t) = \begin{pmatrix} J_{00} & \delta^2\mu_1(\tau)J_{21} & \delta\mu_1(\tau^2)J_{12} \\ \delta\mu_1(\tau^{-1})J_{11} & J_{02} & \delta^2\mu_1(\tau)J_{22} \\ \delta^2\mu_1(\tau^{-2})J_{20} & \delta\mu_1(\tau^{-1})J_{10} & J_{01} \end{pmatrix},$$

where for $a \in k^\times$ and π a character of $k^\times$, $a\pi$ denotes the character $(a\pi)(x) = (a, x)\pi(x)$, and

$$J_{ab} = \sum_{k=0}^{2} \zeta^{-ak} J_{\varepsilon^k\tau^b\mu^{-1}}(u, t)\,.$$

Denoting $(\hat{T}(w)\mathscr{E}^{-1}\mathscr{C}^{-1}f)(t)$ by

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{pmatrix},$$

and noting that $(\tilde{\pi}(w)f)(t)$ is the first component of the function

$$|t|^{1/2}\tilde{\mu}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, 1\right)(\hat{T}(w)\mathscr{E}^{-1}\mathscr{C}^{-1}f)(t)\,,$$

we see that

$$(\tilde{\pi}(w)f)(t) = \begin{cases} |t|^{1/2}\mu_1(t)h_1(t), & \text{if } \nu(t) \equiv 0(3) \\ |t|^{1/2}\mu_1(t)(\tau^2, t)\mu_1(\tau^2)h_3(t), & \text{if } \nu(t) \equiv 1(3) \\ |t|^{1/2}\mu_1(t)(\tau, t)\mu_1(\tau)h_2(t), & \text{if } \nu(t) \equiv 2(3)\,. \end{cases}$$

By evaluating the entries of the matrix $J_{\tilde{\mu}w}(u, t)$, we may obtain more explicit expressions for $(\tilde{\pi}(w)f)(t)$. The final form, however, is rather unwieldy and does not seem to reduce to a nice expression as it does in the case of the 2-sheeted cover. It is thus unlikely that there is an analogue of the Weil representation in this case.

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