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# EVEN CANONICAL SURFACES WITH SMALL $K^2$, III

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## Introduction

This is a continuation of [5]. In the present part, we study irregular even surfaces of general type with $K^2 < 4\chi$, where $K$ and $\chi$ denote respectively a canonical divisor and the holomorphic Euler-Poincaré characteristic. As the first main theorem, we show the following:

THEOREM 1. *For any irregular even surface of general type with $K^2 < 4\chi$, the image of the Albanese map is a curve.*

Therefore, Severi's conjecture is true for even surfaces. The proof uses the unramified covering trick and rests heavily on Xiao's theorem: We take a finite unramified covering of an even surface, which is again even, and show that the semi-canonical map induces a pencil. Then we apply [10, Theorem 3] and [11, Theorem 1] to see that the genus of the base of the pencil equals the irregularity.

We next turn our attention to canonical surfaces and give the genus bound on the Albanese pencil. As in [5, I, §5], we employ an argument modeled on [9] (see also [7]) to show the following:

THEOREM 2. *For any irregular even canonical surface with $K^2 < 4\chi$, the Albanese pencil is a trigonal pencil of genus $g \leq 6$. Furthermore,*

$$K^2 \geq \begin{cases} 3\chi + 10(q-1) & \text{if } g = 3, \\ \dfrac{8g(g-1)}{g^2+7g-16}\left(\chi + 8\left(1 - \dfrac{2}{g}\right)(g-1)\right) & \text{if } 4 \leq g \leq 6. \end{cases}$$

As we noticed in [5, I, §8], the existence of a trigonal pencil on a canonical surface implies that the canonical image cannot be cut out by hyperquadrics. Thus Theorem 2 and [5, I, Lemma 8.2 and Theorem 8.3] give us

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THEOREM 3. *The canonical image of an irregular even canonical surface with* $K^2 < 4\chi$ *is not cut out by hyperquadrics. If the numerical characters further satisfy* $K^2 \leq 4p_g + q - 12$, *then the quadric hull of the surface is of dimension* 3. *In particular, Reid's conjecture* [8, p.541] *is true for irregular even surfaces with* $q \leq 4$.

## 1. Inequality

In this section, we shall show some inequalities generalizing one in [5, I, §5] by using a method modeled on [9]. See also [7].

Let $S$ be an even surface of general type. Note that it is automatically minimal, and we have a line bundle $L$ on $S$ satisfying $K = 2L$. Such a line bundle $L$ is called a semi-canonical bundle of $S$. Suppose that we have a fibration $f: S \to B$ over a non-singular projective curve $B$ of genus $b$. Let $g$ be the genus of a general fibre $D$ of $f$. Since $B$ is a curve, $f_*L$ is a locally free sheaf [2, Corollary 1.7].

LEMMA 1.1. $h^0(L) = h^0(f_*L)$ *and* $\chi(L) = 2\chi(f_*L) - \mathrm{length}(R^1 f_*L)_{tor}$.

*Proof.* By the relative duality theorem, the dual of $R^1 f_*L$ is isomorphic to $f_*L \otimes \omega_B^{\otimes(-1)}$ since $K = 2L$. Then, using the Leray spectral sequence $E_2^{p,q} = H^p(R^q f_*L) \Rightarrow H^{p+q}(L)$, we get the assertion. Q.E.D.

Let $\mathcal{E}$ be the subbundle of $f_*L$ generically generated by global sections. Then $h^0(B, \mathcal{E}) = h^0(L)$. Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

be the Harder-Narashimhan filtration of $\mathcal{E}$ [1], that is, the unique filtration of $\mathcal{E}$ by subbundles $\mathcal{E}_i$ which satisfies

(i) $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable,
(ii) $\mu_i > \mu_{i+1}$, where $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \deg(\mathcal{E}_i/\mathcal{E}_{i-1})/\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$.

Let $\phi_i: \mathbf{P}(\mathcal{E}_i) \to B$ be the associated projective bundle. We denote by $H_i$ and $F$ a (relatively ample) tautological divisor and a fibre of $\phi_i$, respectively. By [4, Lemma 4.6], the $\mathbf{Q}$-divisors $H_i - \mu_i F$ and $H_i - \mu_1 F$ are respectively nef and pseudo-effective.

The natural sheaf homomorphism $f^*\mathcal{E} \to f^*f_*L \to L$ induces a rational map $\psi_i: S \to \mathbf{P}(\mathcal{E}_i)$ such that $f = \phi_i \circ \psi_i$. Let $\rho_i: X_i \to S$ be a composite of blowing-ups which eliminates the indeterminacy of $\psi_i$, and let $\tilde{\psi}_i: X_i \to \mathbf{P}(\mathcal{E}_i)$ be the induced holomorphic map. We denote by $\tilde{M}_i$ the pull-back to $X_i$ of $H_i$ via $\tilde{\psi}_i$.

Then $\tilde{M}_i - \mu_i\rho_i^*D$ is nef, since so is $H_i - \mu_i F$. Put $M_i = (\rho_i)_*\tilde{M}_i$. Then $M_i - \mu_i D$ is also nef. Note that we have $L \equiv M_i + Z_i$ with an effective divisor $Z_i$, where the symbol $\equiv$ means the numerical equivalence. Put $d_i = M_i D$. When $i = l$, we drop the subscript $l$ and write $d$, $M$, $Z$, instead of $d_l$, $M_l$, $Z_l$ for the sake of simplicity.

Put $a_i = g - 1 - d_i = DZ_i$. Then the multiplicity of any irreducible component of $Z_i$ having positive intersection with $D$ is at most $a_i$. Using this and the nefness of $K_{S/B}$, we can show that $a_iK_{S/B} + M_i - \mu_i D + Z_i$ is a nef $\mathbf{Q}$-divisor as in [9, Lemma 4 and Corollary 1] (see also [7, Lemma 2.1]). Since $K = 2L$, we get the following:

LEMMA 1.2. *For each* $i$, $(2a_i + 1)L - (\mu_i + 2a_i(b - 1))D$ *is nef.*

LEMMA 1.3. *Let the notation and assumption be as above.*
(1) *If* $\mathrm{rk}(\mathscr{E}) = 1$, *then*

$$L^2 \geq \frac{2(g-1)}{2g-1}\{g\deg(\mathscr{E}) + (g-1)(b-1)\}.$$

(2) *If* $\mathrm{rk}(\mathscr{E}) = 2$, *then*

$$L^2 \geq \frac{g-1}{2g-d-1}\{g\deg(\mathscr{E}) + 2(g-d-1)(b-1)\}.$$

*Proof.* Since $L - \mu_1 D$ is pseudo-effective, we have $((2a_i + 1)L - (\mu_i + 2a_i(b-1))D)(L - \mu_1 D) \geq 0$ which is equivalent to

$$(2a_i + 1)L^2 \geq (g-1)((2a_i+1)\mu_1 + \mu_i + 2a_i(b-1)). \tag{1.1}$$

If $\mathscr{E}$ is a line bundle, then (1.1) for $i = 1$ gives (1) since $d = 0$ in this case.

Assume that $\mathscr{E}$ is of rank 2. When $\mathscr{E}$ is semi-stable, we put $\mu_1 = \mu_2 = \mu(\mathscr{E})$ in the following calculation. If $d = g - 1$, then we get (2) using(1.1) for $i = 2$. Hence we assume that $a := g - 1 - d > 0$. We have

$$\begin{aligned} L^2 &= L(M - \mu_2 D) + (g-1)\mu_2 + LZ \\ &= (M - \mu_1 D + Z_1)(M - \mu_2 D) + d\mu_1 + (g-1)\mu_2 + LZ \\ &= (g - 1 - a)\mu_1 + (g-1)\mu_2 + LZ. \end{aligned}$$

Since $(2a+1)L - (\mu_2 + 2a(b-1))D$ is nef, we have $(2a+1)LZ \geq a(\mu_2 + 2a(b-1))$.
Hence

(1.2) $(2a+1)L^2 \geq (2a+1)(g-1-a)\mu_1 + ((2a+1)(g-1)+a)\mu_2 + 2a^2(b-1)$

from which we get (2) if $(2g-1)\mu_2 \geq (2a+1)\mu_1 + 2d(b-1)$. Otherwise, (1.1) for $i=2$ is sufficient to imply (2), since $\deg(\mathscr{E}) = \mu_1 + \mu_2$. Q.E.D.

In order to estimate $\deg(\mathscr{E})$, we introduce a quotient bundle $\mathscr{F}$ of $\mathscr{E}$ according to [9]: Let $\mathscr{F}'$ be the subbundle of $\mathscr{E}^* := \mathrm{Hom}(\mathscr{E}, \omega_B)$ generically generated by $H^0(\mathscr{E}^*)$, and put $\mathscr{F} = (\mathscr{F}')^*$. Then $\mathscr{F}$ and $\mathscr{F}^*$ are both nef, and we have $h^1(\mathscr{E}) = h^1(\mathscr{F}) = h^0(\mathscr{F}^*)$. Hence Clifford's theorem [9] gives us

$$h^1(\mathscr{E}) \leq \frac{1}{2}\deg(\mathscr{F}^*) + \mathrm{rk}(\mathscr{F}^*) = b\,\mathrm{rk}(\mathscr{F}) - \frac{1}{2}\deg(\mathscr{F}).$$

Since $h^0(\mathscr{E}) = h^0(L)$, the Riemann-Roch theorem gives us the following:

LEMMA 1.4. *If* $\mathrm{rk}(\mathscr{F}) = \mathrm{rk}(\mathscr{E})$, *then* $\deg(\mathscr{E}) \geq 2(h^0(L) - \mathrm{rk}(\mathscr{E}))$. *If* $\mathrm{rk}(\mathscr{F}) < \mathrm{rk}(\mathscr{E})$, *then* $\deg(\mathscr{E}) \geq h^0(L) - \mathrm{rk}(\mathscr{E}) + b(\mathrm{rk}(\mathscr{E}) - \mathrm{rk}(\mathscr{F})) + \deg(\mathscr{F})/2$.

## 2. Proof of Theorem 1

In what follows, let $S$ be an irregular even surface of general type satisfying $K^2 < 4\chi(\mathcal{O}_S)$, and let $L$ be a semi-canonical bundle. In this section, we show Theorem 1.

Since $S$ is an irregular surface, we can find an $m$-torsion element $\eta \in \mathrm{Pic}^0(S)$ for any positive integer $m$. Let $\pi_m : S_m \to S$ be the cyclic $m$-sheeted unramified covering associated with $\eta$. If $K_m$ denotes the canonical bundle of $S_m$, then $K_m = \pi_m^* K$. Therefore, $S_m$ is also an even surface of general type with a semi-canonical bundle $L_m = \pi_m^* L$. By the universality of the Albanese map, we have a commutative diagram

$$\begin{array}{ccc} S_m & \to & \mathrm{Alb}(S_m) \\ \pi_m \downarrow & & \downarrow \\ S & \to & \mathrm{Alb}(S), \end{array}$$

where the horizontal maps are the Albanese maps. Hence, in order to prove Theorem 1, it suffices to show that the Albanese image of $S_m$ is a curve for some $m$.

By the Riemann-Roch theorem, we have

$$\chi(L_m) = 2h^0(L_m) - h^1(L_m) = -\frac{L_m^2}{2} + \chi_m, \tag{2.1}$$

where we put $\chi_m = \chi(\mathcal{O}_{S_m})$. Since $K^2 = 4L^2$, we have $K^2 \leq 4\chi - 4$. Hence

$$K_m^2 = mK^2 \leq 4m(\chi - 1) = 4\chi_m - 4m. \tag{2.2}$$

Put $n = n(m) = h^0(L_m) - 1$. Since $S_m$ is an even surface of general type. $L_m^2$ is a positive even integer. Hence there is an integer $k$ satisfying $L_m^2 = 4n - 2k$. By (2.2), we have $\chi_m \geq L_m^2 + m = 4n - 2k + m$. Then it follows from (2.1) that $k \geq m - 2 + h^1(L_m) \geq m - 2$. Thus

$$L_m^2 \leq 4n + 4 - 2m. \tag{2.3}$$

Since $L_m^2$ is a positive even integer, we have $n > 0$ when $m \geq 2$. Assuming that $m$ is sufficiently large, we consider the semi-canonical map, that is, the rational map $\Phi_m : S_m \rightarrow \mathbf{P}^n$ associated with $|L_m|$.

*Case* 1. Assume that $|L_m|$ is composed of a pencil for some $m \geq 4$. We have $L_m \equiv \nu D_m + G_m$, where $\{D_m\}$ is a pencil of irreducible curves, $G_m$ is the fixed part of $|L_m|$ and $\nu$ is a positive integer with $\nu \geq n$.

We show that $\{D_m\}$ is a non-linear pencil. Since $L_m^2 \geq 2$, we have $n > 1$ by (2.3). We have

$$4n + 4 - 2m \geq L_m^2 = \nu L_m D_m + L_m G_m \geq n L_m D_m.$$

Hence $L_m D_m \leq 3$. Since $S_m$ is an even surface, $D_m^2$ is a non-negative even integer. From

$$3 \geq L_m D_m = \nu D_m^2 + D_m G_m \geq n D_m^2 \geq 2D_m^2$$

it follows $D_m^2 = 0$. Therefore, $\{D_m\}$ is a pencil of curves of genus $g_m \leq 4$. If it were a linear pencil, then the numerical characters must satisfy

$$K_m^2 \geq 4\chi_m - 4(g_m - 1) \tag{2.4}$$

by a result of Xiao [11]. Since $m \geq 4$ and $g_m \leq 4$, this is impossible by (2.2).

Hence $\{D_m\}$ defines a holomorphic map $f_m : S_m \rightarrow B_m$ onto a non-singular projective curve $B_m$ of positive genus. Then, by [11] again, we know that $B_m$ is a curve of genus $q_m := q(S_m)$. It follows that the Albanese image of $S_m$ is a curve, since we have a commutative diagram

$$\begin{array}{ccc} S_m & \rightarrow & \mathrm{Alb}(S_m) \\ f_m \downarrow & & \downarrow \\ B_m & \rightarrow & \mathrm{Jac}(B_m), \end{array} \tag{2.5}$$

where $\mathrm{Jac}(B_m)$ denotes the Jacobian of $B_m$.

*Case* 2. Suppose that $|L_m|$ is not composed of a pencil and $m \geq 8$. Since we have $L_m^2 < 4n - 6$ by (2.3), it follows from [5, I, Lemma 2.1] that the semi-canonical map $\Phi_m$ is not birational onto its image. Let $V_m$ be the image of $\Phi_m$. Since $\deg V_m \geq n - 1$, we have $L_m^2 \geq (\deg \Phi_m)(\deg V_m) \geq (\deg \Phi_m)(n-1)$. It follows from (2.3) that $\deg \Phi_m = 2, 3$. Furthermore, since

$$\deg V_m \leq \frac{1}{2} L_m^2 \leq 2n + 2 - m < 2n - 2,$$

$V_m$ is birationally equivalent to a ruled surface as is well-known (see e.g., [3, Theorem 1.1]).

Assume that $\deg \Phi_m = 2$. Since $V_m$ is a ruled surface, $\Phi_m$ induces on $S_m$ a pencil of hyperelliptic curves. Since we have $K_m^2 < 4\chi_m - 28$ by (2.2), [10, Theorem 3] shows that the Albanese image of $S_m$ is a curve.

Assume that $\deg \Phi_m = 3$. We claim that $V_m$ is not a rational surface. Assume the contrary. We can show that $V_m$ is not isomorphic to $\mathbf{P}^2$ as in [5, I, Lemma 4.3]. Since we have $\deg V_m < (4/3)(n-2)$, it follows from [9, Lemma 1] that $V_m$ is ruled by straight lines. Then, as in [5, I, Theorem 5.1], one can show that a ruling of $V_m$ induces on $S_m$ a linear pencil of curves of genus $g_m \leq 6$ without base points. On the other hand, it follows from [11] that the numerical characters of $S_m$ must satisfy the inequality (2.4), which is impossible by (2.2). Thus we have shown that $V_m$ is birationally equivalent to an irrational ruled surface.

We consider the unique ruling $V_m \to B_m$, and let $f_m$ be the composite $S_m \to V_m \to B_m$. Since $B_m$ is not $\mathbf{P}^1$, $f_m$ is a holomorphic map. It follows from [11] that $B_m$ is of genus $q_m$, and we can identify $f_m$ with the Albanese map of $S_m$.

In sum, Theorem 1 has been established.

## 3. Proof of Theorem 2

In this section, we shall prove Theorem 2. For this purpose, we freely use the notation in the previous sections. Recall that a minimal surface is called a canonical surface if its canonical map is a birational map onto the image. We remark the following:

LEMMA 3.1. *Let $\tilde{X}$ be any finite unramified covering of a canonical surface $X$. Then $\tilde{X}$ has no pencils of hyperelliptic curves.*

*Proof.* If $\tilde{X}$ has a hyperelliptic pencil, then its canonical image is a ruled surface or a curve. This is impossible, since it dominates the canonical image of $X$. Q.E.D.

Now, let $S$ be an irregular even canonical surface with $K^2 < 4\chi$. As we saw in Section 2, for a sufficiently large integer $m$, the semi-canonical map induces a fibration $f_m : S_m \to B_m$ of genus $g_m$, where $B_m$ is of genus $q_m > 0$. Lemma 3.1 in particular implies that $\deg \Phi_m = 3$ in Case 2 of Section 2.

LEMMA 3.2. *Assume that $m \geq 4$ and the semi-canonical map of $S_m$ is composed of a pencil. Then $g_m = 3$ or 4, and*

$$L_m^2 \geq \frac{2g_m(g_m - 1)}{2g_m - 1} h^0(L_m) + 2(g_m - 1)(q_m - 1).$$

*Assume that $m \geq 7$ and the semi-canonical map of $S_m$ is of degree 3 onto its image. Then $4 \leq g_m \leq 6$ and*

$$L_m^2 \geq \begin{cases} 3(\chi(L_m)/2 + 2(q_m - 1)) & \text{if } g_m = 4, \\ \dfrac{g_m(g_m - 1)}{2(g_m - 2)} h^0(L_m) + 2(g_m - 1)(q_m - 1)) & \text{if } g_m = 5, 6. \end{cases}$$

*Proof.* As in Section 1, let $\tilde{\mathscr{E}}$ be the subbundle of $(f_m)_*L_m$ generically generated by $H^0(B_m, (f_m)_*L_m)$.

Assume that the semi-canonical map of $S_m$ is composed of a pencil. Then $\tilde{\mathscr{E}}$ is a line bundle. If it is special, then $\deg(\tilde{\mathscr{E}}) \geq 2h^0(L_m) - 2 = 2n$ by Clifford's theorem, and Lemma 1.3, (1) shows

$$L_m^2 \geq \frac{2(g_m - 1)}{2g_m - 1} (2g_m n + (g_m - 1)(q_m - 1)) > 4n,$$

which is impossible by (2.3). Hence $\tilde{\mathscr{E}}$ is non-special and $\deg(\tilde{\mathscr{E}}) = h^0(L_m) + q_m - 1$. We get the disired inequality from Lemma 1.3. Since $f_m$ must be a non-hyperelliptic fibration, we have $g_m \geq 3$.

Assume that the semi-canonical map of $S_m$ is degree 3 onto the image $V_m$. If $m \geq 7$, then $\deg V_m < (4/3)(n - 2)$. From [9. Lemma 1] it follows that $V_m$ is ruled by straight lines. This means that $\Phi_m$ maps $D_m$ onto a straight line, and the restriction map $H^0(L_m) \to H^0(L_m|_{D_m})$ is of rank 2. It follows that $\tilde{\mathscr{E}}$ is of rank 2 and, putting $d = 3$, Lemma 1.3 gives us

$$L_m^2 \geq \frac{g_m - 1}{2(g_m - 2)} (g_m \deg(\tilde{\mathscr{E}}) + 2(g_m - 4)(q_m - 1)). \tag{3.1}$$

We remark that $g_m - 1 \geq d = 3$.

Assume that $g_m = 4$. Since $D_m$ is non-hyperelliptic and $L_m D_m = 3$, Clifford's theorem shows $h^0(L_m|_{D_m}) \leq 2$. Therefore, the restriction map $H^0(L_m) \to H^0(L_m|_{D_m})$ is surjective. It follows that $\tilde{\mathscr{E}} \simeq (f_m)_* L_m$. Hence $\deg(\tilde{\mathscr{E}}) = \deg((f_m)_* L_m) \geq \chi(L_m)/2 + 2(q_m - 1)$ by Lemma 1.1. The desired inequality follows from Lemma 1.3.

Assume that $g_m > 4$. Let $\mathscr{F}$ be the quotient bundle of $\tilde{\mathscr{E}}$ defined as in Section 1. If $\mathrm{rk}(\mathscr{F}) = 2$, then $\deg(\tilde{\mathscr{E}}) \geq 2(h^0(L_m) - 2)$ by Lemma 1.4. It follows from (3.1) that $L_m^2 \geq 6(n-1)$ which is impossible by (2.3). If $\mathrm{rk}(\mathscr{F}) = 1$, then it follows from Lamma 1.4 that $\deg(\tilde{\mathscr{E}}) \geq h^0(L_m) - 2 + q_m + \deg(\mathscr{F})/2$. Let $\mathscr{G}$ be the kernel of $\tilde{\mathscr{E}} \to \mathscr{F}$. Since $\mathscr{F}$ is special, we have $\deg(\mathscr{F}) \leq 2q_m - 2$. Hence

$$\mu_1(\tilde{\mathscr{E}}) \geq \deg(\mathscr{G}) \geq h^0(L_m) - 2 + q_m - \deg(\mathscr{F})/2 \geq n.$$

Using (1.1) for $i = 1$, we get

$$L_m^2 \geq \frac{2(g_m - 1)}{2g_m - 1}(g_m n + (g_m - 1)(q_m - 1))$$

which is impossible for $g_m > 4$. If $\mathscr{F} = 0$, then $\deg(\tilde{\mathscr{E}}) = h^0(L_m) + 2(q_m - 1)$ by the Riemann-Roch theorem. Hence we get the desired inequality from Lemma 1.3. Then, by (2.3), we get $g_m \leq 6$. Q.E.D.

Let $f : S \to B$ be the fibration induced by the Albanese map. Since we can assume that $f_m$ is induced by the Albanese map, we get a commutative diagram

$$\begin{array}{ccc} S_m & \to & B_m \\ \pi_m \downarrow & & \downarrow \\ S & \to & B. \end{array}$$

The Galois group $\mathrm{Gal}(S_m/S)$ of $\pi_m : S_m \to S$ acts naturally on $B_m$, and we can identify $B$ with the quotient $B_m/\mathrm{Gal}(S_m/S)$. We assume that $m$ is a prime number with $m \geq 7$. Then $\varpi : B_m \to B$ is an $m$-sheeted cyclic covering.

LEMMA 3.3. *The covering $\varpi$ is unramified.*

*Proof.* Assume that $\mathrm{Gal}(S_m/S)$ has a fixed point $t \in B_m$. Then the induced map $f_m^{-1}(t) \to f^{-1}(\varpi(t))$ is $m$-sheeted. Since $g_m \leq 6$, we have

$$10 \geq K_m f_m^{-1}(t) = mKf^{-1}(\varpi(t)) \geq 4m,$$

which is impossible. Q.E.D.

Since $S_m$ is obtained by the base change $\varpi : B_m \to B$, we have $g_m = g$, where $g$ denotes the genus of a general fibre $D$ of $f : S \to B$. We remark that $D$ is trigonal, since so is $D_m$. Therefore, $f$ is a trigonal fibration of genus $g \leq 6$.

We finish the proof of Theorem 2 with the following:

LEMMA 3.4. *Let $S$ be an irregular even canonical surface with $K^2 < 4\chi$, and let $g$ denote the genus of the Albanese pencil. Then $3 \leq g \leq 6$ and the following inequalities hold.*

$$K^2 \geq \begin{cases} 3\chi + 10(q-1) & \text{if } g = 3, \\ \dfrac{8g(g-1)}{g^2+7g-16}\left(\chi + 8\left(1-\dfrac{2}{g}\right)(q-1)\right) & \text{if } 4 \leq g \leq 6. \end{cases}$$

*In other words, the slope of the Albanese pencil is not less than* $3$, $24/7$, $40/11$ *or* $120/31$ *according to* $g = 3, 4, 5$ *or* $6$, *respectively.*

*Proof.* Since $\varpi : B_m \to B$ is an $m$-sheeted unramified covering, we have $q_m - 1 = m(q-1)$ by Riemann-Hurwitz's formula.

Assume that $\Phi_m$ is composed of a pencil. It follows from (2.1) that $2h^0(L_m) \geq -L_m^2/2 + \chi_m$. From this and Lemma 3.2, (1), we get

$$4L_m^2 \geq \frac{2g(g-1)}{2g-1}(-L_m^2 + 2\chi_m) + 8(g-1)(q_m - 1).$$

Since $L_m^2 = mL^2$, $X_m = m\chi$ and $q_m - 1 = m(q-1)$, we can drop the subscript $m$ from the above inequality. Then, since $K^2 = 4L^2$, we get

$$K^2 \geq \frac{8g(g-1)}{g^2+3g-2}\left(\chi + 2\left(2-\frac{1}{g}\right)(q-1)\right). \tag{3.2}$$

Assume that $\deg \Phi_m = 3$. Then, similarly as above, we get

$$K^2 \geq \frac{8g(g-1)}{g^2+7g-16}\left(\chi + 8\left(1-\frac{2}{g}\right)(q-1)\right) \tag{3.3}$$

by using (2.1) and Lemma 3.2. Note that (3.2) is stronger than (3.3) when $g = 4$.

Q.E.D.

*Remark* 3.5. When $3 \leq g \leq 5$, the assertion on the slope in Lemma 3.4 holds for any (relatively minimal, non-isotrivial) non-hyperelliptic fibrations, as we have shown in [6].

The lower bound on $K^2$ in Lemma 3.4 is sharp for $g = 3, 4$. Indeed, for

$g \leq 4$, even canonical surfaces which attain the lower bound can be constructed as in [5, I, §8]. The following example shows that the bound is also sharp for $g = 6$.

EXAMPLE 1. For simplicity, we denote the pull-back of objects by the same symbol. Let $B$ be a non-singular projective curve of genus $q$ and let $L_0$ be a line bundle of degree $e > 0$ on $B$. We consider a ruled surface $R = \mathbf{P}(\mathcal{O}_B(6L_0) \oplus \mathcal{O}_B)$ and a $\mathbf{P}^1$-bundle $P = \mathbf{P}(\mathcal{O}_R(2\Delta_0 + 14L_0) \oplus \mathcal{O}_R)$ over $R$, where $\Delta_0$ is the minimal section of $R$. Let $H$ be a tautological divisor on $P$, and take a general member $S \in |3H + \Delta_0|$. We choose sections $X_0$ and $X_1$ of $[H]$ and $[H - 2\Delta_0 - 14L_0]$ respectively so that $(X_0, X_1)$ forms a system of homogeneous coordinates on fibres of $P \to R$. Then $S$ is defined in $P$ by

$$\zeta(X_0^3 + \phi_1 X_0^2 X_1 + \phi_2 X_0 X_1^2) = \phi_3 X_1^3,$$

where $\zeta$, $\phi_1$, $\phi_2$ and $\phi_3$ are respectively general sections of $[\Delta_0]$, $[2\Delta_0 + 14L_0]$, $[4\Delta_0 + 28L_0]$ and $7[\Delta_0 + 6L_0]$ on $R$. Hence we can assume that $S$ is irreducible and non-singular. Furthermore, it has a natural trigonal fibration $f : S \to B$ of genus 6. We remark that $\phi_3$ is a non-zero constant on $\Delta_0$. From the above equation, we know that $f$ has a section $B_0$ defined in $P$ by $\zeta = X_1 = 0$, and $\Delta_0$ is linearly equivalent to $3(X_1)$ on $S$. By the adjunction formula, the canonical bundle of $S$ is induced by $H + \Delta_0 + 8L_0 + K_B$ which is equivalent to $10(X_1) + 22L_0 + K_B$ on $S$. If $B$ is a hyperelliptic curve and if we choose $q - 1$ distinct Weierstrass points $p_1, \ldots, p_{q-1}$, then $K_B = 2\sum p_i$ and $S$ is an even surface with a semi-canonical bundle $[5(X_1) + 11L_0 + \sum p_i]$. By a standard calculation, we have $p_g(S) = 62e + 6(q - 1)$, $q(S) = q$ and $K_S^2 = 240e + 40(q - 1)$. Therefore, when $e$ is sufficiently large, $S$ is an even canonical surface with a trigonal fibration $f : S \to B$ whose slope is $120/31$.

The next example shows that we cannot weaken the asumption $K^2 < 4\chi$ in Theorems 1 and 2.

EXAMPLE 2. Let $A$ be a principally polarized abelian surface, and let $\Theta$ be a theta divisor on $A$. For $k \geq 2$, choose a general member $C \in |4k\Theta|$, and let $S$ be the double covering of $A$ with branch locus $C$. Then $S$ is an even surface, since the canonical bundle is induced by $2k\Theta$. Furthermore, the numerical characters satisfy $K_S^2 = 16k^2$, $p_g = 4k^2 + 1$ and $q = 2$ (hence $K_S^2 = 4\chi(\mathcal{O}_S)$). The Albanese map of $S$ is clearly surjective. It is easy to see that the canonical image of $S$ is cut out by hyperquadrics.

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