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GENERALIZED RADON TRANSFORM AND LÉVY'S BROWNIAN MOTION, II*)

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§ 1. Introduction

As a continuation of the author's paper [19], we shall investigate the null spaces of a dual Radon transform R^*, in connection with a Lévy's Brownian motion X with parameter space (R^n, d). We shall follow the notation used in (I), [19].

We begin with a brief review of the general framework behind the representation of Chentsov type:

$$X(x) = \int_{B_x} W(dh) = W(B_x), \tag{1}$$

with $B_x := \{h \in H;\ x \in h\}$. It consists of the following:

(i) A Lévy's Brownian motion $X = \{X(x);\ x \in M\}$ with mean 0 and variance $d(x, y) = E[(X(x) - X(y))^2]$, where $d(x, y)$ is an L^1-embeddable (semi-)metric on M;

(ii) A Gaussian random measure $W = \{W(dh);\ h \in H\}$ based on a measure space (H, ν) such that $H \subset 2^M$ and ν is a positive measure on H satisfying $\nu(B_x) < \infty$ and

$$d(x, y) = \nu(B_x \triangle B_y) = \int_H \pi_h(x, y)\nu(dh) \qquad \text{for all } x, y \in M, \tag{2}$$

where

$$\pi_h(x, y) := |\chi_h(x) - \chi_h(y)| = |\chi_{B_x}(h) - \chi_{B_y}(h)|.$$

As a bridge connecting the metric space (M, d) and the measure space (H, ν), the equation (2) guarantees the existence of a representation of the form (1) for a Lévy's Brownian motion X with parameter space (M, d).

The representation (1) of Chentsov type played in (I) (and will play

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also in the present (II)) an important role, and led us to introduce a pair of integral transformations; one is the *generalized Radon transform*,

$$(3) \qquad (Rf)(h) := \int_h f(x) m(dx) , \qquad f \in L^1(M, m) ,$$

and the other is the *dual Radon transform*

$$(4) \qquad (R^*g)(x) := \int_{B_x} g(h)\nu(dh), \qquad g \in L^2(H, \nu) .$$

DEFINITION 1. For each subset $A \subset M$, we define

$$(5) \quad N_1(A) := \{g \in L^2(H, \nu);\ (R^*g)(x) \equiv 0 \text{ on } A\} = [\chi_{B_x}(h);\ x \in A]^\perp .$$

This closed subspace of $L^2(H, \nu)$ is called the *null space of R^* relative to the subset A.*

The study of such null spaces $N_1(A)$ is of great importance for the following reason. For each Lévy's Brownian motion X with parameter space (M, d), we have a representation of the form (1). Consider an increasing family of closed linear spans $[X(x); x \in A_\rho]$ corresponding to each increasing family of subsets A_ρ with $\cup_{0<\rho<\infty} A_\rho = M$. Just as in the well-known theory of canonical representations of Gaussian processes, we wish to give a description of these $[X(x); x \in A_\rho]$ in terms of a Gaussian random measure W; they are all contained in the big closed subspace

$$\left\{\int_H g(h) W(dh);\ g \in L^2(H, \nu)\right\}$$

of $L^2(\Omega, P)$. Since one can easily see that

$$(6) \qquad [X(x);\ x \in A] = \left\{\int_H g(h) W(dh);\ g \in N_1(A)^\perp\right\}$$

for every $A \subset M$, our problem is to determine completely the null space $N_1(A_\rho)$ of the dual Radon transform R^*.

So the main purpose of this paper is to investigate the null spaces $N_1(A_\rho)$ for a certain increasing family of closed subsets A_ρ of M, such as $A_\rho = V_\rho$ in the case $M = R^n$, where V_ρ denotes the closed ball of radius ρ about the origin O, $0 < \rho < \infty$. Examples of L^1-embeddable metrics d on R^n in which we have succeeded in finding a complete description of $N_1(V_\rho)$ as well as of $[X(x); x \in V_\rho]$ will be explained below.

Sections 3 and 4 concern rotation-invariant distances d on $M = R^n$ which are derived, via the equation (2), from the following choice of H:

$H = \{h_{t,\omega};\ t > 0,\ \omega \in S^{n-1}\}$ is the set of all half-spaces
$h_{t,\omega}\colon = \{x \in R^n;\ (x, \omega) > t\}$ not containing the origin O.

The Euclidean distance $|x - y|$ is a familiar example of such a distance.

The generalized Radon transform $(Rf)(h_{t,\omega})$ is then given by the integral of f over the half-space $h_{t,\omega}$ and hence closely related to the classical Radon transform. This observation leads us to apply the fruitful theory of the classical Radon transform (see, for example, [9], [12] and [16]) and solve the problem concerning the null spaces of R^*. In fact, by using the theorem of Ludwig [16] (cf. [20] and [21]), we are able to find a complete description of $N_1(V_\rho)$ (Theorem 7) as well as that of $[X(x);\ x \in V_\rho]$ (Theorem 8).

Our result on the structure of $[X(x);\ x \in V_\rho]$ can be restated in terms of mutually independent Gaussian processes $M_{m,k}(t)$ introduced by McKean [17]:

$$M_{m,k}(t)\colon = \int_{S^{n-1}} X(t\omega) S_{m,k}(\omega)\sigma(d\omega)\,, \qquad t > 0\,, \tag{7}$$

where σ denotes the uniform probability measure on the unit sphere S^{n-1} and $\{S_{m,k}(\omega);\ (m, k) \in \Delta\}$, $\Delta\colon = \{(m, k);\ m \geq 0 \text{ and } 1 \leq k \leq h(m)\}$, is taken to be a CONS in $L^2(S^{n-1}, \sigma)$ consisting of spherical harmonics. The basic representation (1) of X yields

$$M_{m,k}(t) = \int_0^t \lambda_m(u/t) dB_{m,k}(u)\,, \tag{8}$$

where the kernel $\lambda_m(t)$ is expressed in terms of the Gegenbauer polynomial $C_m^q(u)$ of degree m with $q\colon = (n - 2)/2$:

$$\lambda_m(t) = (\text{const.}) \int_t^1 C_m^q(u)\,(1 - u^2)^{q-1/2} du\,.$$

It turns out that the representation (8) of $M_{m,k}(t)$ is canonical only for $m \leq 2$ (Theorem 10). Moreover, for $m \geq 3$, we determine the dimension of $[B_{m,k}(t);\ t \leq \rho] \ominus [M_{m,k}(t);\ t \leq \rho]$ (orthogonal complement in $L^2(\Omega, P)$) which can be regarded as the degree of non-canonicality of (8). In this, way, our Theorems 8 and 10 might be viewed as a development (or refinement) of the result in [17] proved for a Brownian motion with n-dimensional parameter.

In Section 2 we shall give various kinds of L^1-embeddable metrics d on R^n. Some of them should be mentioned here.

The first kind of d depends on the choice of a bounded subset $K \subset R^n$ such that $|K| > 0$ and $O \in K$. Take the following measure space (H_K, ν_K):

$$H_K: = \{h_{\alpha,p}: = \{x \in R^n;\ \alpha(x-p) \in K\} = \alpha^{-1}K + p;\ \alpha \in SO(n)/\Sigma_K, p \in R^n\}$$

and

$$d\nu_K(h_{\alpha,p}): = c d\alpha dp\,, \qquad c > 0\,,$$

where $\Sigma_K: = \{\alpha \in SO(n);\ \alpha K = K\}$ and $d\alpha$ denotes the normalized Haar measure on $SO(n)/\Sigma_K$. Then, the equation (2) gives us an L^1-embeddable metric d_K invariant under every rigid motion on R^n:

$$\begin{aligned} d_K(x, y) &= c\int_{SO(n)/\Sigma K} |(\alpha^{-1}K + x - y) \triangle \alpha^{-1}K|\, d\alpha \\ &= c\int_{SO(n)} |(K + \alpha(x-y)) \triangle K|\, d\alpha = r_K(|x-y|)\,. \end{aligned}$$

The typical choice of $K = V_\rho$ allows us to compute the explicit form of r_{V_ρ} and get a large class of invariant distances by forming a superposition of the family $\{d_{V_\rho};\ 0 < \rho < \infty\}$ (cf. Section 2, 2–1). This idea of superposition is due to Takenaka [24] who gave a nice account of representations of self-similar Gaussian random fields.

It deserves mention that the generalized Radon transform

$$(Rf)(h_{\alpha,p}) = \int_K f(\alpha^{-1}x + p)dx\,, \qquad h_{\alpha,p} \in H_K\,,$$

was discussed in connection with the Pompeiu problem (cf. [26]).

The next kind of d is of the form $\|x - y\|$, where $\|x\|$ is a norm of negative type ([6] and [8]). Such a norm is characterized as the support function of a special convex body in R^n called a *zonoid* ([5]), and therefore admits of the following expression in terms of a bounded symmetric positive measure τ on S^{n-1}:

$$\|x\| = \int_{S^{n-1}} |(x, \omega)|\tau(d\omega)\,. \tag{9}$$

With the help of this well-known expression, the measure space (H, ν) combined with $\|x - y\|$ via (2) is naturally taken to be

$$\nu(dh_{t,\omega}) = dt\tau(d\omega) \qquad \text{on the set } H \text{ of half-spaces } h_{t,\omega}\,.$$

Note that rotation-invariance of τ yields the Euclidean distance $|x - y|$ up to a constant multiple.

It is worthwhile to remark that every Lévy's Brownian motion X with parameter space $(R^n, \|x-y\|)$ possesses a notable property: For each line L in R^n, restrict the whole parameter space R^n to the one-dimensional set L; then the Gaussian process $X_{|L} = \{X(x);\ x \in L\}$ coincides with a standard Brownian motion. In order to get at his definition of Brownian motion with n-dimensional parameter, Lévy [15] added one more simple condition that the probability law of $X(x) - X(O)$ is invariant under every rotation $\in SO(n)$. The class of Lévy's Brownian motions corresponding to norms of negative type is thus thought of as a nice extension of Lévy's original one.

§ 2. L^1-embeddable metrics on Euclidean space

This section is devoted to the study of the equation (2) connecting a metric space (R^n, d) with a measure space (H, ν). Indeed we describe a variety of L^1-embeddable metrics d on R^n and corresponding measures ν on $H \subset 2^{R^n}$. Among them, we should like to mention the following class of rotation-invariant distances:

$$d(x, y) = c|x-y| + \int_0^\infty \mu(dt) \int_{S^{n-1}} |e^{t(x,\omega)} - e^{t(y,\omega)}| \sigma(d\omega), \tag{10}$$

where $c \geq 0$ and μ is a non-negative measure on $(0, \infty)$ such that

$$\int_0^\infty te^{at}\mu(dt) < \infty \qquad \text{for any } a > 0.$$

This class will be further discussed in Sections 3 and 4.

2-1. The first type of an L^1-embeddable metric d on R^n is derived from the d_K in Section 1 with the choice of $K = V_{u/2}$. For each $u > 0$, we set

$$H_u := \{k_p := V_{u/2} + p;\ p \in R^n\} \quad \text{and} \quad \nu_u(dk_p) := dp/2|S^{n-1}|(u/2)^{n-1},$$

to get the desired distance

$$d_u(x, y) := \int_{H_u} \pi_{k_p}(x, y)\nu_u(dk_p) = r_u(|x-y|),$$

where

$$\begin{aligned} r_u(t) &= |(V_{u/2} + te_1) \triangle V_{u/2}|/2|S^{n-1}|(u/2)^{n-1} \\ &= u\int_0^{\min(t/u,1)} (1-v^2)^{(n-1)/2} dv, \qquad e_1 = (1, 0, \cdots, 0) \in R^n. \end{aligned} \tag{11}$$

Observe that $\lim_{u\to\infty} r_u(t) = r_\infty(t) = t$ for each $t > 0$. Hence we put $d_\infty(x, y) := |x - y|$.

Having found the family $\{d_u;\ 0 < u \leq \infty\}$, we now form its superposition by means of a positive measure $G(du)$ on $(0, \infty]$:

$$d(x, y) := \int_{(0,\infty]} d_u(x, y) G(du)\,. \tag{12}$$

The corresponding measure space (H, ν) is obviously taken as follows:

$$H = \{k_{u,p} := V_{u/2} + p;\ 0 < u < \infty,\ p \in R^n\} \cup \{h_{t,\omega};\ t > 0,\ \omega \in S^{n-1}\}$$

(disjoint union) and

$$\nu(dk_{u,p}) = \frac{2^{n-2}}{|S^{n-1}|} u^{-n+1} G(du)dp, \qquad \nu(dh_{t,\omega}) = G(\{\infty\}) \frac{(n-1)|S^{n-1}|}{|S^{n-2}|} dt\sigma(d\omega)\,.$$

Here is a brief comment on the choice of (H, ν). Even if $\nu(B_x) = \infty$ for some $x \in R^n$, the equation (2) still has a meaning under the condition that $\nu(B_x \triangle B_y) < \infty$ for all $x, y \in R^n$. We therefore impose the condition $\int_{(0,\infty]} \min(u, 1) G(du) < \infty$ on the measure G. In order to get at the stronger conclusion that $\nu(B_x) < \infty$ for all $x \in R^n$, it suffices to change every element $h \in H$ containing the origin O for its complement h^c, so that B_O is empty and $\nu(B_x) = \nu(B_x \triangle B_O) < \infty$. This manipulation was explained in (I), Section 2.

The above distance (12) is invariant under every rigid motion on R^n and takes the form $r(|x - y|)$ with

$$r(t) = \int_{(0,\infty]} r_u(t) G(du)\,. \tag{12$'$}$$

It follows that

$$r'(t) = \int_{(t,\infty]} (1 - t^2/u^2)^{(n-1)/2} G(du)\,. \tag{13}$$

In the one-dimensional case, this expression (13) immediately shows the following

Proposition 1. *Suppose $r(t)$ is a continuous function on $[0, \infty)$, $r(0) = 0$ and has the right derive $r'_+(t) \geq 0$ which is non-increasing on $(0, \infty)$ and satisfies $\left|\int_0^1 t dr'_+(t)\right| < \infty$. Then the distance $d(x, y) := r(|x - y|)$ on R^1 is L^1-embeddable.*

For $n \geq 2$, we devote our attention to the case where $G(du)$ is absolutely continuous on $(0, \infty)$ with density $g(u)$ and $G(\{\infty\}) = 0$. The equality (13) becomes

$$r'(t) = \int_t^\infty (1 - t^2/u^2)^{(n-1)/2} g(u) du \,, \tag{13'}$$

which coincides with the classical Radon transform $\hat{f}(\delta h_{t,\omega})$ applied to the radial function $f(y) := g(|y|)/|S^{n+1}||y|^n$ on R^{n+2}, i.e., the integral of f over the hyperplane $\delta h_{t,\omega} := \{y \in R^{n+2}; (y, \omega) = t\}$ in R^{n+2} ([9], p. 103). By appeal to the inversion formula ([9], p. 120), we get

$$g(u) = d_n \int_u^\infty \left\{\left(-\frac{d}{dt}\right)^{n+1} r'(t)\right\} (t^2 - u^2)^{(n-1)/2} dt \,, \tag{14}$$

with

$$d_n := \frac{2^{n-1}}{\pi} \left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^2 .$$

We consider the functions $\psi_\lambda(t) := (1 - e^{-\lambda t})/\lambda$, $\lambda > 0$; every $\psi_\lambda(t)$ satisfies $(-d/dt)^{n+1}\psi'_\lambda(t) \geq 0$ for all $n \geq 2$. By (14), the L^1-embeddable metric $\psi_\lambda(|x-y|)$ on R^n is of the form (12) with the corresponding density

$$g_\lambda(u) = d_n \lambda^{n+1} \int_u^\infty e^{-\lambda t} (t^2 - u^2)^{(n-1)/2} dt \,.$$

Thus, the method of superposition gives us the following

PROPOSITION 2 (cf. [2] and [3]). *Suppose a function $r(t)$ on $[0, \infty)$ is expressed in the form*

$$r(t) = ct + \int_0^\infty \psi_\lambda(t) \gamma(d\lambda) \,, \tag{15}$$

where $c \geq 0$ and γ is a non-negative measure on $(0, \infty)$ such that

$$\int_0^\infty \min(1, \lambda^{-1}) \gamma(d\lambda) < \infty \,.$$

Then the distance $d(x, y) := r(|x-y|)$ on R^n is L^1-embeddable.

2-2. The second type of d is an extension of the norm $||x-y||$ admitting of the expression (9).

PROPOSITION 3. *Suppose $r(t)$ is a function described in Proposition 1. Then the distance*

$$(16) \qquad d(x, y) := \int_{S^{n-1}} r(|x - y, \omega)|\tau(d\omega) \qquad on\ R^n$$

is L^1-embeddable.

Proof. The proof is carried out by constructing a measure space (H, ν) combined with (16) via the equation (2). Since $r(t)$ is of the form

$$(17) \qquad r(t) = ct + \int_0^\infty \min(t, u)G(du), \qquad c := G(\{\infty\}),$$

it is convenient to divide d into two parts:

$$d_1(x, y) := c\int_{S^{n-1}} |(x - y, \omega)|\tau(d\omega) = c||x - y||,$$

and

$$d_2(x, y) := \int_0^\infty G(du)\int_{S^{n-1}} \min(|(x - y, \omega)|, u)\tau(d\omega).$$

We have already described a measure space (H_1, ν_1) for the first part d_1 in Section 1. On the other hand, a measure space (H_2, ν_2) combined with d_2 is easily found; it is

$H_2 := \{k_{u,t,\omega} := \{x \in R^n;\ |(x, \omega) - t| < u/2\};\ 0 < u < \infty,\ t \in R^1$ and $\omega \in S^{n-1}\}$ equipped with $\nu_2(dk_{u,t,\omega}) := G(du)dt\tau(d\omega)/2$.

The proof is thus completed.

For a given norm $||x||$ of negative type, we consider the distance $||x - y||^\alpha$, $0 < \alpha < 1$. It is known ([8] and [14]) that $||x - y||^\alpha$ can be expressed in the form (16) with $r(t) = t^\alpha$. Hence the method of superposition again shows that the distance $d(x, y) := \psi(||x - y||)$ on R^n is L^1-embeddable if $\psi(t) = \int_{(0,1]} t^\alpha m(d\alpha)$, where m is a bounded positive measure on $(0, 1]$.

2-3. In connection with the theory of continuous functions $\phi(x)$ of negative type on the semigroup $(R^n, +)$ ([4]), we proceed to discuss a new class of L^1-embeddable metrics on R^n.

First recall the known expression of ϕ ([4], p. 220):

$$\phi(x) = a + (b, x) - Q(x) + \int_{R^n\setminus\{0\}} \left(1 - e^{(x,\xi)} + \frac{(x, \xi)}{1 + |\xi|^2}\right)\gamma(d\xi),$$

where $a \in R^1$, $b \in R^n$, Q is a non-negative quadratic form on R^n and γ is a non-negative measure on $R^n\setminus\{0\}$ such that

$$\int_{0<|\xi|<1} |\xi|^2 \gamma(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|>1} e^{(x,\xi)}\gamma(d\xi) < \infty \quad \text{for all } x \in R^n .$$

Set $d(x, y) := 2\phi(x+y) - \phi(2x) - \phi(2y)$, to get

$$d(x, y) = Q(x-y) + \int_{R^n\setminus\{0\}} (e^{(x,\xi)} - e^{(y,\xi)})^2 \gamma(d\xi) .$$

This form of d gaurantees the existence of a centered Gaussian random field $X = \{X(x);\ x \in R^n\}$ such that $d(x, y) = E[(X(x) - X(y))^2]$.

We are ready to state the following

PROPOSITION 4. *Suppose $r(t)$ is a function described in Proposition* 1, *and define a distance on R^n by*

$$\text{(18)} \qquad d(x, y) := \int_{R^n\setminus\{0\}} r(|e^{(x,\xi)} - e^{(y,\xi)}|)\gamma(d\xi) ,$$

where γ is a positive measure on $R^n\setminus\{0\}$ such that

$$\int_{0<|\xi|<1} r(|\xi|)\gamma(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|>1} e^{(x,\xi)}\gamma(d\xi) < \infty$$

for all $x \in R^n$. Then d is L^1-embeddable.

Proof. In view of the general form (17) of r, it suffices to treat the two special cases: (i) $r(t) = t$ and (ii) $r(t) = \min(t, u)$, $0 < u < \infty$.

(i) The case $r(t) = t$. A measure space corresponding to (18) is given by

$$\nu(dh_{t,\omega}) = \int_{R^n\setminus\{0\}} \gamma(d\xi)\{|\xi| e^{|\xi|t} dt \delta_{\xi/|\xi|}(d\omega)\}$$

on the set H of half-spaces $h_{t,\omega}$, $t > 0$ and $\omega \in S^{n-1}$, where δ_a denotes the Dirac measure at the point $a \in S^{n-1}$.

(ii) The case $r(t) = \min(t, u)$. Consider the following subset parametrized by $(t, \xi) \in R^1 \times R^n$:

$$\tilde{k}_{t,\xi} := \{x \in R^n ;\, |e^{(x,\xi)} - t| < u/2\} .$$

Then it is easy to verify that the measure

$$\nu_u(d\tilde{k}_{t,\xi}) := dt\gamma(d\xi)/2 \qquad \text{on } H_u := \{\tilde{k}_{t,\xi};\ t \in R^1,\ \xi \in R^n\}$$

yields the desired distance (18) in this second case, which completes the proof.

If a rotation-invariant distance of the form (18) is requested, we must take a rotation-invariant measure γ, which is of the form

$$\gamma(d\xi) = d\mu(|\xi|)d\sigma(\xi/|\xi|) \qquad \text{with a positive measure } \mu \text{ on } (0, \infty)$$

such that $\int_0^\infty r(t)e^{at}d\mu(t) < \infty$ for all $a > 0$. It also deserves mention that one can derive the distance $\|x - y\|^\alpha$ in Section 2–2 as the limit of distances of the form (18) with $r(t) = t^\alpha$, $0 < \alpha \leq 1$. Indeed, for each $\rho > 0$, take the measure $\gamma_\rho(d\xi) := d\tau(\xi/\rho)/\rho^\alpha$ concentrated on the sphere δV_ρ of radius ρ; then one can see that

$$\lim_{\rho \downarrow 0} \int_{\delta V_\rho} |e^{(x,\xi)} - e^{(y,\xi)}|^\alpha \gamma_\rho(d\xi) = \int_{S^{n-1}} |(x - y, \omega)|^\alpha \tau(d\omega) = \|x - y\|^\alpha .$$

2-4. Let X be a centered Gaussian random field with homogeneous increments ([25]). Then the variance $d(x, y) := E[(X(x) - X(y))^2]$ takes the analogous form

$$d(x, y) = Q(x - y) + \int_{R^n \setminus \{0\}} |e^{i(x,\xi)} - e^{i(y,\xi)}|^2 \gamma(d\xi) ,$$

where γ is a spectral measure on $R^n \setminus \{0\}$ satisfying $\int \min(|\xi|^2, 1)\gamma(d\xi) < \infty$. On the lines of Proposition 4, we can prove the following

PROPOSITION 5. *Suppose $r(t)$ is a function described in* [19], *Proposition* 2. *Set*

$$d(x, y) := \int_{R^n \setminus \{0\}} r(d_G(e^{i(x,\xi)}, e^{i(y,\xi)}))\gamma(d\xi) , \tag{19}$$

where d_G denotes the geodesic distance on the unit circle $S^1 = \{z \in C : |z| = 1\}$ and γ is a symmetric positive measure on $R^n \setminus \{0\}$ such that

$$\int_{R^n \setminus \{0\}} \min(r(|\xi|), 1)\gamma(d\xi) < \infty .$$

Then the distance d on R^n is L^1-embeddable.

§ 3. Null spaces of dual Radon transforms

In this section we are concerned with every rotation-invariant distance d on R^n of the form (10). The corresponding measure space is then taken to be the set H of half-spaces $h_{t,\omega}$ equipped with the rotation-invariant measure

$$\nu(dh_{t,\omega}) = \left\{c + \int_0^\infty ue^{tu}\mu(du)\right\}dt\sigma(d\omega)\,. \tag{20}$$

Our aim is to determine the null space $N_1(V_\rho)$ of the dual Radon transform R^* on this $L^2(H, \nu)$ (see (5)). In view of the relation (6) for a Lévy's Brownian motion X with parameter space (R^n, d), our result on $N_1(V_\rho)$ will show a gap between the two closed subspaces $[X(x); x \in V_\rho]$ and $[W(dh_{t,\omega}); h_{t,\omega} \in H(\rho)]$ in $L^2(\Omega, P)$, where $H(\rho) := \{h_{t,\omega} \in H;\ 0 < t \leq \rho,\ \omega \in S^{n-1}\}$ is the set of all half-spaces intersecting V_ρ.

3-1. We shall start with a brief discussion of the restriction $X_{|V_\rho}$ of the whole parameter space R^n to the closed ball V_ρ. Since $B_x \subset H(\rho)$ for every $x \in V_\rho$, the complement of $H(\rho)$ is of no importance. That is, a measure space combined with the distance $d_{|V_\rho}$ on V_ρ via (2) is given by

$$\tilde{H}_\rho := \{\tilde{h}_{t,\omega} := h_{t,\omega} \cap V_\rho;\ h_{t,\omega} \in H(\rho)\} \qquad \text{and } d\tilde{\nu}(\tilde{h}_{t,\omega}) = d\nu(h_{t,\omega})\,,$$

which is isomorphic to the original $(H(\rho), \nu)$.

The relevant dual Radon transform R_ρ^* is, therefore, considered to be a Hilbert-Schmidt operator from $L^2(H(\rho), \nu)$ to $L^2(V_\rho, dx)$, although both R_ρ^* and R^* take the same form

$$\int_{B_x} g(h_{t,\omega})\nu(dh_{t,\omega}), \qquad g \in L^2(H(\rho), \nu)\,.$$

As was shown in (I), Theorem 5, the singular value decomposition of R_ρ^* is expressed by means of $\lambda_{\rho,i} > 0$, $f_{\rho,i}(x)$ and $g_{\rho,i}(h_{t,\omega})$, $i \in I_\rho$:

$$(R_\rho^* g)(x) = \sum_{i \in I_\rho} \lambda_{\rho,i}(g, g_{\rho,i})_{L^2(H(\rho),\nu)} f_{\rho,i}(x)\,,$$

where $\{f_{\rho,i};\ i \in I_\rho\}$ (resp. $\{g_{\rho,i};\ i \in I_\rho\}$) forms an ONS in $L^2(V_\rho, dx)$ (resp. $L^2(H(\rho), \nu)$).

The Gaussian system $X_{|V_\rho}$ now admits of the Karhunen-Loève expansion

$$X(x) = \sum_{i \in I_\rho} \lambda_{\rho,i}\xi_{\rho,i} f_{\rho,i}(x)\,, \qquad x \in V_\rho\,, \tag{21}$$

where the system

$$\xi_\rho = \left\{\xi_{\rho,i} := \int_{H(\rho)} g_{\rho,i}(h_{t,\omega})W(dh_{t,\omega});\ i \in I_\rho\right\}$$

is an i.i.d. sequence of standard Gaussian random variables. Moreover we have

$$(22)\qquad [X(x); x \in V_\rho] = [\xi_{\rho,i};\ i \in I_\rho] = \left\{\int_{H(\rho)} g(h_{t,\omega}) W(dh_{t,\omega});\ g \in N_\rho^\perp\right\},$$

with the null space N_ρ of R_ρ^*:

$$N_\rho: = \{g \in L^2(H(\rho), \nu);\ (R_\rho^* g)(x) \equiv 0 \text{ on } V_\rho\}.$$

Note that $N_1(V_\rho) = N_\rho \oplus L^2(H(\rho)^c, \nu)$, which implies that (22) coincides with (6) for $A = V_\rho$.

3-2. We are now going to determine the null space N_ρ of R_ρ^*, $0 < \rho < \infty$.

For that purpose we need

LEMMA 6. *We have an expansion*

$$(23)\qquad \begin{aligned} \chi_{B_x}(h_{t,\omega}) &= \sum_{m=0}^{\infty} \lambda_m(t/|x|) \sum_{k=1}^{h(m)} S_{m,k}(x/|x|) S_{m,k}(\omega) \\ &= \sum_{m=0}^{\infty} \lambda_m(t/|x|) h(m) \Phi_m^q((x, \omega)/|x|), \end{aligned}$$

where $\Phi_m^q(t): = C_m^q(t)/C_m^q(1)$ *with* $q: = (n-2)/2$, *and*

$$(24)\qquad \lambda_m(t) = \frac{|S^{n-2}|}{|S^{n-1}|} \chi_{(0,1]}(t) \int_t^1 \Phi_m^q(u)(1-u^2)^{q-1/2} du.$$

Furthermore we have

$$(25)\qquad \lambda_m(t) = \frac{|S^{n-2}|}{|S^{n-1}|(n-1)} \Phi_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}$$

for $m \geq 1$ *and* $0 < t < 1$.

Proof. Since $\chi_{B_x}(h_{t,\omega}) = \chi_{h_{t,\omega}}(x) = \chi_{(t/|x|,1]}((x', \omega))$, $x': = x/|x|$, the above assertions for the variables $\omega, x' \in S^{n-1}$ coincide with (I), Lemma 7 stated in terms of the variables $x, y \in S^n$.

Now, take an arbitrary function g from $L^2(H(\rho), \nu)$. Such a function is written in the form

$$g(h_{t,\omega}) = \sum_{(m,k) \in \Delta} g_{m,k}(t) S_{m,k}(\omega),$$

where

$$g_{m,k}(t): = \int_{S^{n-1}} g(h_{t,\omega}) S_{m,k}(\omega) \sigma(d\omega), \qquad 0 < t \leq \rho.$$

The density in the expression (20) of ν is simply denoted by

(20′) $$v(t):=c+\int_0^\infty ue^{ut}\mu(du).$$

Then all functions $g_{m,k}(t)$ belong to $L^2((0,\rho],\, v(t)dt)$, because

$$\sum_{(m,k)\in\Delta}\int_0^\rho g_{m,k}^2(t)v(t)dt=\|g\|^2_{L^2(H(\rho),\nu)}<\infty .$$

Lemma 6 implies that

$$(R_\rho^* g)(x)=\sum_{(m,k)\in\Delta}S_{m,k}(x/|x|)\int_0^{|x|}\lambda_m(t/|x|)g_{m,k}(t)v(t)dt .$$

We now assume that $g\in N_\rho$. Then we have

(26) $$\int_0^u \lambda_m(t/u)g_{m,k}(t)v(t)dt\equiv 0\,,\qquad 0<u\le\rho\,,$$

for every $(m,k)\in\Delta$.

In case $m=0$, we make use of (24) to get

$(26)_0$ $$\int_0^u (1-t^2/u^2)^{(n-3)/2}G_{0,1}(t)dt\equiv 0,\qquad 0<u\le\rho\,,$$

where we have put

$$G_{0,1}(t):=\int_0^t g_{0,1}(s)v(s)ds .$$

As is well known ([12]), p. 14), the integral equation $(26)_0$ yields the unique solution $G_{0,1}(t)\equiv 0$, i.e., $g_{0,1}(t)\equiv 0$ on $(0,\rho]$.

The equation (26) for $m\ge 1$ takes a different form: By (25), we have

$(26)_m$ $$\begin{aligned}&\int_0^1 C_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}G_{m,k}(ut)dt\\ &\quad=\int_{-1}^1 C_{m-1}^{q+1}(t)(1-t^2)^{q+1/2}G_{m,k}(ut)dt/2\equiv 0\,,\quad 0<u\le\rho,\end{aligned}$$

where

$$G_{m,k}(t)=g_{m,k}(t)v(t)\quad\text{for}\quad 0<t\le\rho\quad\text{and}\quad G_{m,k}(t)=(-1)^{m-1}g_{m,k}(-t)v(-t)$$

for $-\rho\le t<0$. The theorem in Ludwig [16] for the Gegenbauer transform (see also [20] and [21]) now concludes that $G_{m,k}(t)$, $t>0$, is a polynomial of the form $\sum_{j=1}^{[(m-1)/2]}a_{m,k,j}\,t^{m-1-2j}$ with some coefficients $a_{m,k,j}\in R^1$, We have thus proved that g in N_ρ is necessarily of the form

$$\sum_{(m,k,j)\in J}a_{m,k,j}p_{m,k,j},\qquad\text{where } p_{m,k,j}(h_{t,\omega}):=S_{m,k}(\omega)t^{m-1-2j}/v(t)$$

and $J := \{(m, k, j) \in Z^3;\ m \geq 3,\ 1 \leq k \leq h(m)$ and $1 \leq j \leq [(m-1)/2]\}$.

Conversely, the functions $p_{m,k,j}(h_{t,\omega})$, $(m, k, j) \in J$, form an orthogonal system in $L^2(H(\rho), \nu)$ and we can check that every $p_{m,k,j}$ belongs to the null space N_ρ.

What we have proved is summarized below.

THEOREM 7. *Let R_ρ^* be the dual Radon transform on $L^2(H(\rho), \nu)$, where ν is a measure of the form* (20). *Then we have*

$$N_\rho = [p_{m,k,j}(h_{t,\omega});\ (m, k, j) \in J]\,.$$

In other words, a function g belongs to N_ρ if and only if g is expressed in the form

$$g(h_{t,\omega}) = \sum_{(m,k,j)\in J} a_{m,k,j} S_{m,k}(\omega) t^{m-1-2j}/v(t)\,. \tag{27}$$

Let ρ go to infinity in the above theorem. Then we obtain, as a by-product of Theorem 7, a complete description of the full null space of R^*:

$$N_\infty := \{g \in L^2(H, \nu);\ (R^* g)(x) \equiv 0,\ x \in R^n\}\,.$$

THEOREM 7′. *If the measure μ in* (20′) *is equal to* 0 (*in other words, if $d(x, y) = c|x-y|$, $c > 0$*), *then $N_\infty = \{0\}$, i.e., R^* is injective on $L^2(H, \nu)$. While, if μ is positive we have $N_\infty = [p_{m,k,j}(h_{t,\omega});\ (m, k, j) \in J]$.*

3-3. We are now in a position to state noteworthy consequences of the preceding results. By virtue of the relation (22), our conclusion follows from Theorems 7 and 7′.

THEOREM 8. *Let X be a Lévy's Brownian motion with parameter space (R^n, d), where d is of the form* (10). *Then we have, for $0 < \rho < \infty$,*

$$[X(x) \colon x \in V_\rho] = \Big\{\iint_{H(\rho)} g(h) W(dh);\ g \in L^2(H(\rho), \nu) \text{ satisfying}$$
$$\int_0^\rho t^{m-1-2j} dt \int_{S^{n-1}} S_{m,k}(\omega) g(h_{t,\omega}) \sigma(d\omega) = 0$$
$$\text{for all } (m, k, j) \in J\Big\}\,.$$

For the case $\rho = \infty$, we have

$$[X(x);\ x \in R^n] = \Big\{\iint_H g(h) W(dh);\ g \in L^2(H, \nu)\Big\},$$
$$\text{if} \quad d(x, y) = c|x-y|, \quad c > 0\,,$$

and

$$[X(x);\, x\in R^n] = \left\{\int_H g(h)W(dh);\ g\in L^2(H,\nu) \text{ satisfying } \int_0^\infty t^{m-1-2j}dt\int_{S^{n-1}} S_{m,k}(\omega)g(h_{t,\omega})\sigma(d\omega) = 0 \text{ for all } (m,k,j)\in J\right\},$$

if d is given by (10) *with positive μ.*

3-4. With a suitable choice of $\alpha(x)>0$ satisfying $\int_{R^n}\nu(B_x)\alpha(x)dx<\infty$, the Hilbert-Schmidt operator $R\circ T_\alpha$ from $L^2(R^n, \alpha(x)dx)$ to $L^2(H,\nu)$ was discussed in connection with a factorization of the covariance operator of X ((I), Theorem 3). As a counterpart of the exterior Radon transform (cf. [21] and [22]), it would be interesting to study the exterior halfspace transform

$$(R\circ T_\alpha f)(h_{t,\omega}) := \int_{h_{t,\omega}} f(x)\alpha(x)dx\,, \qquad f\in L^2(V_\rho^c, \alpha(x)dx)\,, \tag{28}$$

where the resultant function $R\circ T_\alpha f$ is considered to be in $L^2(H(\rho)^c, \nu)$.

Under the assumption that α is a radial function, $\alpha(x)=\alpha(|x|)$ on V_ρ^c, we can determine the null space of $R\circ T_\alpha$:

$$N_\rho(\alpha) := \{f\in L^2(V_\rho^c, \alpha(|x|)dx);\ (R\circ T_\alpha f)(h_{t,\omega})\equiv 0 \text{ on } H(\rho)^c\}\,.$$

First observe that, for a given $f\in L^2(V_\rho^c, \alpha(|x|)dx)$,

$$(R\circ T_\alpha f)(h_{t,\omega}) = (R_\alpha^*\tilde f)(\omega/t)\,, \qquad t>\rho \text{ and } \omega\in S^{n-1}\,, \tag{29}$$

where $\tilde f(h_{t,\omega}) := f(\omega/t)\in L^2(H(\rho^{-1}), \nu_\alpha)$, ν_α being a measure on $H(\rho^{-1})$ defined by

$$\nu_\alpha(dh_{t,\omega}) := |S^{n-1}|t^{-n-1}\alpha(1/t)dt\sigma(d\omega)\,,$$

and R_α^* is the dual Radon transform defined on $L^2(H(\rho^{-1}), \nu_\alpha)$. On the lines of Theorem 7, we can prove the following

Proposition 9. *For $0<\rho<\infty$, we have*

$$N_\rho(\alpha) = [f_{m,k,j}(x);\ (m,k,j)\in J(\alpha)]\,,$$

where

$$f_{m,k,j}(x) := S_{m,k}(x/|x|)|x|^{-m-n+2j}/\alpha(|x|), \qquad |x|\ge\rho\,,$$

and

$$J(\alpha):=\left\{(m,k,j)\in J;\int_\rho^\infty t^{-2m-n+4j-1}(\alpha(t))^{-1}dt<\infty\right\}.$$

§ 4. The McKean processes

As in Section 3, we shall assume that $X=\{X(x);\ x\in R^n\}$ is a Lévy's Brownian motion with parameter space (R^n, d), where d is a rotation-invariant distance of the form (10). Since the representations (8) of the McKean processes $M_{m,k}(t)$, $(m,k)\in\Delta$, follow from the original representation (1) of X, we can answer, as a byproduct of Theorem 8, the basic question concerning the canonical property of (8).

We being by applying Lemma 6 to the representation (1) of X; we get

$$\begin{aligned}(30)\qquad X(x)&=\sum_{(m,k)\in\Delta}S_{m,k}(x/|x|)\int_{H(|x|)}\lambda_m(t/|x|)S_{m,k}(\omega)\,W(dh_{t,\omega})\\&=\sum_{(m,k)\in\Delta}S_{m,k}(x/|x|)\int_0^{|x|}\lambda_m(t/|x|)dB_{m,k}(t)\,,\end{aligned}$$

where the Gaussian processes $B_{m,k}(t)$, $(m,k)\in\Delta$, are defined by

$$(31)\qquad B_{m,k}(t):=\int_{H(t)}S_{m,k}(\omega)W(dh_{u,\omega})\,,\qquad t>0\,.$$

Observe that

$$\begin{aligned}E[B_{m,k}(t)B_{m',k'}(t')]&=\int_{H(t)\cap H(t')}S_{m,k}(\omega)S_{m',k'}(\omega)\nu(dh_{u,\omega})\\&=\delta_{(m,k),(m',k')}\int_0^{\min(t,t')}v(u)du\,,\end{aligned}$$

where $v(u)$ was given by (20′). This shows that the processes $B_{m,k}(t)$ are mutually independent Gaussian additive processes with common spectral density $v(t)=E[(B_{m,k}(dt))^2]/dt$.

In view of the expression (30) of X, we are naturally led to the following

DEFINITION 2 (cf. [17]). The Gaussian process

$$(7)\qquad M_{m,k}(t):=\int_{S^{n-1}}X(t\omega)S_{m,k}(\omega)\sigma(d\omega)\,,\qquad t>0\,,$$

is called the *McKean process with index* (m,k), $(m,k)\in\Delta$. In the case $m=0$, $M_{0,1}(t)$ has a more familiar name, the *$M(t)$-process* (cf. [15]).

With this definition, the expression (30) is rewritten as follows:

$$X(x) = \sum_{(m,k)\in\Delta} S_{m,k}(x/|x|)M_{m,k}(|x|)\,,$$

and

$$M_{m,k}(t) = \int_0^t \lambda_m(u/t)dB_{m,k}(u)\,, \tag{8}$$

where the kernel $\lambda_m(t)$ was computed in Lemma 6.

Now, Theorem 8 is rephrased in terms of these Gaussian processes $M_{m,k}(t)$ and $B_{m,k}(t)$, $(m, k) \in \Delta$.

THEOREM 10. (i) *In the case* $m \leq 2$, *the representation* (8) *of* $M_{m,k}(t)$ *is canonical, i.e.,*

$$[M_{m,k}(t); t \leq \rho] = [B_{m,k}(t);\ t \leq \rho] \qquad \textit{for every } \rho > 0.$$

(ii) *In the case* $m \geq 3$, *the representation* (8) *of* $M_{m,k}(t)$ *is not canonical. Furthermore we have*

$$\begin{aligned}&[B_{m,k}(t);\ t \leq \rho] \ominus [M_{m,k}(t);\ t \leq \rho]\\ &\quad = \left[\int_0^\rho t^{m-1-2j}(v(t))^{-1}dB_{m,k}(t);\ 1 \leq j \leq [(m-1)/2]\right]\end{aligned}$$

for every $0 < \rho < \infty$, *and*

$$\begin{aligned}&[B_{m,k}(t);\ t > 0] \ominus [M_{m,k}(t);\ t > 0]\\ &\quad = \begin{cases}\{0\} & \textit{if } d(x, y) = c|x-y|\,,\quad c > 0\,,\\ \left[\int_0^\infty t^{m-1-2j}(v(t))^{-1}dB_{m,k}(t);\ 1 \leq j \leq [(m-1)/2]\right], & \end{cases}\end{aligned}$$

otherwise.

Concluding remarks. (i) Our discussions in Sections 3 and 4 can be extended to the case with other parameter spaces such as $M = S^n$ (n-sphere) or H^n (n-dimensional real hyperbolic space). In particular, consider a familiar Lévy's Brownian motion X with parameter space (M, d_G), d_G being the usual geodesic distance on $M = S^n$ or H^n (cf. [18] and [23]). Such an X admits of a nice representation ([23]) analogous to (1) for a Brownian motion with n-dimensional parameter. By making use of this known representation of X, we can show that Theorems 8 and 10 have respective counterparts in these two cases of (S^n, d_G) and (H^n, d_G). The details are omitted.

(ii) In their study of conformal invariance of white noise, Hida, Lee and Lee [13] introduced a generalized Gaussian random field $Y = \{Y(x);\ x \in R^n,\ 0 < |x| < 1\}$ defined by

$$Y(x) = \int_{B_x} F(x, h_{t,\omega}) W(dh_{t,\omega}), \tag{32}$$

where the kernel F is given by

$$F(x, h_{t,\omega}) = a(x) t^{-n+1} / \{(x, \omega) - t|x|\}, \tag{33}$$

and $W = \{W(dh_{t,\omega}); h_{t,\omega} \in H(1)\}$ is a Gaussian random measure (white noise) with variance $\nu(dh_{t,\omega}) := t^{n-1} dt \sigma(d\omega)$, ν being a measure on the set $H(1)$ of half-spaces $h_{t,\omega}$, $0 < t < 1$ and $\omega \in S^{n-1}$.

This representation (32) of Y might be thought of as a multi-dimensional version of canonical representations of Gaussian processes, and takes a more general form than the representation (1) of Chentsov type (which corresponds to the choice of $F(x, h_{t,\omega}) \equiv 1$). This generality would cause us many difficulties in investigating the integral transformation R_F^* associated with (32):

$$(R_F^* g)(x) := \int_{B_x} F(x, h_{t,\omega}) g(h_{t,\omega}) \nu(dh_{t,\omega}), \tag{34}$$

g being in a suitable class of functions on $H(1)$. But in the persent situation where the kernel F is specified by (33) with the additional condition that $a(x) > 0$, we can prove analogous results on the null spaces $N_\rho(F)$ of R_F^*, $0 < \rho < 1$:

$$N_\rho(F) := \{g(h_{t,\omega});\ \operatorname{supp} g \subset H(\rho) \quad \text{and } (R_F^* g)(x) \equiv 0,\ 0 < |x| \leq \rho\}.$$

Indeed, similar arguments to Section 3-2 lead us to the following conclusion:

$$N_\rho(F) = [g_{m,k,j}(h_{t,\omega});\ m \geq 2,\ 1 \leq k \leq h(m) \text{ and } 1 \leq j \leq [m/2]],$$

where we put

$$g_{m,k,j}(h_{t,\omega}) := t^{m-2j} \chi_{(0,\rho]}(t) S_{m,k}(\omega).$$

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