I. Kubo and Y. Yokoi
Nagoya Math. J.
Vol. 115 (1989), 139–149

A REMARK ON THE SPACE OF TESTING RANDOM VARIABLES IN THE WHITE NOISE CALCULUS

IZUMI KUBO AND YOSHITAKA YOKOI

Dedicated to Professor Takeyuki Hida on the occasion of his sixtieth birthday

§ 1. Introduction

The first author and S. Takenaka introduced the structure of a Gel'fand triplet $\mathscr{H} \subset (L^2) \subset \mathscr{H}^*$ into Hida's calculus on generalized Brownian functionals [4–7]. They showed that the space $\mathscr{H}$ of testing random variables has nice properties. For example, $\mathscr{H}$ is closed under multiplication of two elements in $\mathscr{H}$, each element of $\mathscr{H}$ is a continuous functional on the basic space $\mathscr{E}^*$, in addition it can be considered as an analytic functional, and moreover $\exp[t\Delta_V]$ (Δ_V is Volterra's Laplacian) is real analytic in $t \in \boldsymbol{R}$ as a one-parameter group of operators on $\mathscr{H}$, etc.

In this paper, we will prove, by a method different from [4–7], that each element of $\mathscr{H}$ is continuous on the basic space $\mathscr{E}^*$ and by using this result we will show that the evaluation map δ_x: $\varphi \mapsto \varphi(x)$ $(x \in \mathscr{E}^*)$ belongs to $\mathscr{H}^*$. The norm of δ_x will also be estimated.

The fact that δ_x belongs to $\mathscr{H}^*$ is very useful in the argument of positive functionals [8].

§ 2. Gel'fand triplets

Here we will summarize fundamental facts about three Gel'fand triplets $\mathscr{F} \subsetneqq \mathscr{F}^{(0)} \subsetneqq \mathscr{F}^*$, $\exp[\hat{\otimes}\mathscr{E}] \subsetneqq \exp[\hat{\otimes}E_0] \subsetneqq \exp[\hat{\otimes}\mathscr{E}^*]$ and $\mathscr{H} \subsetneqq (L^2) \subset \mathscr{H}^*$, which were introduced and discussed in [4–7, 9], for later use. Let T be a separable topological space with a topological Borel field $\mathscr{B}$ and ν be a σ-finite measure on T without atoms. We suppose that there exists a Gel'fand triplet (or a rigged Hilbert space) $\mathscr{E} \subset L^2(T, \nu) \subset \mathscr{E}^*$ (cf. [3]). Namely, the space $\mathscr{E}$ of testing functions on T is topologized by the pro-

Received August 28, 1987.

jective limit of Hilbert spaces $\{E_p\}_{p\in Z}$ with inner products $\{(\xi, \eta)_p; \xi, \eta \in \mathscr{E}\}_{p\in Z}$ such that

(G.1) $(\xi, \eta)_0 \equiv \int_T \xi(t)\eta(t)d\nu(t)$,

(G.2) the norms $\{\|\xi\|_p = ((\xi, \xi)_p)^{1/2}\}_{p\in Z}$ are consistent and increasing,

(G.3) E_{-p} is the dual space of E_p $(p \geq 0)$, and

(G.4) for any p there exists q $(>p)$ such that the injection mapping $\iota_{p,q}$: $E_q \to E_p$ is of Hilbert-Schmidt type.

The dual space $\mathscr{E}^*$ of $\mathscr{E}$ is the inductive limit of E_{-p} as $p \to \infty$. We denote by $\langle\cdot, \cdot\rangle$ the canonical bilinear forms between any dual pairs. Then obviously, $\langle\xi, \eta\rangle = (\xi, \eta)_0$ holds if $\xi, \eta \in \mathscr{E}$.

Further let us assume the following [A.1] and [A.2].

[A.1] There exists a constant $\rho \in (0, 1)$ such that

$$\rho\|\xi\|_{p+1} \geq \|\xi\|_p \qquad \text{for any } \xi \in \mathscr{E} \text{ and any } p \in \boldsymbol{Z}. \tag{2.1}$$

[A.2] *The evaluation map* δ_t: $\xi \mapsto \xi(t)$ gives a continuous map $t \mapsto \delta_t$ from T into E_{-1} with

$$\|\delta\|^2 \equiv \int_T \|\delta_t\|_{-1}^2 d\nu(t) < \infty. \tag{2.2}$$

Then [A.1] assures suitable analytical properties of nonlinear functionals which appear in these Gel'fand triplets. [A.2] assures that each testing function $\xi(t) \in \mathscr{E}$ is continuous and that the injection $\iota_{0,1}$ is of Hilbert-Schmidt type.

Since $\mathscr{E} \subsetneqq E_0 = L^2(T, \nu) \subsetneqq \mathscr{E}^*$ is a Gel'fand triplet, by Bochner-Minlos' theorem, we can find a probability measure μ on $\mathscr{E}^*$ such that

$$\int_{\mathscr{E}^*} \exp[i\langle x, \xi\rangle]d\mu(x) = \exp\left[-\frac{1}{2}\|\xi\|_0^2\right]. \tag{2.3}$$

Notice that the measure μ is full on E_{-1}, i.e. $\mu(E_{-1}) = 1$ by (2.3). Let us denote $L^2(\mathscr{E}^*, \mu)$ simply by (L^2).

Let $E_p^{\hat{\otimes}n}$ be the n-fold symmetric tensor product of E_p. By virtue of (G.2), we have natural inclusions $E_{p+1}^{\hat{\otimes}n} \subsetneqq E_p^{\hat{\otimes}n}$. Let $\mathscr{E}^{\hat{\otimes}n}$ denote the projective limit of $E_p^{\hat{\otimes}n}$ and $\mathscr{E}^{*\hat{\otimes}n}$ the inductive limit of $E_{-p}^{\hat{\otimes}n}$ as $p \to \infty$. We always associate *the inductive limit convex topology* with the inductive limit space. Here we remark the following Lemma, which implies the continuity of the mapping $\mathscr{E}^* \ni x \mapsto x^{\hat{\otimes}n} \in \mathscr{E}^{*\hat{\otimes}n}$.

LEMMA 2.1. *Fix a* $y \in \mathscr{E}^*$, *e.g.* $y \in E_{-q}$ *for some* $q \geq 0$, *and a neighbourhood* W *which is given by the absolutely convex envelope of the sets* $\{z \in E_{-p};\ \|z\|_{-p} < \gamma_p\}$, $p \geq q$ *with given* γ_p, $0 < \gamma_p \leq 1$. *Then for any* $x \in W + y$, *there exists a finite number of positive numbers* α_p, $q \leq p \leq N$, *with* $\sum_{p=q}^{N} \alpha_p \leq 1$ *such that* $x^{\hat{\otimes} n}$ *is expressed in the form*

$$x^{\hat{\otimes} n} = y^{\hat{\otimes} n} + \sum_{p=q}^{N} v_{n,p} \qquad \textit{with } \|v_{n,p}\|_{E_{-p}^{\hat{\otimes} n}} < n(1 + \|y\|_{-p})^{n-1}\alpha_p \gamma_p \tag{2.4}$$

for any $n \geq 1$.

Proof. Since any $x \in W + y$ can be written as $x = y + \sum_{p=q}^{N} \alpha_p z_p$ with $\sum_{p=q}^{N} \alpha_p \leq 1$, $\alpha_p > 0$ and $\|z_p\|_{-p} < \gamma_p$,

$$v_{n,p} \equiv \sum_{k=1}^{N} \binom{n}{k} \sum_{\max(p_1,\cdots,p_k)=p} \alpha_{p_1} \cdots \alpha_{p_k} z_{p_1} \hat{\otimes} \cdots \hat{\otimes} z_{p_k} \hat{\otimes} y^{\hat{\otimes}(n-k)}$$

$p \geq q$, satisfy the requirement. □

The orthogonal direct sum

$$\exp[\hat{\otimes} E_p] \equiv \sum_{n=0}^{\infty} \oplus (n!)^{1/2} E_p^{\hat{\otimes} n} \tag{2.5}$$

with inner product

$$((f_n)_{n\geq 0}, (g_n)_{n\geq 0})_{\exp[\hat{\otimes} E_p]} = \sum_{n=0}^{\infty} n!(f_n, g_n)_{E_p^{\hat{\otimes} n}} \tag{2.6}$$

is called *a Fock's space*. Its dual space is $\exp[\hat{\otimes} E_{-p}]$ with the canonical bilinear form

$$\langle (G_n)_{n\geq 0}, (f_n)_{n\geq 0} \rangle = \sum_{n=0}^{\infty} n! \langle G_n, f_n \rangle \tag{2.7}$$

for $(G_n)_{n\geq 0} \in \exp[\hat{\otimes} E_{-p}]$ and $(f_n)_{n\geq 0} \in \exp[\hat{\otimes} E_p]$, $(p \geq 0)$. Again by virtue of (G.2), we have natural inclusions $\exp[\hat{\otimes} E_{p+1}] \subsetneqq \exp[\hat{\otimes} E_p]$ for $p \in \mathbf{Z}$. We denote by $\exp[\hat{\otimes} \mathscr{E}]$ the projective limit of $\exp[\hat{\otimes} E_p]$ and by $\exp[\hat{\otimes} \mathscr{E}^*]$ the inductive limit of $\exp[\hat{\otimes} E_{-p}]$ as $p \to \infty$, respectively.

PROPOSITION 2.2. (a) *The triplet* $\exp[\hat{\otimes} \mathscr{E}] \subsetneqq \exp[\hat{\otimes} E_0] \subsetneqq \exp[\hat{\otimes} \mathscr{E}^*]$ *is a Gel'fand triplet.*
(b) *The mapping from* $\mathscr{E}^*$ *to* $\exp[\hat{\otimes} \mathscr{E}]$ *defined by*

$$\mathscr{E}^* \ni x \longmapsto \exp[\hat{\otimes} x] \equiv \sum_{n=0}^{\infty} \oplus \frac{1}{n!} x^{\hat{\otimes} n} \in \exp[\hat{\otimes} \mathscr{E}^*]$$

is continuous.

(c) *For* $(g_n)_{n\geq 0} \in \exp[\hat{\otimes}\mathscr{E}]$, *define a functional* $\Psi(x)$ *on* $\mathscr{E}^*$ *by*

$$\Psi(x) \equiv \sum_{n=0}^{\infty} \langle g_n, x^{\hat{\otimes} n} \rangle .$$

Then $\Psi(x)$ *is a continuous functional on* $\mathscr{E}$.

(d) *For* $(G_n)_{n\geq 0} \in \exp[\mathscr{E}^*]$, *define a functional* $U(\xi)$ *on* $\mathscr{E}$ *by*

$$U(\xi) \equiv \sum_{n=0}^{\infty} \langle G_n, \xi^{\hat{\otimes} n} \rangle . \tag{2.8}$$

Then $U(\xi)$ *is a continuous functional on* $\mathscr{E}$.

Proof. (a) is seen in [4] by (2.1). (b) Fix a $y \in \mathscr{E}$ and let q be a natural number such that $y \in E_{-q}$. For a given absolutely convex neighbourhood V of the origin of $\exp[\mathscr{E}^*]$ of the form

$$V = \operatorname{conv}\Big(\bigcup_{p\geq q} \{z;\ \|z\|_{\exp[\hat{\otimes}E_{-p}]} < \varepsilon_p\}\Big) ,$$

put $\gamma_p \equiv \min\{\varepsilon_p \exp[-(1+\|y\|_{-p})^2],\ 1\}$ and let W be the neighbourhood in Lemma 2.1. Then by (2.4), for $x \in W + y$ we have the expression

$$\exp[\hat{\otimes}x] - \exp[\hat{\otimes}y] = \sum_{q\leq p\leq N} \Big(\sum_{n=1}^{\infty} \oplus \frac{1}{n!} v_{n,p}\Big)$$

with norms

$$\Big\|\sum_{n=1}^{\infty} \oplus \frac{1}{n!} v_{n,p}\Big\|_{\exp[\hat{\otimes}E_{-p}]} = \Big(\sum_{n=1}^{\infty} \frac{n!}{(n!)^2} \|v_{n,p}\|^2_{E_{-p}^{\hat{\otimes} n}}\Big)^{1/2} < \alpha_p \varepsilon_p .$$

Hence $\exp[\hat{\otimes}x] \in V + \exp[\hat{\otimes}y]$ for any $x \in W + y$. Thus (b) is proved. By (b), (c) is obvious since $(g_n)_{n\geq 0}$ is a continuous linear functional on $\exp[\hat{\otimes}\mathscr{E}^*]$ and since $\Psi(x) = \langle (g_n)_{n\geq 0}, \exp[\hat{\otimes}x]\rangle$, (d) is easier to prove. □

Let $\mathscr{F}$ (resp, $\mathscr{F}^{(p)}$, $\mathscr{F}^*$) be the image space of $\exp[\hat{\otimes}\mathscr{E}]$ (resp. $\exp[\hat{\otimes}E_p]$, $\exp[\hat{\otimes}\mathscr{E}^*]$) under the mapping (2.8) and introduce a topology from the original space. Then $\mathscr{F}^{(p)}$ is the reproducig kernel Hilbert space with the reproducing kernel $\exp[(\xi, \eta)_{-p}]$. The following Propositions are shown in [4].

PROPOSITION 2.3. (a) $\mathscr{F} \subset \mathscr{F}^{(0)} \subset \mathscr{F}^*$ *is a Gel'fand triplet.*

(b) *Let* ξ, *and* ζ *be in* $\mathscr{E}$ *and* n, m *be non-negative integers. Then* $\langle \xi, \eta\rangle^m$ *and* $\langle \xi, \zeta\rangle^n$ *belong to* $\mathscr{F}^{(p)}$ *and satisfy the equality*

$$(\langle \xi, \eta\rangle^m, \langle \xi, \zeta\rangle^n)_{\mathscr{F}^{(p)}} = \delta_{m,n} n!(\eta, \zeta)_p^n \qquad \textit{for any } p \in \mathbf{Z} .$$

PROPOSITION 2.4. *For each fixed* $\xi \in \mathscr{E}$, *write*

$$f(\xi) = f(\xi; x) \equiv \exp\left[\langle x, \xi\rangle - \frac{1}{2}\|\xi\|_0^2\right]. \tag{2.9}$$

Then the mapping $\mathscr{S}$ *defined by*

$$(\mathscr{S}\varphi)(\xi) \equiv \int_{\mathscr{E}^*} \varphi(x) f(\xi, x) d\mu(x) = \int_{\mathscr{E}^*} \varphi(x+\xi) d\mu(x) \tag{2.10}$$

is an isomorphism from (L^2) *onto* $\mathscr{F}^{(0)}$. *Especially,*

$$(\mathscr{S}f(\eta))(\xi) = \exp[\langle \eta, \xi\rangle] \qquad \text{for any } \xi, \eta \in \mathscr{E} \tag{2.11}$$

and

$$\mathscr{S}: H_n(\langle x, \eta\rangle; \|\eta\|^2) \longmapsto \langle \xi, \eta\rangle^n, \tag{2.12}$$

where $H_n(z; \gamma)$ $(n = 0, 1, 2, \cdots)$ *are the Hermite polynomials with parameter* γ *defined by the generating function* $\exp\left[\omega z - \frac{\gamma}{2}\omega^2\right]$;

$$\sum_{n=0}^{\infty} \frac{1}{n!}\omega^n H_n(z; \gamma) \equiv \exp\left[\omega z - \frac{\gamma}{2}\omega^2\right]. \tag{2.13}$$

Put $\mathscr{H}^{(p)} \equiv \mathscr{S}^{-1}(\mathscr{F}^{(p)})$ for $p \geq 0$ and $\mathscr{H} \equiv \mathscr{S}^{-1}(\mathscr{F})$ and introduce inner products by

$$(\varphi, \psi)_{\mathscr{H}^{(p)}} \equiv (\mathscr{S}\varphi, \mathscr{S}\psi)_{\mathscr{F}^{(p)}}$$

in $\mathscr{H}^{(p)}$. Let $\mathscr{H}^{(-p)}$ be the dual of $\mathscr{H}^{(p)}$ for $p \geq 1$, and $\mathscr{H}$ (resp. $\mathscr{H}^*$) be the projective (resp. inductive) limit of $\mathscr{H}^{(p)}$. We call $\mathscr{H}$ the space of *testing random variables* and $\mathscr{H}^*$ the space of *generalized random variables.*

PROPOSITION 2.5. *For any* $\xi \in \mathscr{E}$, $f(\xi; x)$ *is in* $\mathscr{H}$ *and the mapping* $\mathscr{S}$ *is extended on* $\mathscr{H}^*$ *by*

$$(\mathscr{S}\Psi)(\xi) = \langle \Psi(x), f(\xi; x)\rangle. \tag{2.14}$$

Then $\mathscr{S}$ *gives the isomorphism from* $\mathscr{H} \subsetneqq (L^2) \subsetneqq \mathscr{H}^*$ *to* $\mathscr{F} \subsetneqq \mathscr{F}^{(0)} \subsetneqq \mathscr{F}^*$. *Namely,* $\mathscr{H}^{(p)}$ *is isomorphic to* $\mathscr{F}^{(p)}$ *through* $\mathscr{S}$ *for any* $p \in \boldsymbol{Z}$.

PROPOSITION 2.6. *For* $p \geq 0$, *the isomorphism*

$$\exp[\hat{\otimes} E_p] \ni (f_n)_{n\geq 0} \longmapsto \varphi \in \mathscr{H}^{(p)}$$

is given by

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \qquad \|\varphi\|_{\mathscr{H}^{(p)}} = \|(f_n)_{n\geq 0}\|_{\exp[\hat{\otimes} E_p]}, \tag{2.15}$$

where $I_n(f_n)$ *is the multiple Wiener-Itô integral*

$$I_n(f_n) = \int \cdots \int_{T^n} f_n(t_1, \cdots, t_n) W(dt_1) \cdots W(dt_n) \tag{2.16}$$

with respect to the Gaussian white noise $W(dt)$ *given by the relation*

$$\langle x, \xi \rangle = \int_T \xi(t) W(dt, x) \ a.s. \ x \in \mathscr{E}^* \ (\mu). \tag{2.17}$$

§ 3. The space $\mathscr{H}$ of testing ramdom variables

In [4–7], it was shown that the multiplication $\varphi, \psi \mapsto \varphi \cdot \psi$ is continuous as the mapping from $\mathscr{H} \times \mathscr{H}$ into $\mathscr{H}$. Further each element of $\varphi \in \mathscr{H}$ is continuous functional on $\mathscr{E}^*$. More surprising thing is that each $U(\xi) \in \mathscr{F}$ can be extended to a continuous functional $\tilde{U}(x)$ on $\mathscr{E}^*$ and the class $\{\tilde{U}(x); U(\xi) \in \mathscr{F}\}$ coincides with $\mathscr{H}$. Those results were proved in a very complicated way with the help of Volterra's Laplacian.

Here we prove the continuity in $x \in \mathscr{E}^*$ for every functional $\varphi(x) \in \mathscr{H}$ and the continuity of the evaluation map:

$$\delta_x: \mathscr{H} \ni \varphi \longmapsto \varphi(x) \in \boldsymbol{R}, \tag{3.1}$$

directly by using basic results.

Firstly, we prove that the multiple Wiener-Itô integral $I_n(f_n)$ has a continuous version as a functional on $\mathscr{E}^*$ if f_n is a good function.

THEOREM 3.1. *For* $f_n \in \mathscr{E}^{\hat{\otimes} n}$,

$$I_n(f_n)(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n! 2^{-k}}{(n-2k)! k!} \langle x^{\hat{\otimes}(n-2k)}, f_{n|n-2k} \rangle \quad a.s. \ x \in \mathscr{E}^*, \tag{3.2}$$

where

$$\begin{aligned} &f_{n|n-2k}(t_1, \cdots t_{n-2k}) \\ &\quad \equiv \int \cdots \int_{T^k} f_n(t_1, \cdots, t_{n-2k}, u_1, u_1, \cdots, u_k, u_k) d\nu(u_1) \cdots d\nu(u_k). \end{aligned} \tag{3.3}$$

Proof. We denote by $\mathscr{I}_n(f_n)$ the right hand side of (3.2) for $f_n \in \mathscr{E}^{\hat{\otimes} n}$. Then it is a continuous (non-linear) functional of $x \in \mathscr{E}^*$ because of Lemma 2.1 and of the following estimation:

$$\begin{aligned} &\|f_{n|n-2k}\|_{E_p^{\hat{\otimes}(n-2k)}} \\ &\quad \leqq \int \cdots \int_{T^k} \|f_n(t_1, \cdots, t_{n-2k}, u_1, u_1, \cdots, u_k, u_k)\|_{E_p^{\hat{\otimes}(n-2k)}} d\nu(u_1) \cdots d\nu(u_k) \end{aligned} \tag{3.4}$$

$$\leq \int \cdots \int_{T^k} \|f_n\|_{E_p^{\hat{\otimes} n}} \|\delta_{u_1}\|_{-1}^2 \cdots \|\delta_{u_k}\|_{-1}^2 \rho^{2(p-1)k} d\nu(u_1) \cdots d\nu(u_k)$$
$$\leq \|f_n\|_{E_p^{\hat{\otimes} n}} (\|\delta\| \rho^{p-1})^{2k} .$$

Consequently, for $x \in E_{-p}$, we have

$$(3.5) \qquad \begin{aligned} |\mathscr{I}_n(f_n)(x)| &\leq \sum_{k=0}^{[n/2]} \frac{n! 2^{-k}}{(n-2k)!k!} \|x\|_{-p}^{n-2k} \|f_{n|n-2k}\|_{E_p^{\hat{\otimes}(n-2k)}} \\ &\leq \sum_{k=0}^{[n/2]} \frac{n! 2^{-k}}{(n-2k)!k!} \|x\|_{-p}^{n-2k} (\|\delta\| \rho^{p-1})^{2k} \|f_n\|_{E_p^{\hat{\otimes} n}} \\ &\leq \sqrt{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \|x\|_{-p}^{n-k} (\|\delta\| \rho^{p-1})^k \|f_n\|_{E_p^{\hat{\otimes} n}} \\ &\leq \sqrt{n!}\, (\|x\|_{-p} + \|\delta\| \rho^{p-1})^n \|f_n\|_{E_p^{\hat{\otimes} n}} , \end{aligned}$$

by $2^{-k}/k! = (2k-1)!!/(2k)! \leq \sqrt{n!}/(2k)!$ for $2k \leq n$. Since $\mathscr{I}_n(f_n)$ is linear in f_n, $\mathscr{I}_n(f_n^{(j)})$ converges to $\mathscr{I}_n(f_n)$ uniformly on any bounded set B of $\mathscr{E}^*$, if $f_n^{(j)} \to f_n$ in $\mathscr{E}^{\hat{\otimes} n}$.

First consider the case $f_n = \eta(t_1) \cdots \eta(t_n)$. Then the equality $I_n(f_n) = H_n(\langle x, \eta \rangle, \|\eta\|_0^2)$ is well known (actually it is shown by Propositions 2.4 and 2.6). Since the equality

$$\begin{aligned} \langle x^{\hat{\otimes}(n-2k)}, f_{n|n-2k} \rangle &= \langle x^{\hat{\otimes}(n-2k)}, \|\eta\|_0^{2k} \eta(t_1) \cdots \eta(t_{n-2k}) \rangle \\ &= \|\eta\|_0^{2k} \langle x, \eta \rangle^{n-2k} \end{aligned}$$

holds, (3.2) is obvious in this case by the formula of the Hermite polynomials;

$$H_n(z; \gamma) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n! (\gamma/2)^k}{(n-2k)!k!} z^{n-2k} \quad \text{(see p. 193 [11]).}$$

For a general f_n in $\mathscr{E}^{\hat{\otimes} n}$, there exists a sequence of the form $\{f_n^{(j)} = \sum_l c_{j,l} (\eta_l^{(j)})^{\hat{\otimes} n}\}_{j=1}^\infty$ which converges to f_n in $\mathscr{E}^{\hat{\otimes} n}$. Then $I_n(f_n^{(j)}) = \mathscr{I}(f_n^{(j)})$ holds a.s. $x \in \mathscr{E}^*$ and $\mathscr{I}_n(f_n^{(j)})$ converges to $\mathscr{I}_n(f_n)$ for every $x \in \mathscr{E}^*$. Since

$$\|I_n(f_n^{(j)}) - I_n(f_n)\|_{(L^2)} = \sqrt{n!} \|f_n^{(j)} - f_n\|_{E_0^{\hat{\otimes} n}} ,$$

a suitable subsequence of $I_n(f_n^{(j)})$ converges to $I_n(f_n)$ a.s. This implies that $I_n(f_n) = \mathscr{I}_n(f_n)$ a.s. $x \in \mathscr{E}^*$. □

Now we are ready to prove our main theorem:

THEOREM 3.2. *For any $\varphi \in \mathscr{H}$, φ has a continuous version $\varphi(x)$ and it is bounded on each bounded set of $\mathscr{E}^*$. Moreover the evaluation map*

δ_x: $\varphi \to \varphi(x)$ is a continuous linear functional on $\mathscr{H}$, i.e., $\delta_x \in \mathscr{H}^*$ for any $x \in \mathscr{E}^*$.

Proof. For $\varphi \in \mathscr{H}$, let $(f_n)_{n\geq 0}$ be the element of $\exp[\hat{\otimes}\mathscr{E}]$ satisfying (2.15) in Proposition 2.6. Put

$$g_m \equiv \sum_{k=0}^{\infty}(-1)^k \frac{(m+2k)!\,2^{-k}}{m!\,k!} f_{m+2k|m}\,.$$

Then $(g_m)_{m\geq 0}$ belongs to $\exp[\hat{\otimes}\mathscr{E}]$, because

$$\begin{aligned}
\|(g_m)_{m\geq 0}\|_{\exp[\hat{\otimes}E_{-p}]} &\leq \sum_{m=0}^{\infty}\sqrt{m!}\,\|g_m\|_{E_{-p}^{\hat{\otimes}m}} \\
&\leq \sum_{m=0}^{\infty}\sqrt{m!}\left(\sum_{k=0}^{\infty}\frac{(m+2k)!\,2^{-k}}{m!\,k!}(\|\delta\|\rho^{p-1})^{2k}\|f_{m+2k}\|_{E_{-p}^{\hat{\otimes}(m+2k)}}\right) \\
&\leq \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{\sqrt{n!}\,2^{-k}}{\sqrt{(n-2k)}\,k!}(\|\delta\|\rho^{p-1})^{2k}\sqrt{n!}\,\|f_n\|_{E_{-p-r}^{\hat{\otimes}n}}\rho^{rn} \\
&\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\sqrt{n!}}{\sqrt{(n-k)}\,k!}\|\delta\|^k\sqrt{n!}\,\|f_n\|_{E_{-p-r}^{\hat{\otimes}n}}\rho^{rn} \\
&\leq \sum_{n=0}^{\infty}(1+\|\delta\|)^n\rho^{rn}\sqrt{n!}\,\|f_n\|_{E_{-p-r}^{\hat{\otimes}n}} \\
&\leq (1-(1+\|\delta\|)^2\rho^{2r})^{-1/2}\|(f_n)_{n\geq 0}\|_{\exp[\hat{\otimes}E_{-p-r}]}
\end{aligned}$$

for sufficiently large r as $(1+\|\delta\|)\rho^r < 1$, by $\sqrt{(2k)!} \leq 2^k k!$ and $1 \leq \frac{n!}{(n-k)!\,k!}$. By Theorem 3.1 and the definition of $\mathscr{I}_n(f_n)$, we see that

$$\tilde{\varphi}(x) \equiv \sum_{n=0}^{\infty}\mathscr{I}_n(f_n)(x) = \langle (g_m)_{m\geq 0}, \exp[\hat{\otimes}x]\rangle \tag{3.6}$$

and $\varphi(x) = \tilde{\varphi}(x)$ a.s. μ. By Proposition 2.2 (c), $\tilde{\varphi}(x)$ is a continuous functional on $\mathscr{E}^*$. By (3.5),

$$|\tilde{\varphi}(x)| \leq \left|\sum_{n=0}^{\infty}\mathscr{I}_n(f_n)(x)\right| \leq (1-(\|x\|_{-p}+\|\delta\|\rho^{p-1})^2)^{-1/2}\|\varphi\|_{\mathscr{H}^{(p)}}$$

holds for sufficiently large p as $\|x\|_{-p}+\|\delta\|\rho^{p-1} < 1$. This shows that the evaluation map δ_x belongs to $\mathscr{H}^*$. □

From now on, $\varphi(x)$ (for $\varphi \in \mathscr{H}$) is always considered as the continuous version.

§ 4. The evaluation map δ_x

We have seen that δ_y belongs to $\mathscr{H}^*$, if $y \in \mathscr{E}^*$. Therefore δ_y must belong to $\mathscr{H}^{(-p)}$ for some $p = p(y) \geq 0$ and its image under $\mathscr{S}$ can be observed. By (2.14) in Proposition 2.5, we have

$$(\mathscr{S}\delta_y)(\xi) = \langle \delta_y, f(\xi;\cdot)\rangle = f(\xi;y) \qquad \text{for } \xi \in \mathscr{E}. \tag{4.1}$$

Since $\mathscr{S}$ is an isomorphism from $\mathscr{H}^{(-p)}$ to $\mathscr{F}^{(-p)}$, we can estimate the norm of δ_y by computing $\|f(\xi;y)\|_{\mathscr{F}^{(-p)}}$ directly.

Suppose that $y \in E_{-p}$, $p \geq 1$. Since the injection $\iota_{0,p}$ is of Hilbert-Schmidt type, there exists a c.o.n.s. $\{\zeta_j\}_{j=1}^{\infty}$ of E_0 such that $\{\zeta_j\}_{j=1}^{\infty} \subset E_p$ and $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$ for $\lambda_j^2 \equiv \|\zeta_j\|_{-p}^2$. For $\xi \in \mathscr{E}$, we have

$$\begin{aligned} f(\xi;y) &= \exp\left[\langle y,\xi\rangle - \frac{1}{2}\|\xi\|_0^2\right] = \sum_{j=1}^{\infty}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\langle \zeta_j^{\hat{\otimes} n}, \xi^{\hat{\otimes} n}\rangle H_n(\langle y,\zeta_j\rangle)\right) \\ &= \sum_{n=0}^{\infty}\sum_{n=n_1+\cdots+n_j+\cdots}\prod_{j=1}^{\infty}\frac{1}{n_j!}\langle \zeta_j^{\hat{\otimes} n_j}, \xi^{\hat{\otimes} n_j}\rangle H_{n_j}(\langle y,\zeta_j\rangle). \end{aligned}$$

Hence we have, for y any $z \in E_{-p}$,

$$\begin{aligned} &(f(\cdot\,;y), f(\cdot\,;z))_{\mathscr{F}^{(-p)}} \\ &= \sum_{n=0}^{\infty}\sum\prod_{j=1}^{\infty}\frac{1}{n_j!}\lambda_j^{2n_j}H_{n_j}(\langle y,\zeta_j\rangle)\cdot H_{n_j}(\langle z,\zeta_j\rangle) \\ &= \prod_{j=1}^{\infty}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\lambda_j^{2n}H_n(\langle y,\zeta_j\rangle)\cdot H_n(\langle z,\zeta_j\rangle)\right) \\ &= \prod_{j=1}^{\infty}(1-\lambda_j^4)^{-1/2} \\ &\quad\times \prod_{j=1}^{\infty}\exp\left[-\frac{1}{2}\frac{\lambda_j^4\langle y,\zeta_j\rangle^2 - 2\lambda_j^2\langle y,\zeta_j\rangle\langle z,\zeta_j\rangle + \lambda_j^4\langle z,\zeta_j\rangle^2}{1-\lambda_j^4}\right] \\ &\leq \prod_{j=1}^{\infty}(1-\lambda_j^4)^{-1/2}\exp\left[\frac{1}{2}(\|y\|_{-p}^2 + \|z\|_{-p}^2)\right] \end{aligned} \tag{4.2}$$

by Proposition 2.3 and the formula

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(u)H_n(v) = (1-t^2)^{-1/2}\exp\left[-\frac{1}{2}\frac{t^2u^2 - 2tuv + t^2v^2}{1-t^2}\right] \tag{4.3}$$

with $H_n(u) = H_n(u;1)$ (see [11] p. 194]. In particular,

$$\begin{aligned} \|f(\cdot\,;y)\|_{\mathscr{F}^{(p)}}^2 &= \prod_{j=1}^{\infty}\left((1-\lambda_j^4)^{-1/2}\exp\left[\frac{\lambda_j^2\langle y,\zeta_j\rangle^2}{1+\lambda_j^2}\right]\right) \\ &\leq \prod_{j=1}^{\infty}(1-\lambda_j^4)^{-1/2}\exp\left[\|y\|_{-p}^2\right]. \end{aligned} \tag{4.4}$$

Summarizing the above computations, we have:

Theorem 4.1. *The generalized random variable δ_y has the following* *properties*;

(a) $(\mathscr{S}\delta_y)(\xi) = f(\xi; y) = \exp\left[\langle y, \xi\rangle - \frac{1}{2}\|\xi\|_0^2\right]$,

(b) $(\delta_y, \delta_z)_{\mathscr{H}^{(-p)}} = \prod_{j=1}^{\infty}(1-\lambda_j^4)^{-1/2}$

$$\times \prod_{j=1}^{\infty}\exp\left[-\frac{1}{2}\frac{\lambda_j^4\langle y, \zeta_j\rangle^2 - 2\lambda_j^2\langle y, \zeta_j\rangle\langle z, \zeta_j\rangle + \lambda_j^4\langle z, \zeta_j\rangle^2}{1-\lambda_j^4}\right],$$

(c) $\|\delta_y\|_{\mathscr{H}^{(-p)}} \le \exp\left[\frac{1}{2}\|\iota_{0,p}\|_{H.S.}^2\right]\exp\left[\frac{1}{2}\|y\|_{-p}^2\right]$ *if* $y \in E_{-p}$,

(d) $\int_{\mathscr{E}^*}\|\delta_y\|_{\mathscr{H}^{(-p)}}^2 d\mu(y) = \|\iota_{(L^2),\mathscr{H}^{(p)}}\|_{H.S.}^2$.

Proof. The only thing we still have to prove is (d). By (2.2) the injection $\iota_{0,1}$ from E_1 into E_0 is of Hilbert-Schmidt type. By Sazonov's theorem, the support of the measure μ is E_{-1}. Hence the integral in (d) is taken over E_{-1}. Since $\{\langle y, \zeta_j\rangle; j = 1, 2, \cdots\}$ are independent of each other with respect to μ, we can easily calculate;

(4.5) $$\int_{E_{-1}}\|\delta_y\|_{\mathscr{H}^{(-p)}}^2 d\mu(y) = \prod_{j=1}^{\infty}(1-\lambda_j^2)^{-1}.$$

The left hand side is equal to the Hilbert-Schmidt operator norm of the injection $\iota_{(L^2),\mathscr{H}^{(p)}}$ by the proof of Proposition 3.6 in [9]. □

In [7], the renormalization : : has been introduced. By the notation used in it we may write

(4.6) $$\delta_y(x) = :\exp\left[\langle y, x\rangle\cdot - \frac{1}{2}\int_T (x(t)\cdot)^2 d\nu(t)\right]:1,$$

because the right hand side is defined by

$$\mathscr{S}^{-1}\left(\exp\left[\langle y, \xi\rangle - \frac{1}{2}\int_T \xi(t)^2 d\nu(t)\right]\right).$$

Acknowledgement. The authors give their hearty thanks to the referee for his suggestions to improve the manuscript.

References

[1] Hida, T., Analysis of Brownian functionals, Carleton Math. Lec. Notes, No. 13, 2nd Ed. (1978).
[2] Gel'fand, I. M. and Shilov, G. E., Generalized functions, Vol. 2. Academic Press.
[3] Gel'fand, I. M. and Vilenkin, N. Ya., Generalized functions, Vol. 4. Academic Press.

[4] Kubo, I. and Takenaka, S., Calculus on Gaussian white noises I, Proc. Japan Acad., **56**, Ser. A, No. 8, (1980), 376–380.
[5] ——, Calculus on Gaussian white noises II, Proc. Japan Acad., **56**, Ser. A, No. 9, (1980), 411–416.
[6] ——, Calculus on Gaussian white noises III, Proc. Japan Acad., **57**, Ser. A, No. 9 (1981), 433–437.
[7] ——, Calculus on Gaussian white noises IV, Proc. Japan Acad., **58**, Ser. A, No. 9 (1982), 186–189.
[8] Yokoi, Y., Private notes for positivity of generalized Brownian functionals.
[9] Ito, Y. and Kubo, I., Calculus on Gaussian and Poisson white noises, (submitted to Nagoya Math. J.).
[10] Kuo, H.-H., Gaussian measures in Banach spaces, Lect. Notes in Math. Vol. 463, Springer-Verlag, (1975).
[11] Erdélyi, A., Higher transcendental functions, Vol. 2 (1953), McGraw-Hill.

I. Kubo
Faculty of Integrated Arts and Sciences
Hiroshima University
Hiroshima 730, Japan

Y. Yokoi
Department of Mathematics
Faculty of General Education
Kumamoto University
Kumamoto 860, Japan