T. Ono
Nagoya Math. J.
Vol. 107 (1987), 121-133

ON SOME CLASS NUMBER RELATIONS FOR GALOIS EXTENSIONS

TAKASHI ONO

Introduction

Let k be an algebraic number field of finite degree over $\boldsymbol{Q}$, the field of rationals, and K be an extension of finite degree over k. By the use of the class number of algebraic tori, we can introduce an arithmetical invariant $E(K/k)$ for the extension K/k. When $k = \boldsymbol{Q}$ and K is quadratic over $\boldsymbol{Q}$, the formula of Gauss on the genera of binary quadratic forms, i.e. the formula $h_K^+ = h_K^* 2^{t_K - 1}$, where $h_K^+ =$ the class number of K in the narrow sense, $h_K^* =$ the number of classes is a genus of the norm form of $K/\boldsymbol{Q}$ and $t_K =$ the number of distinct prime factors of the discriminant Δ_K of K, may be considered as an equality between $E(K/\boldsymbol{Q})$ and other arithmetical invariants of K.

In this paper, we shall obtain, for any galois extension K/k, an equality between $E(K/k)$ and some elementary cohomological invariants of K/k. Therefore our formula may be viewed as a generalization of the formula of Gauss. When K/k is cyclic, our formula for $E(K/k)$ bears a resemblance to a formula which gives the number of ambiguous classes of ideals but, in my opinion, $E(K/k)$ is easier than the other unless $k = \boldsymbol{Q}$ and $K/\boldsymbol{Q}$ is quadratic.[1)]

Roughly speaking, the general formula shows that there is another balance between the class number and the group of (local or global) units of number fields. Our proof which depends on class number formulas, isogenies and Tamagawa numbers of tori, is winding. We feel that it should eventually be replaced by a direct proof using methods provided, perhaps, by the algebraic K-theory.

The main results of this paper have been summarized in Proc. Japan

Received April 7, 1986.

1) See, e.g. the formula on page 406, line 12 of [C$_1$]. See also the 3rd paragraph of the Introduction of [C$_2$] where Chevalley alludes to an application of algebraic groups to the arithmetic of number fields.

Acad., 61A, 311–312 (1985).

§ 1. Definition of $E(K/k)$

To begin with, we recall the definition of the class number of a torus. Let k be an algebraic number field of finite degree over $\boldsymbol{Q}$, k_v be the completion of k at a place v of k. When v is non-archimedean, we often put $v = \mathfrak{p}$ and denote by $\mathfrak{o}_\mathfrak{p}$ the maximal compact subring of $k_\mathfrak{p}$. We denote by S_∞ the set of all archimedean places of k. For $v \in S_\infty$ we occasionally put $\mathfrak{o}_v = k_v$ for notational convenience. The adele ring of k will be written k_A. In general, for a ring with 1, we denote by $R^\times$ the group of units, i.e. the group of invertible elements of R. We write G_m for the multiplicative group of the universal domain.

Let T be a torus defined over k. We refer to [O$_1$], [O$_2$] for standard facts on tori defined over number fields. Denote by $T(k)$, $T(k_v)$ the subgroups of T of points rational over k, k_v, respectively. The adele group of T over k will be written $T(k_A)$. The unique maximal compact subgroup of $T(k_\mathfrak{p})$ is described as:

$$T(\mathfrak{o}_\mathfrak{p}) = \{x \in T(k_\mathfrak{p});\ \xi(x) \in \mathfrak{o}_\mathfrak{p}^\times \text{ for all } \xi \in \hat{T}(k_\mathfrak{p})\}$$

where $\hat{T} = \mathrm{Hom}(T, G_m)$, the character module of T, and $\hat{T}(k_\mathfrak{p})$ is the submodule of T of characters defined over $k_\mathfrak{p}$. Now, we put

$$T(k_A)_\infty \overset{\text{def}}{=} \prod_{v \in S_\infty} T(k_v) \times \prod_{\mathfrak{p}} T(\mathfrak{o}_\mathfrak{p})$$

and define the class number h_T of T over k by

$$h_T \overset{\text{def}}{=} [T(k_A) \colon T(k)T(k_A)_\infty]\,,$$

where $[A : B]$ means the index of a group A over a subgroup B. We remind the reader that the group $T(\mathfrak{o}_k)$ of units of T over k is defined by $T(\mathfrak{o}_k) = T(k) \cap T(k_A)_\infty$.

Let K be a finite extension of k. When $T = R_{K/k}(G_m)$, the torus obtained from G_m by the restriction of the field definition from K to k, then $h_T = h_K$, the usual class number of K. It is natural to consider the exact sequence of tori over k:

$$0 \longrightarrow R^{(1)}_{K/k}(G_m) \longrightarrow R_{K/k}(G_m) \xrightarrow{N} G_m \longrightarrow 0$$

where N is the norm map for K/k and $R^{(1)}_{K/k}(G_m) = \mathrm{Ker}\, N$. As mentioned above, tori G_m, $R_{K/k}(G_m)$ have class numbers h_k, h_K, respectively. We shall denote by $h_{K/k}$ the class number of the torus $R^{(1)}_{K/k}(G_m)$ and put

$$E(K/k) \overset{\text{def}}{=} \frac{h_K}{h_k h_{K/k}}.$$

Trivially, we have $E(k/k) = 1$. As we shall see later, there are many examples of extensions K/k for which $E(K/k) = 1$.

§ 2. Statement of results

Let K/k be a finite galois extension of an algebraic number field k and $\mathfrak{g}$ be the galois group of K/k: $\mathfrak{g} = G(K/k)$. For each place v of k, we denote by w any place of K which lies above v (written $w|v$) and by $\mathfrak{g}_w$ the galois group of K_w/k_v. We denote by $\mathfrak{O}_K$ the ring of integers of K and so $\mathfrak{O}_K^\times$ means the group of units of K. For a finite group G and a left G-module A, $H^0(G, A)$ will denote the 0-th the Tate cohomology group $H^0(G, A) = A^G/N_G A$ where $A^G = \{x \in A;\ x^\sigma = x \text{ for all } \sigma \in G\}$, $N_G A = \{\sum_{\sigma \in G} x^\sigma;\ x \in A\}$.[1]

THEOREM. *Notation being as above, we have*

$$E(K/k) = \frac{[\mathrm{Ker}(H^0(\mathfrak{g}, K^\times) \longrightarrow H^0(\mathfrak{g}, K_A^\times)] \prod_v [H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]}{[K' : k][H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}, \tag{2.1}$$

where K'/k is the maximal abelian subextension of K/k and in the product $\prod_v [H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]$ we choose, for each v, any w such that $w|v$.[2]

COROLLARY. *When K/k is cyclic, we have*

$$E(K/k) = \frac{\prod_v e_v(K,k)}{[K : k][H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}, \tag{2.2}$$

where $e_v(K/k)$ means the ramification index for K_w/k_v, $w|v$.

§ 3. Isogeny of tori

Here we shall recall a formula due to Shyr on the relative class number of two isogenous tori in terms of their Tamagawa numbers and certain indices. Let T, T^* be tori defined over k and $\lambda: T^* \to T$ be an

1) If $x \in A$ and $\sigma \in G$, then the action of σ on x will be denoted by σx or x^σ. in the latter case, note that $x^{\sigma\tau} = (x^\tau)^\sigma$, $\tau \in G$.

2) Since K/k is galois, the groups $H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)$, $w|v$, are all isomorphic. We denote by [*] the cardinality of a set *.

isogeny, i.e. a surjective homomorphism with finite kernel, over k. This isogeny induces the following homomorphisms:

$$\begin{aligned} \hat{\lambda}(k)&: \hat{T}(k) \longrightarrow \hat{T}^*(k), \\ \lambda(\mathfrak{o}_v)&: T^*(\mathfrak{o}_v) \longrightarrow T(\mathfrak{o}_v) \qquad \text{for all } v, \\ \lambda(\mathfrak{o}_k)&: T^*(\mathfrak{o}_k) \longrightarrow T(\mathfrak{o}_k). \end{aligned}$$

In general, let A, B be abelian groups. A homomorphism $\alpha\colon A \to B$ will be called admissible if $\operatorname{Ker}\alpha$ and $\operatorname{Cok}\alpha$ are both finite. When that is so, following Tate, we put

$$q(\alpha) = \frac{[\operatorname{Cok}\alpha]}{[\operatorname{Ker}\alpha]}.$$

From a general theory of tori, we know that all maps $\hat{\lambda}(k)$, $\lambda(\mathfrak{o}_v)$, $\lambda(\mathfrak{o}_k)$ are admissible and $q(\lambda(\mathfrak{o}_v)) = 1$ for almost all v. In this situation, we have

$$\frac{h_{T^*}}{h_T} = \frac{\tau(T^*) \prod_v q(\lambda(\mathfrak{o}_v))}{\tau(T)q(\hat{\lambda}(k))q(\lambda(\mathfrak{o}_k))}, \tag{3.1}$$

where $\tau(T)$ is the Tamagawa number of T.

Remark. Using the Weil functor $R_{k/\boldsymbol{Q}}$, we may assume that $k = \boldsymbol{Q}$ to prove (3.1). For a torus T over $\boldsymbol{Q}$, there is a formula which expresses h_T in terms of other arithmetical invariants of T which generalizes the well-known class number formula of Dedekind for an algebraic number field (cf. [S1], [O3]). The relation (3.1) follows from that formula for h_T (cf. [S1], Theorem 3.1.1). All these appeared implicitly in [O1] where the relative Tamagawa number of two isogenous tori was the main object of study rather than the class numbers.

§ 4. Proof of Theorem

Let K/k be a finite galois extension of number fields. Let n denote the degree $[K:k]$. As in Section 1, consider the exact sequence of tori over k:

$$0 \longrightarrow T' \xrightarrow{\iota} T \xrightarrow{N} G_m \longrightarrow 0, \tag{4.1}$$

where $T = R_{K/k}(G_m)$, $T' = R^{(1)}_{K/k}(G_m) = \operatorname{Ker} N$ and ι is the embedding map. The dual of (4.1) is the following exact sequence of $\mathfrak{g}$-modules:

$$0 \longleftarrow J \xleftarrow{\hat{\iota}} \boldsymbol{Z}[\mathfrak{g}] \xleftarrow{\hat{N}} \boldsymbol{Z} \longleftarrow 0, \tag{4.2}$$

where $J = \hat{T}' = Z[\mathfrak{g}]/Zs$, $s = \sum_{\sigma\in\mathfrak{g}}\sigma$, $\hat{N}(z) = zs$, $\hat{\iota}(\gamma) = \gamma \bmod Zs$. Call ε the homomorphism over k: $G_m \to T$ such that $\hat{\varepsilon}(\gamma) = S(\gamma) \overset{\text{def}}{=} \sum_{\sigma\in\mathfrak{g}} z_\sigma$ for $\gamma = \sum_{\sigma\in\mathfrak{g}} z_\sigma\sigma \in Z[\mathfrak{g}]$. Since $\hat{\varepsilon}$ is surjective, ε is injective and from now on we shall embed G_m in T by ε which induces the natural embedding of $k^\times$ in $K^\times$. Since $\hat{\varepsilon}\hat{N}(z) = \hat{\varepsilon}(zs) = nz$, we have $N\varepsilon(x) = x^n$, i.e. $Nx = x^n$, for $x \in G_m$.

The sequence (4.1) yields the following isogeny of tori over k:

$$\lambda\colon T' \times G_m \longrightarrow T\,, \tag{4.3}$$

with $\lambda(u, v) = uv$, $u \in T'$, $v \in G_m$. Applying (3.1) with $T^* = T' \times G_m$, we get

$$E(K/k)^{-1} = \frac{h_{T'}h_k}{h_K} = \tau(T')\frac{\prod_v q(\lambda(\mathfrak{o}_v))}{q(\hat{\lambda}(k))q(\lambda(\mathfrak{o}_k))}\,. \tag{4.4}$$

We shall determine $q(\hat{\lambda}(k))$, $q(\lambda(\mathfrak{o}_v))$, $q(\lambda(\mathfrak{o}_k))$ and $\tau(T')$ successively.

(i) $q(\hat{\lambda}(k))$.

The dual of (4.3) is

$$\hat{\lambda}\colon Z[\mathfrak{g}] \longrightarrow J \times Z \tag{4.5}$$

with $\hat{\lambda}(\gamma) = (\gamma \bmod Zs,\ S(\gamma))$. Since $Z[\mathfrak{g}]^{\mathfrak{g}} = Zs$ and $J^{\mathfrak{g}} = \{0\}$, the map $\hat{\lambda}$ induces on the $\mathfrak{g}$-invariant parts the map $\hat{\lambda}(k)\colon Zs \to Z$ given by $\hat{\lambda}(k)(zs) = nz$. Therefore $[\operatorname{Ker}\hat{\lambda}(k)] = 1$ and $[\operatorname{Cok}\hat{\lambda}(k)] = n$ and so

$$q(\hat{\lambda}(k)) = n\,. \tag{4.6}$$

(ii) $q(\lambda(k_v))$, $v \in S_\infty$.

Case 1. $k_v = C$. In this case, we have

$$T(k_v) = R_{K/k}(G_m)(k_v) = \prod_{w|v} K_w^\times = (C^\times)^n \overset{\text{def}}{=} \Gamma\,,$$
$$T'(k_v) = \{u = (u_1, \cdots, u_n) \in \Gamma;\ Nu = u_1 \cdots u_n = 1\} \overset{\text{def}}{=} \Gamma'\,,$$
$$\lambda(k_v)\colon \Gamma' \times C^\times \to \Gamma,\quad \lambda(k_v)(u, v) = uv,\quad u \in \Gamma',\quad v \in C^\times.^{1)}$$

Now, $(u, v) \in \operatorname{Ker}\lambda(k_v) \Leftrightarrow u = v^{-1}$ and $Nu = 1 \Leftrightarrow (u, v) = ((v^{-1}, \cdots, v^{-1}), v)$ and $v^n = 1$. Hence, $[\operatorname{Ker}\lambda(k_v)] = n$. On the other hand, for any $x \in \Gamma$, take $v \in C^\times$ such that $v^n = Nx$ and put $u = v^{-1}x$, then $Nu = 1$ and $x = uv$, i.e. $x = \lambda(k_v)(u, v)$. Hence $[\operatorname{Cok}\lambda(k_v)] = 1$ and we get

1) Note that $C^\times$ is embedded in Γ diagonally.

$$q(\hat{\lambda}(k_v)) = n^{-1}\,. \tag{4.7}$$

Case 2. $k_v = \boldsymbol{R}$, $K_w = \boldsymbol{R}$, $w|v$. In this case, we have

$$T(k_v) = \prod_{w|v} K_w^\times = (\boldsymbol{R}^\times)^n \stackrel{\text{def}}{=} \Gamma\,,$$

$$T'(k_v) = \{u \in \Gamma;\ Nu = 1\} \stackrel{\text{def}}{=} \Gamma'\,,$$

$$\lambda(k_v)\colon \Gamma' \times \boldsymbol{R}^\times \to \Gamma, \quad \lambda(k_v)(u, v) = uv, \quad u \in \Gamma', \quad v \in \boldsymbol{R}^\times\,.$$

Now, $(u, v) \in \operatorname{Ker} \lambda(k_v) \Leftrightarrow u = v^{-1}$ and $Nu = 1 \Leftrightarrow (u, v) = ((v^{-1}, \cdots, v^{-1}), v)$ and $v^n = 1 \Leftrightarrow (u, v) = ((1, \cdots, 1), 1)$ when n is odd and $(u, v) = \pm((1, \cdots, 1), 1)$ where n is even. Hence,

$$[\operatorname{Ker} \lambda(k_v)] = \begin{cases} 1, & n\colon \text{odd}, \\ 2, & n\colon \text{even}. \end{cases}$$

On the other hand, when n is odd, the same argument as in Case 1 shows that $\lambda(k_v)$ is surjective. However, when n is even, that argument shows that $\operatorname{Im} \lambda(k_v) = \{x \in \Gamma;\ Nx > 0\}$ which implies that $\Gamma/\operatorname{Im} \lambda(k_v) \approx \boldsymbol{R}/\boldsymbol{R}_+$. Hence,

$$[\operatorname{Cok} \lambda(k_v)] = \begin{cases} 1, & n\colon \text{odd}, \\ 2, & n\colon \text{even}. \end{cases}$$

Therefore, we get

$$q(\lambda(k_v)) = 1\,. \tag{4.8}$$

Case 3. $k_v = \boldsymbol{R}$, $K_w = \boldsymbol{C}$, $w|v$. In this case, n is even and so we put $n = 2m$. We have

$$T(k_v) = \prod_{w|v} K_w^\times = (\boldsymbol{C}^\times)^m \stackrel{\text{def}}{=} \Gamma\,,$$

$$T'(k_v) = \{u \in \Gamma;\ Nu = N_v u_1 \cdots N_v u_m = 1\} \stackrel{\text{def}}{=} \Gamma'\,,$$

where $N_v a = a\bar{a}$, for $a \in K_w = \boldsymbol{C}$,

$$\lambda(k_v)\colon \Gamma' \times \boldsymbol{R}^\times \to \Gamma, \quad \lambda(k_v)(u, v) = uv, \quad u \in \Gamma', \quad v \in \boldsymbol{R}^\times.$$

Now, $(u, v) \in \operatorname{Ker} \lambda(k_v) \Leftrightarrow u = v^{-1}$ and $Nu = 1 \Leftrightarrow (u, v) = ((v^{-1}, \cdots, v^{-1}), v)$ and $v^{2m} = 1 \Leftrightarrow (u, v) = \pm((1, \cdots, 1), 1)$. Hence, $[\operatorname{Ker} \lambda(k_v)] = 2$. On the other hand, since $Nx = N_v x_1 \cdots N_v x_m > 0$ for all $x \in \Gamma$, we see as before that $[\operatorname{Cok} \lambda(k_v)] = 1$. Therefore we get

$$q(\lambda(k_v)) = \frac{1}{2}\,. \tag{4.9}$$

For k, denote by r_1 (resp. r_2) the number of real (resp. complex) places and by ρ the number of real places which ramify for K/k. From (4.7), (4.8), (4.9), we get

$$(4.10)\qquad \prod_{v\in S_\infty} q(\lambda(k_v)) = n^{-r_2}2^{-\rho} = n^{-r_2}\prod_{u\in S_\infty} e_v(K/k)^{-1} = n^{-r_2}\prod_{v\in S_\infty}[H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]^{-1}$$

where $e_v(K/k)$ means the ramification index of K_x/k_v, $w|v$.

(iii) $q(\lambda(\mathfrak{o}_\mathfrak{p}))$.

Consider the localization $K_\mathfrak{P}/k_\mathfrak{p}$ of K/k, $\mathfrak{P}|\mathfrak{p}$. Put $\mathfrak{g}_\mathfrak{P} = G(K_\mathfrak{P}/k_\mathfrak{p})$. We have $n = [K:k] = e_\mathfrak{p} f_\mathfrak{p} g_\mathfrak{p}$, $n_\mathfrak{p} = e_\mathfrak{p} f_\mathfrak{p} = [K_\mathfrak{P}:k_\mathfrak{p}]$, where $e_\mathfrak{p} = e_\mathfrak{p}(K/k) =$ the ramification index for $K_\mathfrak{P}/k_\mathfrak{p}$, $f_\mathfrak{p} = f_\mathfrak{p}(K/k) =$ the residue class degree for $K_\mathfrak{P}/k_\mathfrak{p}$ and $g_\mathfrak{p} = g_\mathfrak{p}(K/k) =$ the number of distinct prime factors of $\mathfrak{p}$ in K. For simplicity, we often write g for $g_\mathfrak{p}$, etc. Since K/k is galois, all $K_\mathfrak{P}/k_\mathfrak{p}$, $\mathfrak{P}|\mathfrak{p}$, are isomorphic over $k_\mathfrak{p}$ and we use $N_\mathfrak{p}$ for the norm for $K_\mathfrak{P}/k_\mathfrak{p}$ for all $\mathfrak{P}|\mathfrak{p}$. Now we have

$$T(\mathfrak{o}_\mathfrak{p}) = R_{K/k}(G_m)(\mathfrak{o}_\mathfrak{p}) = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{O}_\mathfrak{P}^\times \overset{\text{def}}{=} \Gamma\,,$$
$$T'(\mathfrak{o}_\mathfrak{p}) = \{u = (u_1, \cdots, u_g) \in \Gamma;\ Nu = N_\mathfrak{p}u_1 \cdots N_\mathfrak{p}u_g = 1\} \overset{\text{def}}{=} \Gamma'\,,$$
$$\lambda(\mathfrak{o}_\mathfrak{p})\colon \Gamma' \times \mathfrak{o}_\mathfrak{p}^\times \to \Gamma, \quad \lambda(\mathfrak{o}_\mathfrak{p})(u, v) = uv, \quad u \in \Gamma', \quad v \in \mathfrak{o}_\mathfrak{p}^\times\,.$$

For a field F, we shall denote by $W(F)$ (resp. $W_n(F)$) the group of roots of 1 in F (resp. the group of n-th roots of 1 in F) and by $w(F)$ (resp. $w_n(F)$) the order of the group $W(F)$ (resp. $W_n(F)$). Since $(u, v) \in \operatorname{Ker}\lambda(\mathfrak{o}_\mathfrak{p}) \Leftrightarrow u = v^{-1}$ and $Nu = 1 \Leftrightarrow (u, v) = ((v^{-1}, \cdots, v^{-1}), v)$, $v^g\mathfrak{p}^n\mathfrak{p} = v^n = 1$, we have $[\operatorname{Ker}\lambda(\mathfrak{o}_\mathfrak{p})] = w_n(k_\mathfrak{p})$. On the other hand, in view of the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{N} & \mathfrak{o}_\mathfrak{p}^\times \\ \cup & & \cup \\ \operatorname{Im}\lambda(\mathfrak{o}_\mathfrak{p}) = \Gamma'\mathfrak{o}_\mathfrak{p}^\times & \xrightarrow{\nu} & (\mathfrak{o}_\mathfrak{p}^\times)^n \end{array}, \qquad \nu = N|_{\operatorname{Im}\lambda(\mathfrak{o}_\mathfrak{p})}\,,$$

we have

$$\begin{aligned}[\operatorname{Cok}\lambda(\mathfrak{o}_\mathfrak{p})] &= [\operatorname{Im}N\colon \operatorname{Im}\nu][\operatorname{Ker}N\colon \operatorname{Ker}\nu] = [\operatorname{Im}N\colon \operatorname{Im}\nu][\Gamma'\colon\Gamma'] \\ &= [\operatorname{Im}N\colon \operatorname{Im}\nu] \overset{1)}{=} [N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times : (\mathfrak{o}_\mathfrak{p}^\times)^n] = [\mathfrak{o}_\mathfrak{p}^\times : (\mathfrak{o}_\mathfrak{p}^\times)^n]/[\mathfrak{o}_\mathfrak{p}^\times : N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times] \\ &= w_n(k_\mathfrak{p})p^{\operatorname{ord}_p(n)[k_\mathfrak{p}\cdot Q_p]}/[\mathfrak{o}_\mathfrak{p}^\times : N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times]\end{aligned}$$

because $[\mathfrak{o}_\mathfrak{p}^\times : (\mathfrak{o}_\mathfrak{p}^\times)^n] = w_n(k_\mathfrak{p})|n|_\mathfrak{p}^{-1}$.[2] Hence we get

1) Note that $N\Gamma = N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times$ for any $\mathfrak{P}|\mathfrak{p}$.

2) See [L1] p. 47, Proposition 6.

$$q(\lambda(\mathfrak{o}_\mathfrak{p})) = \frac{p^{\mathrm{ord}_p(n)[k_\mathfrak{p}:\,\boldsymbol{Q}_p]}}{[\mathfrak{o}_\mathfrak{p}^\times : N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times]}\,. \tag{4.11}$$

Since $\sum_\mathfrak{p}[k_\mathfrak{p} : \boldsymbol{Q}_p] = [k : \boldsymbol{Q}] = n_0$ and $\prod_p p^{\mathrm{ord}_p(n)} = n$, multiplying (4.11) for all $\mathfrak{p}$'s, we get

$$\prod_\mathfrak{p} q(\lambda(\mathfrak{o}_\mathfrak{p})) = n^{n_0} \prod_\mathfrak{p} [\mathfrak{o}_p^\times : N_\mathfrak{p}\mathfrak{O}_\mathfrak{P}^\times]^{-1} = n^{n_0} \prod_\mathfrak{p} [H^0(\mathfrak{g}_\mathfrak{P}, \mathfrak{O}_\mathfrak{P}^\times)]^{-1}\,. \tag{4.12}$$

(iv) $q(\lambda(\mathfrak{o}_k))$.

We have

$$T(\mathfrak{o}_k) = R_{K/k}(G_m)(\mathfrak{o}_k) = \mathfrak{O}_K^\times \overset{\mathrm{def}}{=} \Gamma\,,$$

$$T'(\mathfrak{o}_k) = \{u \in \Gamma ; Nu = 1\} \overset{\mathrm{def}}{=} \Gamma'\,,$$

$$\lambda(\mathfrak{o}_k)\colon \Gamma' \times \mathfrak{o}_k^\times \to \Gamma, \quad \lambda(\mathfrak{o}_k)(u, v) = uv, \quad u \in \Gamma', \quad v \in \mathfrak{o}_k^\times\,.$$

Since $(u, v) \in \mathrm{Ker}\,\lambda(\mathfrak{o}_k) \Leftrightarrow u = v^{-1}$ and $Nu = 1 \Leftrightarrow (u, v) = (v^{-1}, v)$, $v^n = 1$, we have $[\mathrm{Ker}\,\lambda(\mathfrak{o}_k)] = w_n(k)$. On the other hand, in view of the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{N} & \mathfrak{o}_k^\times \\ \cup & & \cup \\ \mathrm{Im}\,\lambda(\mathfrak{o}_k) = \Gamma'\mathfrak{o}_k^\times & \xrightarrow{\nu} & (\mathfrak{o}_k^\times)^n \end{array}\,, \qquad \nu = N|_{\mathrm{Im}\,\lambda(\mathfrak{o}_k)}\,,$$

we have

$$\begin{aligned}[\mathrm{Cok}\,\lambda(\mathfrak{o}_k)] &= [\mathrm{Im}\,N : \mathrm{Im}\,\nu][\mathrm{Ker}\,N : \mathrm{Ker}\,\nu] = [N\mathfrak{O}_K^\times : (\mathfrak{o}_k^\times)^n] \\ &= [\mathfrak{o}_k^\times : (\mathfrak{o}_k^\times)^n]/[\mathfrak{o}_k^\times : N\mathfrak{O}_K^\times]\,.\end{aligned}$$

Now, by the Dirichlet's theorem, we have $\mathfrak{o}_k^\times = W(k) \times \boldsymbol{Z}^{r_k}$, $r_k = r_1 + r_2 - 1$, and so we get $[\mathfrak{o}_k^\times : (\mathfrak{o}_k^\times)^n] = [w(k) : w(k)^n]n^{r_k} = w_n(k)n^{r_k}$ because $W(k)/W_n(k) \approx W(k)^n$. Hence,

$$q(\lambda(\mathfrak{o}_k)) = \frac{n^{r_k}}{[\mathfrak{o}_k^\times : N\mathfrak{O}_K^\times]} = n^{r_k}[H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]^{-1}\,. \tag{4.13}$$

From (4.6), (4.10), (4.12), (4.13), we have

$$\begin{aligned}\frac{q(\hat{\lambda}(k))q(\lambda(\mathfrak{o}_k))}{\prod_v q(\lambda(\mathfrak{o}_v))} &= \frac{nn^{r_k}[H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]^{-1}}{n^{-r_2}\prod_{v \in S_\infty}[H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]^{-1} n^{n_0} \prod_\mathfrak{p}[H^0(\mathfrak{g}_\mathfrak{P}, \mathfrak{O}_\mathfrak{P}^\times)]^{-1}} \\ &= n^{1+r_k+r_2-n_0}\frac{\prod_v [H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]}{[H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]} = \frac{\prod_v [H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]}{[H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}\end{aligned}$$

because $n_0 = r_1 + 2r_2$. Substituting this in (4.4), we get

$$E(K/k) = \frac{1}{\tau(T')}\,\frac{\prod_v [H^0(\mathfrak{g}_w, \mathfrak{O}_w^\times)]}{[H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}\,. \tag{4.14}$$

(v) $\tau(T')$.

It remains to determine the Tamagawa number $\tau(T')$, where $T' = R^{(1)}_{K/k}(G_m)$. Consider, as before, the exact sequence of tori defined over k:

$$0 \longrightarrow T' \xrightarrow{\iota} T \xrightarrow{N} G_m \longrightarrow 0, \qquad T = R_{K/k}(G_m)\,.$$

Applying to this sequence Theorem 4.2.1 of [O_2], we get

$$\tau(T') = [\mathrm{Cok}\ \hat{\iota}^{\mathfrak{g}}]q(\mu),^{1)}$$

where μ is the natural homomorphism

$$\begin{array}{ccc} \mu\colon\ k^\times/NK^\times & \longrightarrow & k_A^\times/N(K_A^\times) \\ \| & & \| \\ H^0(\mathfrak{g}, K^\times) & & H^0(\mathfrak{g}, K_A^\times)\,. \end{array}$$

Since $\mathrm{Cok}\,\mu \approx k_A^\times/k^\times NK_A^\times \approx \mathfrak{g}/\mathfrak{g}'$ by the class field theory, where $\mathfrak{g}'$ being the commutator subgroup of $\mathfrak{g}$, we have

$$[\mathrm{Cok}\,\mu] = [K' : k]\,,$$

K'/k being the maximal abelian subextension of K/k. On the other hand, since $\hat{T}'^{\mathfrak{g}} = (Z[\mathfrak{g}]/Zs)^{\mathfrak{g}} = 0$, $\hat{\iota}^{\mathfrak{g}}$ is the zero map and so $[\mathrm{Cok}\,\hat{\iota}^{\mathfrak{g}}] = 1$. We have therefore

$$\tau(T') = \frac{[K' : k]}{[\mathrm{Ker}\,(H^0(\mathfrak{g}, K^\times) \to H^0(\mathfrak{g}, K_A^\times)]}\,. \tag{4.15}$$

Our formula (2.1) in the Theorem follows from (4.14) and (4.15). q.e.d.

When K/k is cyclic, we have $K' = K$ and, by Hasse's norm theorem, $[\mathrm{Ker}(H^0(\mathfrak{g}, K^\times) \to H^0(\mathfrak{g}, K_A^\times)] = 1$. Furthermore, we have $H^0(\mathfrak{g}_{\mathfrak{P}}, \mathfrak{O}_{\mathfrak{P}}^\times) = e_{\mathfrak{p}}(K/k)$,[2)] and (2.2) of the Corollary is proved.

§ 5. Miscellaneous examples

EXAMPLE 1. (K/k: cyclic of prime degree)

Suppose that the extension K/k is cyclic of prime degree p. Then, two non-negative integers $\tau(K/k)$ and $\varepsilon(K/k)$ can be introduced by the equations

1) I want to take this opportunity to make the following corrections in my paper [O_2]: For "$q(\hat{\iota}_k)$" read "[$\mathrm{Cok}\,\hat{\iota}_k$]" on p. 63, line 8 and line 7 from the bottom, For "$q(\hat{\iota}^{\mathfrak{g}})$" read "[$\mathrm{Cok}\,\hat{\iota}^{\mathfrak{g}}$]" on p, 66, line 11 and p. 67, line 3. Suppress "$q(\hat{\iota}^{\mathfrak{g}}) =$" on p. 66, line 2 from the bottom,

2) Cf. [L_1], p. 188, Lemma 4.

$$\prod_v e_v(K/k) = p^{\tau(K/k)}, \qquad H^0(\mathfrak{g}, \mathfrak{O}_K^\times) = (Z/pZ)^{\varepsilon(K/k)}$$

where $\tau(K/k)$ means the number of places of k ramified for K/k. In this situation, the formula (2.2) can be written as

$$E(K/k) = p^{\tau(K/k)-\varepsilon(K/k)-1}. \tag{5.1}$$

EXAMPLE 2. ($K/\boldsymbol{Q}$: quadratic)

Suppose that $k = \boldsymbol{Q}$ and $p = 2$ in Example 1. Then, we have

$$\tau(K/\boldsymbol{Q}) = \begin{cases} t_K + 1, & \Delta_K < 0, \\ t_K, & \Delta_K > 0, \end{cases}$$

where t_K is the number of distinct prime factors of Δ_K. On the other hand, since $H^0(\mathfrak{g}, \mathfrak{O}_K^\times) = \{\pm 1\}/N\mathfrak{O}_K^\times$, we have

$$\varepsilon(K/\boldsymbol{Q}) = \begin{cases} 1, & \Delta_K < 0 \text{ or } \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{1\}, \\ 0, & \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{\pm 1\}. \end{cases}$$

Using notation in (4.1), we have, by (5.1),

$$E(K/\boldsymbol{Q}) = \frac{h_K}{h_{T'}} = \begin{cases} 2^{t_K-1}, & \Delta_K < 0 \text{ or } \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{\pm 1\}, \\ 2^{t_K-2}, & \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{1\}. \end{cases} \tag{5.2}$$

Denote by h_K^+ the class number of K in the narrow sense, then

$$h_K^+ = \begin{cases} h_K, & \Delta_K < 0 \text{ or } \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{\pm 1\}, \\ 2h_K, & \Delta_K > 0 \text{ and } N\mathfrak{O}_K^\times = \{1\}. \end{cases}$$

Therefore (5.2) yields

$$\frac{h_K^+}{h_{T'}} = 2^{t_K-1}. \tag{5.3}$$

Now, observe that $T' \approx O_2^+(N)$, the special orthogonal group of the quadratic form $N = N(x + y\omega)$, $\mathfrak{O}_K = Z + Z\omega$, and that $h_{T'} = h_K^* =$ the number of classes in the principal genus of the quadratic form N.[1] Hence, (5.3) is nothing but the well-known formula of Gauss:

$$h_K^+ = 2^{t_K-1} h_K^*. \tag{5.4}$$

EXAMPLE 3. (Hilbert class fields)

Let p be a prime such that $p \equiv 3 \pmod 4$ and let $k = \boldsymbol{Q}(\sqrt{-p})$.

1) Cf. [O_3], § I, § II, § III.

Since $\Delta_k = -p$, we have $t_k = 1$. By (5.4) with $K = k$, we have $h_k = h_k^+ = h_k^*$ which is known to an odd number. Assume that the ideal class group H_k of k is cyclic. By the way, this is the case for all $p < 100$:

p	3	7	11	19	23	31	43	47	59	67	71	79	83	97
h_k	1	1	1	1	3	3	1	5	3	1	7	5	3	4

[1].

Let K be the Hilbert class field over k. Note that $N\mathfrak{O}_K^\times = \{\pm 1\}$ since $-1 \in \mathfrak{O}_K^\times$ and $N(-1) = (-1)^{h_k} = -1$. Excluding the trivial case $p = 3$, we have $\mathfrak{o}_k^\times = \{\pm 1\}$ and so $H^0(\mathfrak{g}, \mathfrak{O}_K^\times) = 0$. Since we have $e_v(K/k) = 1$ for all v, the formula (2.2) yields

$$E(K/k) = \frac{1}{h_k},$$

which implies that $h_K = h_{K/k}$ for such an extension.

EXAMPLE 4. (CM-fields)

Let K be a totally imaginary quadratic extension of a totally real number field k. It is well-known that h_k divides h_K.[2] The quotient h^- is called the relative class number. From (5.1) we have

$$h^- = h_{K/k} 2^{\tau(K/k) - \varepsilon(K/k) - 1}. \tag{5.5}$$

Here, we have $\tau(K/k) = d/2 + t(K/k)$ where $d = [K : \boldsymbol{Q}]$ and $t(K/k) =$ the number of finite places of k ramified for K/k. On the other hand, $\varepsilon(K/k)$ is determined by the relation

$$[\mathfrak{o}_k^\times : N\mathfrak{O}_K^\times] = 2^{\varepsilon(K/k)}. \tag{5.6}$$

As is well-known, $[\mathfrak{O}_K^\times : W(K)\mathfrak{o}_k^\times] = 1$ or 2. *Assume that this index* $= 1$, *i.e.* $\mathfrak{O}_K^\times = W(K)\mathfrak{o}_k^\times$. Since $Nw = w\bar{w} = |w|^2 = 1$ for $w \in W(K)$, we have $N\mathfrak{O}_K^\times = (\mathfrak{o}_k^\times)^2$. Since k is totally real, $\mathfrak{o}_k^\times = \{\pm 1\} \times \boldsymbol{Z}^{(d/2)-1}$, and so $[\mathfrak{o}_k^\times : (\mathfrak{o}_k^\times)^2] = 2^{d/2}$. Hence $\varepsilon(K/k) = d/2$ and we get

$$h^- = h_{K/k} 2^{t(K/k)-1}, \quad \text{or } E(K/k) = 2^{t(K/k)-1}. \tag{5.7}$$

The above assumption is satisfied when $K = \boldsymbol{Q}(\zeta_{p^a})$, $k = \boldsymbol{Q}(\zeta_{p^a} + \zeta_{p^a}^{-1})$ where

1) Cf. [T1], p. 404.
2) As for standard facts on CM-fields, see [W1], Chapter 4.

ζ_{p^a}, $a \geqq 1$, denotes a primitive p^a-th root of 1, p an odd prime. Since p is the only prime which is ramified for $K/\boldsymbol{Q}$ and it is totally ramified for $K/\boldsymbol{Q}$, we have $t(K/k)=1$. Therefore (5.7) yields $h^- = h_{K/k}$; and so $E(K/k)=1$. In view of recent progresses in the theory of cyclotomic fields, it is nice to know that h^- becomes the class number of the algebraic group $R^{(1)}_{K/k}(G_m)$ which makes sense for an arbitrary galois extension.

EXAMPLE 5. (K: CM-field, $K/\boldsymbol{Q}$: cyclic)

Let $K/\boldsymbol{Q}$ be a cyclic extension of degree d such that K is a CM-field. Let k be the maximal real subfield of K. Then, the formula (2.2) yields

$$E(K/\boldsymbol{Q}) = \frac{2\prod_p e_p(K/\boldsymbol{Q})}{d[\boldsymbol{Z}^\times : N\mathfrak{O}_K^\times]}. \tag{5.8}$$

As in Example 4, *assume that* $\mathfrak{O}_K^\times = W(K)\mathfrak{o}_k^\times$. *Then, we see easily that* $N(W(K)) = N\mathfrak{o}_k^\times = 1$. *Hence*, by (5.8), we have

$$E(K/\boldsymbol{Q}) = d^{-1}\prod_p e_p(K/\boldsymbol{Q}). \tag{5.9}$$

If there is only one prime p which is ramified for $K/\boldsymbol{Q}$ and this p is totally ramified for $K/\boldsymbol{Q}$, then we have, by (5.9),

$$E(K/\boldsymbol{Q}) = 1. \tag{5.10}$$

This is, of course, the case where $K=\boldsymbol{Q}(\zeta_{p^a})$ for an odd prime p.

REFERENCES

[C1] C. Chevalley, Sur la théorie du corps de classes dans les corps finis et les corps locaux. J. Fac. Sci. Tokyo Univ., **2** (1933), 365–476.

[C2] C. Chevalley, Théorie des Groupes de Lie, Tome II, Groupes algébriques, Hermann, Paris, 1951.

[K1] T. Kimura, Algebraic Class Number Formulae for Cyclotomic Fields, Lecture Notes, (in Japanese), Sophia Univ., Tokyo, 1985.

[L1] S. Lang, Algebraic Number Theory, Addison-Wesley, Reading, Mass., 1970.

[L2] S. Lang, Units and class groups in number theory and algebraic geometry, Bull. Am. Math. Soc., **6** (1982), 253–316.

[O1] T. Ono, Arithmetic of algebraic tori, Ann. of Math., **74** (1961), 101–139.

[O2] T. Ono, On the Tamagawa number of algebraic tori, Ann. of Math., **78** (1963), 47–73.

[O3] T. Ono, Arithmetic of Algebraic Groups and its Applications, Lecture Notes, Rikkyo Univ., Tokyo, 1986.

[S1] J.-M. Shyr, Class number formula of algebraic tori with applications to relative class numbers of certain relative quadratic extensions of algebraic number fields, Thesis, The Johns Hopkins Univ., Baltimore, Maryland, 1974.

[S2] J.-M. Shyr, On some class number relations of algebraic tori, Michigan Math. J.,

24 (1977), 365–377.
[T1] T. Takagi, Lectures on Elementary Number Theory, 2nd ed. (in Japanese), Kyoritsu, Tokyo, 1971.
[W1] L. Washington, Introduction to Cyclotomic Fields, Springer New York, 1982.

Department of Mathematics
The Johns Hopkins University
Baltimore, MD 21218
U. S. A.