K. Shiohama and H. Xu
Nagoya Math. J.
Vol. 150 (1998), 105–134

A GENERAL RIGIDITY THEOREM FOR COMPLETE SUBMANIFOLDS

KATSUHIRO SHIOHAMA AND HONGWEI XU

Abstract. Making use of 1-forms and geometric inequalities we prove the rigidity property of complete submanifolds M^n with parallel mean curvature normal in a complete and simply connected Riemannian $(n+p)$-manifold N^{n+p} with positive sectional curvature. For given integers n, p and for a nonnegative constant H we find a positive number $\tau(n,p) \in (0,1)$ with the property that if the sectional curvature of N is pinched in $[\tau(n,p), 1]$, and if the squared norm of the second fundamental form is in a certain interval, then N^{n+p} is isometric to the standard unit $(n+p)$-sphere. As a consequence, such an M is congruent to one of the five models as seen in our Main Theorem.

§0. Introduction

An important problem in differential geometry is the study of relations between the geometric structure and the geometric invariants of Riemannian submanifolds. After the pioneering work of Simons [S] the following result, known as the rigidity theorem for submanifolds containing minimal cases, was proved first by Lawson [L1], Chern-do Carmo-Kobayashi [CDK] and later Li and Li [LL] and finally by the second author [X1], as stated

THEOREM 0.1. *For given constant $H \geq 0$ and positive integers n (≥ 2), p there exists a positive number $C(n,p,H)$ with the following property: If M^n is a closed submanifold in the standard unit $(n+p)$-sphere $\mathbf{S}^{n+p}(1)$ with parallel mean curvature normal field having norm H, and if S is the squared norm of the second fundamental form satisfying*

$$S \leq C(n,p,H),$$

then M is congruent to one of the following:

(1) $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$

Received August 22, 1996.
Research supported in part by Grant-in-Aid for Co-operative Research No. 06302007 from the Ministry of Education, Science and Culture.
The second author was supported by the NNSFC. Grant No. 1957107 (China).

(2) *the isoparametric hypersurface* $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ *in* $\mathbf{S}^{n+1}(1)$,

(3) *one of the Clifford minimal hypersurfaces* $\mathbf{S}^k(\sqrt{\frac{k}{n}}) \times \mathbf{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ *in* $\mathbf{S}^{n+1}(1)$, *for* $k = 1, \cdots, n-1$,

(4) *the Clifford torus* $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2)$ *in* $\mathbf{S}^3(r)$ *with constant mean curvature* H_0, *where* r_1, $r_2 = [2(1+H^2) \pm 2H_0(1+H^2)^{1/2}]^{-1/2}$, $r = (1+H^2-H_0^2)^{-1/2}$, *and* $0 \leq H_0 \leq H$,

(5) *the Veronese surface in* $\mathbf{S}^4(\frac{1}{\sqrt{1+H^2}})$.

Here λ *and* $C(n,p,H)$ *are given by*

$$\lambda = \frac{1}{2(n-1)}[nH + \sqrt{n^2H^2 + 4(n-1)}],$$

and

$$C(n,p,H) = \begin{cases} \alpha(n,H), & \textit{for } p=1, \textit{ or } p=2 \textit{ and } H \neq 0 \\ \min\{\alpha(n,H), \frac{1}{3}(2n+5nH^2)\}, & \\ & \textit{for } p \geq 3, \textit{ or } p=2 \textit{ and } H=0, \end{cases}$$

where

$$\alpha(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Note that the special case where $p=1$ and $H \neq 0$ was proved independently by Alencar and do Carmo in [AdC]. Also note that Theorem 0.1 was obtained under the assumption that the ambient space is the *round sphere*. The existence of parallel mean curvature normal field imposes very nice properties to submanifolds, whatever the ambient spaces are. In fact the second author proved in [X2] the rigidity for compact minimal submanifolds in pinched Riemannian manifolds, as stated

THEOREM 0.2. ([X2]) *For given positive integers* $n \geq 2$, p *there exists a number* $\delta(n,p)$ *with* $0 < \delta(n,p) < 1$ *with the following properties: If* M^n *is an oriented closed minimal submanifold in a complete simply connected manifold* N^{n+p} *whose sectional curvature* K_N *satisfies* $\delta(n,p) \leq K_N \leq 1$ *and if*

$$\beta(n,p)(1-c) \leq S \leq n - \frac{n}{3}\operatorname{sgn}(p-1) - \gamma(n,p)(1-c),$$

where $c := \inf K_N$, *then* N *is isometric to* $\mathbf{S}^{n+p}(1)$. *Moreover* M *is congruent to one of the following*:

(1) $\mathbf{S}^n(1)$,

(2) *the Clifford minimal hypersurfaces* $\mathbf{S}^k(\sqrt{\frac{k}{n}}) \times \mathbf{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ *in* $\mathbf{S}^{n+1}(1)$, *for* $k = 1, \cdots, n-1$,

(3) *the Veronese surface in* $\mathbf{S}^4(1)$.

Here $\beta(n,p)$, $\gamma(n,p)$ *and* $\delta(n,p)$ *are given as*

$$\begin{aligned}
\beta(n,p) &:= \frac{1}{12}[p\,n(n-1)(52n-50)]^{1/2},\\
\gamma(n,p) &:= n + \frac{2}{3}(p-1)(n-1)^{1/2} + \frac{1}{12}[p\,n(n-1)(52n-50)]^{1/2},\\
\delta(n,p) &:= 1 - n(3 - \operatorname{sgn}(p-1))\{3n + 2(p-1)(n-1)^{1/2}\\
&\qquad + \frac{\sqrt{2}}{2}[p\,n(n-1)(26n-25)]^{1/2}\}^{-1}.
\end{aligned}$$

The purpose of this paper is to relax the special closed submanifolds in the above results to complete submanifolds with parallel mean curvature normal fields, and the ambient space to general Riemannian manifold N^{n+p}. Thus we obtain the generalization of Theorems 0.1 and 0.2.

MAIN THEOREM. *For given positive integers* n (≥ 2), p *and a non-negative constant* H *there exists a number* $\tau(n,p)$ *such that* $0 < \tau(n,p) < 1$ *with the following properties: If* M^n *is an oriented complete submanifold with parallel mean curvature normal field with its norm* H *in a complete and simply connected* $(n+p)$*-dimensional Riemannian manifold with* $\tau(n,p) \leq K_N \leq 1$, *and if*

$$\begin{aligned}
&nH^2 + A_1(n,p)(1-c) + A_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}\\
&\quad \leq S \leq C(n,p,H) - B_1(n,p)(1-c) - B_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4},
\end{aligned}$$

where $c := \inf K_N$, *then* N *is isometric to* $\mathbf{S}^{n+p}(1)$. *Moreover,*

1. *If* $\sup_M S < \alpha(n,H)$, *then* M *is congruent to either* $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$ *or the Veronese surface in* $\mathbf{S}^4(\frac{1}{\sqrt{1+H^2}})$.

2. *If* M *is compact, then* M *is congruent to one of the following*:

(1) $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$,

(2) *the isoparametric hypersurface* $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ *in* $\mathbf{S}^{n+1}(1)$,

(3) *one of the Clifford minimal hypersurfaces* $\mathbf{S}^k(\sqrt{\frac{k}{n}}) \times \mathbf{S}^{n-k}(\sqrt{\frac{n-k}{n}})$ *in* $\mathbf{S}^{n+1}(1)$, *for* $k = 1, \cdots, n-1$,

(4) *the Clifford torus* $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2)$ *in* $\mathbf{S}^3(r)$ *with constant mean curvature* H_0, *where* $r_1,\ r_2 = [2(1+H^2) \pm 2H_0(1+H^2)^{1/2}]^{-1/2}$, $r = (1+H^2-H_0^2)^{-1/2}$, *and* $0 \le H_0 \le H$,

(5) *the Veronese surface in* $\mathbf{S}^4(\frac{1}{\sqrt{1+H^2}})$.

Here $C(n,p,H)$, $\alpha(n,H)$ *and* λ *are defined as in Theorem* 0.1.

The constants $\tau(n,p)$, $A_1(n,p)$, $A_2(n,p)$, $B_1(n,p)$ and $B_2(n,p)$ are precisely given in the proof (see §3). In the case where M is not oriented, we can obtain similar results by using the Riemannian double cover. Our proof method is quite different from the previous ones developed in Theorems 0.1 and 0.2. In contrast to the proofs of Theorems 0.1 and 0.2, our proof does not require the generalized maximum principle and the generalized Simons inequality, but the use of two distinct differential 1-forms and integral inequalities for the semi-norms of the second fundamental form of M. The crucial point of our proof is to verify that $c = 1$. The closed minimal case in our Main Theorem has already been established in Theorem 0.2, and hence $H \neq 0$ *is assumed throughout* §*2 and* §*3*. In due course of the proof of our Main Theorem we obtain geometric inequalities (see Theorems 2.4, 2.8, etc.) by which the rigidity results for compact cases are obtained (see Theorems 3.1, 3.3 and 3.4). In complete case we show by computations that the Ricci curvature is bounded below by a positive constant, and hence it reduces to compact case.

The paper is organized as follows. Local formulas and propositions are prepared in §1. In §2 we present two geometric inequalities for the semi-norms of the second fundamental form. In §3 we investigate the rigidity of closed submanifolds with parallel mean curvature normal field in a pinched manifold. In §4 we discuss complete submanifold with parallel mean curvature normal field and complete the proof of our Main Theorem.

The second author would like to express his thanks to Professor P. Li for his valuable suggestions.

§1. Preliminaries

Throughout this paper let M^n be an n-dimensional connected and complete Riemannian manifold isometrically immersed into an $(n+p)$-dimensional Riemannian manifold N^{n+p}. The following convention of indices are used throughout.

$$1 \leq A, B, C, \cdots \leq n+p, \quad 1 \leq i, j, k, \cdots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

Choose an orthonormal frame field $\{e_A\}$ in a neighborhood of a point $p \in M$ such that the e_i's span the tangent space T_pM to M at p. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of N. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.$$

Let $a(x)$, $b(x)$ for $x \in N$ be the minimum and maximum of K_N at that point and c, d the infimum and supremum of $a, b\colon N \to \mathbf{R}$ over N. The curvature tensors of N, M are denoted by K_{ABCD}, R_{ijkl}, and the normal curvature tensor of M by $R_{\alpha\beta kl}$ respectively. Let R be the scalar curvature of M. The second fundamental form of M is denoted by h and the mean curvature normal field by ξ. Set $H := \|\xi\|$, the mean curvature of M and S the squared norm of h. We then have

$$h = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \tag{1.1}$$

$$\xi = \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha, \tag{1.2}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{1.3}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta). \tag{1.4}$$

The scalar curvature R of M is given by

$$R = \sum_{i,j} K_{ijij} + n^2 H^2 - S. \tag{1.5}$$

DEFINITION 1.1. We say that M admits parallel mean curvature normal field iff ξ is parallel in the normal bundle over M.

We assume that M admits a parallel mean curvature normal field and that $H \neq 0$. Choose e_{n+1} such that it is parallel to ξ and $\operatorname{tr} H_{n+1} := \operatorname{tr}(h_{ij}^{n+1}) = nH$. Then, setting H_α the $(n \times n)$-matrix with (i,j) component h_{ij}^α, we observe that $\operatorname{tr} H_\beta = 0$ for all $\beta \geq n+2$. The squared norm S of the second fundamental form is divided into two parts as follows.

$$S_H := \operatorname{tr} H_{n+1}^2, \qquad S_I := \sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2.$$

The following proposition is immediate from the definition, and the proof is omitted here.

PROPOSITION 1.2. *If M admits parallel mean curvature normal field ξ, then either $H \equiv 0$ or H is non-zero constant and $H_\alpha H_{n+1} = H_{n+1} H_\alpha + (K_{n+1\alpha ij})_{n \times n}$.*

Denoting the covariant derivatives of h_{ij}^α by h_{ijk}^α and h_{ijkl}^α respectively, we have

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_s h_{sj}^\alpha \omega_{is} + \sum_s h_{is}^\alpha \omega_{js} + \sum_\beta h_{ij}^\beta \omega_{\alpha\beta},$$

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_s h_{sjk}^\alpha \omega_{is} + \sum_s h_{isk}^\alpha \omega_{js} + \sum_s h_{ijs}^\alpha \omega_{ks} + \sum_\beta h_{ijk}^\beta \omega_{\alpha\beta}.$$

We then have

$$h_{ijk}^\alpha - h_{ikj}^\alpha = -K_{\alpha ijk}, \tag{1.6}$$

and the Ricci formula

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_s h_{sj}^\alpha R_{sikl} + \sum_s h_{is}^\alpha R_{sjkl} + \sum_\beta h_{ij}^\beta R_{\beta\alpha kl}. \tag{1.7}$$

Let $K_{\alpha ijkl}$ be the covariant derivative of $K_{\alpha ijk}$ as the section of $T^\perp(M) \otimes T^*(M) \otimes T^*(M) \otimes T^*(M)$ and $K_{ABCD;E}$ the covariant derivative of K_{ABCD} as the curvature tensor of N. Restricted to M we have

$$\begin{aligned} \sum_l K_{\alpha ijkl}\omega_l &= dK_{\alpha ijk} + \sum_s K_{\alpha sjk}\omega_{is} \\ &\quad + \sum_s K_{\alpha isk}\omega_{js} + \sum_s K_{\alpha ijs}\omega_{ks} + \sum_\beta K_{\beta ijk}\omega_{\alpha\beta}, \end{aligned} \tag{1.8}$$

and

$$
(1.9) \qquad K_{\alpha ijk;l} = K_{\alpha ijkl} - \sum_{\beta} K_{\alpha\beta jk} h_{il}^{\beta} - \sum_{\beta} K_{\alpha i\beta k} h_{jl}^{\beta} - \sum_{\beta} K_{\alpha ij\beta} h_{kl}^{\beta} + \sum_{m} K_{mijk} h_{ml}^{\alpha}.
$$

DEFINITION 1.3. *M is called a submanifold with parallel second fundamental form iff $h_{ijk}^{\alpha} \equiv 0$ for all i, j, k and α. N is by definition a locally symmetric space iff $K_{ABCD;E} \equiv 0$ for all A, B, C, D, E.*

The Laplacian Δh_{ij}^{α} of the second fundamental form h is defined by $\Delta h_{ij}^{\alpha} := \sum_k h_{ijkk}^{\alpha}$. We set $\nabla_k h_{ij}^{\alpha} := h_{ijk}^{\alpha}$. For an $(n \times n)$-matrix $A = (a_{ij})$ we denote by $N(A)$ the squared norm of A, i.e., $N(A) := \operatorname{tr}(A^t A) = \sum_{i,j} a_{ij}^2$. Then $N(A) = N(TA^tT)$ holds for every orthogonal $(n \times n)$-matrix T.

PROPOSITION 1.4. (see [CDK], [LL]) *For symmetric matrices A_{n+1}, $\cdots, A_{n+p}$ let $S_{\alpha\beta} := \operatorname{tr}(A_\alpha A_\beta)$, $S_\alpha := S_{\alpha\alpha} = N(A_\alpha)$ and $S := \sum_\alpha S_\alpha$. Then*

$$
\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \le (1 + \frac{1}{2}\operatorname{sgn}(p-1))S^2,
$$

where $\operatorname{sgn}(\cdot)$ *is the standard sign function. Moreover, equality holds if and only if at most two matrices A_α and A_β are non-zero and they can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}_\alpha$ and $\tilde{A}_\beta$ respectively where*

$$
\tilde{A}_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \tilde{A}_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.
$$

The following Proposition is seen in [G]. The proof is omitted here.

PROPOSITION 1.5. *Let N be an $(n+p)$-dimensional Riemannian manifold. If $a \le K_N \le b$ is satisfied at a point, then the following estimates hold at that point.*

(1) $|K_{ACBC}| \le \frac{1}{2}(b-a)$, *for* $A \ne B$,

(2) $|K_{ABCD}| \leq \frac{2}{3}(b-a)$, *for all distinct indices* A, B, C, D.

PROPOSITION 1.6. *Let* $a_1, \cdots, a_n$ *and* $b_1, \cdots, b_n$ *be real numbers satisfying* $\sum_i a_i = \sum_i b_i = 0$, $\sum_i a_i^2 = a$ *and* $\sum_i b_i^2 = b$. *Then*

$$|\sum_i a_i b_i^2| \leq (n-2)[n(n-1)]^{-1/2} a^{1/2} b,$$

where equality holds if and only if either $ab = 0$, *or at least* $n-1$ *pairs of numbers of* (a_i, b_i)*'s are the same.*

Proof. We only need to check the case where $n \geq 3$ and $a = b = 1$. Consider the function

$$f(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_i x_i y_i^2 \tag{1.10}$$

subject to the constraint conditions

$$\sum_i x_i = \sum_i y_i = 0, \qquad \sum_i x_i^2 = \sum_i y_i^2 = 1. \tag{1.11}$$

The Lagrange multiplier is employed for the proof by setting

$$\begin{aligned} &F(x_1, \cdots, x_n, y_1, \cdots, y_n, \lambda, \mu, \nu, \sigma) \\ &\quad = \sum_i x_i y_i^2 + \lambda\Big(\sum_i x_i^2 - 1\Big) + \mu\Big(\sum_i y_i^2 - 1\Big) + \nu \sum_i x_i + \sigma \sum_i y_i. \end{aligned}$$

If $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ is a critical point of f with the critical value T_0 under the constraint conditions (1.11), we then have

$$y_i^2 + 2\lambda x_i + \nu = 0, \tag{1.12}$$

$$2x_i y_i + 2\mu y_i + \sigma = 0. \tag{1.13}$$

From (1.11) and the above relations,

$$\nu = -\frac{1}{n}, \qquad \lambda = \frac{1}{2}\mu = -\frac{1}{2}\sum_i x_i y_i^2 = -\frac{1}{2}T_0. \tag{1.14}$$

Combining (1.12) and (1.14),

$$T_0^2 = -2\lambda \sum_i x_i y_i^2 = \sum_i y_i^4 - \frac{1}{n}. \tag{1.15}$$

Now we compute the maximum of the function

$$g(z_1, \cdots, z_n) := \sum_i z_i^4 - \frac{1}{n}$$

under the constraint conditions

$$\sum_i z_i = 0, \qquad \sum_i z_i^2 = 1. \tag{1.16}$$

From (1.16) we get

$$\sum_{1 \le i < j \le n-1} z_i z_j = z_n^2 - \frac{1}{2}. \tag{1.17}$$

Making use of (1.16), (1.17) and the Schwarz inequality,

$$\begin{aligned} g(z_1, \cdots, z_n) & \\ &= \left(\sum_{i=1}^{n-1} z_i^2\right)^2 - 2 \sum_{1 \le i < j \le n-1} z_i^2 z_j^2 + z_n^4 - \frac{1}{n} \\ &\le (1 - z_n^2)^2 - \frac{4}{(n-1)(n-2)} \left(\sum_{1 \le i < j \le n-1} z_i z_j\right)^2 + z_n^4 - \frac{1}{n} \\ &= \frac{2n(n-3)}{(n-1)(n-2)} z_n^4 - \frac{2n(n-3)}{(n-1)(n-2)} z_n^2 + \frac{n-1}{n} - \frac{1}{(n-1)(n-2)}. \end{aligned}$$

Note that

$$z_n^2 \le \frac{n-1}{n},$$

where equality holds if and only if

$$z_1, \cdots, z_{n-1} = \pm \frac{1}{\sqrt{n(n-1)}}.$$

Therefore we have

$$g(z_1, \cdots, z_n) \le (n-2)^2 [n(n-1)]^{-1}, \tag{1.18}$$

where equality holds if and only if at least $n-1$ numbers of z_i's are equal. From (1.15) and (1.18)

$$\left|\sum_i x_i y_i^2\right| \le \max\{|M_0|, |m_0|\} \le (n-2)[n(n-1)]^{-1/2}, \tag{1.19}$$

where M_0 and m_0 are the maximum and minimum of f under the constraint condition (1.11) respectively. The above computation implies that the equalities in (1.19) hold if and only if at least $n-1$ pairs of the (x_i, y_i)'s are equal. □

COROLLARY 1.7. *If $a_1, \cdots, a_n$ are n real numbers with $\sum_i a_i = 0$, then*

$$\Big|\sum_i a_i^3\Big| \le (n-2)[n(n-1)]^{-1/2}\Big(\sum_i a_i^2\Big)^{3/2}, \tag{1.20}$$

$$\sum_i a_i^4 \le \frac{n^2-3n+3}{n(n-1)}\Big(\sum_i a_i^2\Big)^2. \tag{1.21}$$

Moreover the following (1), (2) *and* (3) *are equivalent.*

(1) *The equality in* (1.20) *holds.*

(2) *The equality in* (1.21) *holds.*

(3) *At least $n-1$ numbers of the a_i's are equal.*

§2. Geometric inequalities for the second fundamental form

Throughout this section *let M^n be an oriented closed submanifold of dimension n in an $(n+p)$-dimensional manifold N^{n+p} with parallel mean curvature normal field.* The squared norm of the second fundamental form is divided into S_H and S_I, which we shall call the semi-norms of the second fundamental form. The geometric inequalities for the semi-norms are provided in this section. In Theorems 2.4 and 2.8 we need not assume the completeness of N.

From (1.6), (1.7) and Proposition 1.2,

$$\Delta h_{ij}^{n+1} = -\sum_k (K_{n+1kikj} + K_{n+1ijkk}) + \sum_{k,m}(h_{mk}^{n+1}R_{mijk} + h_{im}^{n+1}R_{mkjk}).$$

Substituting (1.3) into the above, we get

$$\begin{aligned}\Delta h_{ij}^{n+1} = &-\sum_k (K_{n+1kikj} + K_{n+1ijkk}) + \sum_{m,k}(h_{mi}^{n+1}K_{mkjk} + h_{mk}^{n+1}K_{mijk})\\ &+ \sum_{m,k,\alpha}(h_{mi}^{n+1}h_{mj}^{\alpha}h_{kk}^{\alpha} + h_{km}^{n+1}h_{ki}^{\alpha}h_{mj}^{\alpha} - h_{km}^{n+1}h_{km}^{\alpha}h_{ij}^{\alpha} - h_{mi}^{n+1}h_{mk}^{\alpha}h_{kj}^{\alpha}).\end{aligned}$$

Therefore

$$\frac{1}{2}\Delta S_H = \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j,k} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = A + B + C. \tag{2.1}$$

Here we set

$$\begin{aligned}
A &:= nH \operatorname{tr} H_{n+1}^3 - (\operatorname{tr} H_{n+1}^2)^2 - \sum_{\beta\neq n+1}[\operatorname{tr}(H_{n+1}H_\beta)]^2,\\
B &:= \sum_{i,j,k,m}(h_{ij}^{n+1}h_{mj}^{n+1}K_{mkik} + h_{ij}^{n+1}h_{mk}^{n+1}K_{mijk}),\\
C &:= \sum_{i,j,k}(h_{ijk}^{n+1})^2 - \sum_{i,j,k}(h_{ij}^{n+1}K_{n+1kikj} + h_{ij}^{n+1}K_{n+1ijkk})\\
&\quad + \sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}H_\beta)^2 - \sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}^2H_\beta^2).
\end{aligned}$$

Several lemmas will be needed for the proof of our geometric inequalities. We first find the lower bound for A in terms of H, S_H and S.

LEMMA 2.1.

$$A \geq (S_H - nH^2)\Big[2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_H - nH^2)^{1/2}\Big].$$

Proof. Let $\{e_i\}$ be an orthonormal frame at a point on M such that the matrix H_{n+1} takes the diagonal form and such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$ for all i, j. Set

$$f_k := \sum_i(\lambda_i^{n+1})^k, \qquad B_k := \sum_i(\mu_i^{n+1})^k, \qquad \mu_i^{n+1} := H - \lambda_i^{n+1},$$

for $i = 1, \cdots, n$.

Then

$$B_1 = 0, \qquad B_2 = S_H - nH^2, \tag{2.2}$$

and

$$B_3 = 3HS_H - 2nH^3 - f_3. \tag{2.3}$$

From (2.2), (2.3) and Corollary 1.7, we get

$$A = nHf_3 - S_H^2 - \sum_{\beta\neq n+1}\Big[\sum_{i,j}(h_{ij}^{n+1} - H\delta_{ij})\,h_{ij}^\beta\Big]^2$$

$$
\begin{aligned}
&\geq nH\Big[3HS_H - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}B_2^{3/2}\Big] - S_H^2 - \sum_{\beta\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\beta}\Big)^2 \\
&\geq (S_H - nH^2)\Big[2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}HB_2^{1/2}\Big] + nH^2S_H - S_H^2 - B_2S_I \\
&= (S_H - nH^2)\Big[2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_H - nH^2)^{1/2}\Big].
\end{aligned}
$$

This proves Lemma 2.1. □

The lower bound of B in terms of a, H and S_H is obtained as follows.

LEMMA 2.2.

$$B \geq na(S_H - nH^2).$$

Proof. It follows that

$$
\begin{aligned}
B &= \sum_{i,k}(\lambda_i^{n+1})^2K_{ikik} + \sum_{i,k}\lambda_i^{n+1}\lambda_k^{n+1}K_{kiik} \\
&= \frac{1}{2}\sum_{i,k}(\lambda_i^{n+1} - \lambda_k^{n+1})^2K_{ikik} \geq \frac{1}{2}a\sum_{i,k}(\lambda_i^{n+1} - \lambda_k^{n+1})^2 \\
&= na(S_H - nH^2).
\end{aligned}
$$

This proves Lemma 2.2. □

The integral of C is estimated as

LEMMA 2.3.

$$\int_M C\,dM \geq -\frac{1}{72}n(n-1)(26n+16p-41)\int_M (b-a)^2\,dM.$$

Proof. First of all we note that

$$
\begin{aligned}
-\sum_{i,j,k}&(h_{ik}^{n+1}K_{n+1jijk} + h_{ij}^{n+1}K_{n+1ijkk}) \\
&= -\sum_{i,j,k}\nabla_k(h_{ik}^{n+1}K_{n+1jij} + h_{ij}^{n+1}K_{n+1ijk}) \\
&\qquad + \sum_{i,j,k}(h_{ikk}^{n+1}K_{n+1jij} + h_{ijk}^{n+1}K_{n+1ijk}).
\end{aligned}
$$

We define a differentiable 1-form ω as follows.

$$\omega = \sum_{i,j,k}(h_{ik}^{n+1}K_{n+1jij} + h_{ij}^{n+1}K_{n+1ijk})\omega_k.$$

We then get

$$\operatorname{div}\omega = \sum_{i,j,k}\nabla_k(h_{ik}^{n+1}K_{n+1jij} + h_{ij}^{n+1}K_{n+1ijk}).$$

Thus,

$$\begin{aligned}C = &\sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j,k}(h_{ikk}^{n+1}K_{n+1jij} + h_{ijk}^{n+1}K_{n+1ijk})\\ &- \operatorname{div}\omega + \sum_{i,j,m,\beta\neq n+1} h_{ij}^{n+1}h_{mj}^{\beta}K_{\beta n+1mi}.\end{aligned}$$

Since the mean curvature normal field of M is parallel,

$$\sum_i h_{iij}^{n+1} = 0, \qquad \text{for all } j. \tag{2.4}$$

From (1.6), (2.4) and Proposition 1.5,

$$\begin{aligned}\sum_{i,j,k} h_{ikk}^{n+1}K_{n+1jij} &= \sum_{i,j,k}(h_{kki}^{n+1} - K_{n+1kik})K_{n+1jij}\\ &\geq -\sum_i\Big(\sum_j K_{n+1jij}\Big)^2\\ &\geq -\frac{1}{4}n(n-1)^2(b-a)^2.\end{aligned} \tag{2.5}$$

On the other hand, $\sum_{i,j,k}(h_{ijk}^{n+1} + \frac{1}{2}K_{n+1ijk})^2 \geq 0$ implies that

$$\begin{aligned}&\sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j,k} h_{ijk}^{n+1}K_{n+1ijk}\\ &\qquad\geq -\frac{1}{4}\sum_{i,j,k}K_{n+1ijk}^2\\ &\qquad\geq -\frac{1}{4}\sum_{i\neq j\neq k\neq i}K_{n+1ijk}^2 - \frac{1}{2}\sum_{i\neq j}K_{n+1iji}^2\\ &\qquad\geq -\frac{1}{72}n(n-1)(8n-7)(b-a)^2.\end{aligned} \tag{2.6}$$

Also from (1.4),

$$\begin{aligned}\sum_{i,j,m,\beta\neq n+1} & h_{ij}^{n+1}h_{mj}^{\beta}K_{\beta n+1mi}\\ &= \frac{1}{2}\sum_{i,j,m,\beta\neq n+1}(h_{ij}^{n+1}h_{mj}^{\beta}+h_{ij}^{\beta}h_{mj}^{n+1})K_{\beta n+1mi}-\frac{1}{2}\sum_{i,m,\beta\neq n+1}K_{\beta n+1mi}^2\\ &\geq -\frac{2}{9}(p-1)n(n-1)(b-a)^2.\end{aligned}$$

Here we note that $h_{ij}^{n+1}h_{mj}^{\beta}+h_{ij}^{\beta}h_{mj}^{n+1}$ is symmetric with respect to i and m, while $K_{\beta n+1mi}$ is anti-symmetric with respect to i and m. Therefore

$$\sum_{i,j,m,\beta\neq n+1}(h_{ij}^{n+1}h_{mj}^{\beta}+h_{ij}^{\beta}h_{mj}^{n+1})K_{\beta n+1mi}=0.$$

Thus we have

$$C \geq -\frac{1}{72}n(n-1)(26n+16p-41)(b-a)^2-\operatorname{div}\omega. \tag{2.7}$$

By using Green's divergence theorem we get

$$\int_M C\,dM \geq -\frac{1}{72}n(n-1)(26n+16p-41)\int_M (b-a)^2\,dM.$$

□

We are now in position to establish the following geometric inequality by setting the constant

$$D(n,p) := \frac{1}{72}n(n-1)(26n+16p-41).$$

THEOREM 2.4. *Let $a(x)$ and $b(x)$ for a point $x\in N^{n+p}$ be the minimum and maximum of the sectional curvature of N at the point respectively. Then*

$$\begin{aligned}\int_M \{(S_H-nH^2)\Big[na+2nH^2-S-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_H-nH^2)^{1/2}\Big]&\\ -D(n,p)(b-a)^2\}\,dM\leq 0.&\end{aligned}\tag{2.8}$$

Proof. It follows from Lemmas 2.1, 2.2 and (2.1) that

$$\frac{1}{2}\Delta S_H \geq (S_H - nH^2)\Big[na + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_H - nH^2)^{1/2}\Big] + C. \tag{2.9}$$

Integrating both sides of (2.9) and applying Lemma 2.3, we conclude the proof. ∎

If $p > 1$, then (1.3), (1.6) and (1.7) imply that for all $\beta \neq n+1$,

$$\begin{aligned}\Delta h^\beta_{ij} = &-\sum_k (K_{\beta kikj} + K_{\beta ijkk}) + \sum_{k,m}(h^\beta_{mk}K_{mijk} + h^\beta_{im}K_{mkjk})\\ &+ \sum_{\alpha,m,k}(h^\beta_{mi}h^\alpha_{mj}h^\alpha_{kk} + h^\beta_{mk}h^\alpha_{ki}h^\alpha_{mj} - h^\beta_{mi}h^\alpha_{mk}h^\alpha_{kj} - h^\beta_{km}h^\alpha_{km}h^\alpha_{ij})\\ &+ \sum_{k,\alpha\neq n+1} h^\alpha_{ki}K_{\alpha\beta jk} + \sum_{k,l,\alpha\neq n+1}(h^\alpha_{jl}h^\beta_{lk} - h^\alpha_{kl}h^\beta_{lj})h^\alpha_{ki}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{1}{2}\Delta S_I &= \sum_{i,j,k,\beta\neq n+1}(h^\beta_{ijk})^2 + \sum_{i,j,\beta\neq n+1} h^\beta_{ij}\Delta h^\beta_{ij}\\ &= W + X + Y + Z,\end{aligned} \tag{2.10}$$

where we set

$$W := -\sum_{\alpha,\beta\neq n+1} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)]^2,$$

$$X := nH\sum_{\beta\neq n+1}\mathrm{tr}(H_{n+1}H_\beta^2) - \sum_{\beta\neq n+1}[\mathrm{tr}(H_{n+1}H_\beta)]^2, \tag{2.11}$$

$$\begin{aligned}Y := &\sum_{i,j,k,m,\beta\neq n+1}(h^\beta_{ij}h^\beta_{jm}K_{mkik} + h^\beta_{mk}h^\beta_{ij}K_{mijk})\\ &+ \sum_{i,j,k,\alpha,\beta\neq n+1} h^\alpha_{ik}h^\beta_{ij}K_{\alpha\beta jk},\end{aligned} \tag{2.12}$$

$$\begin{aligned}Z := &\sum_{i,j,k,\beta\neq n+1}(h^\beta_{ijk})^2 - \sum_{i,j,k,\beta\neq n+1}(h^\beta_{ij}K_{\beta kikj} + h^\beta_{ij}K_{\beta ijkk})\\ &+ \sum_{\beta\neq n+1}\mathrm{tr}(H_{n+1}H_\beta)^2 - \sum_{\beta\neq n+1}\mathrm{tr}(H_{n+1}^2H_\beta^2).\end{aligned} \tag{2.13}$$

From Proposition 1.4 we find a lower bound for the first term of right hand side of (2.10) as

$$W \geq -(1 + \frac{1}{2}\operatorname{sgn}(p-2))\Big(\sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2\Big)^2. \tag{2.14}$$

To estimate the lower bounds for the other terms on the right hand side of (2.10) we need the following Lemmas.

LEMMA 2.5.

$$X \geq \Big[2nH^2 - (n-2)n^{1/2}(n-1)^{-1/2}H(\operatorname{tr} H_{n+1}^2 - nH^2)^{1/2} - \operatorname{tr} H_{n+1}^2\Big] \sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2.$$

Proof. We rewrite (2.11) as

$$\begin{aligned} X = nH &\sum_{\beta \neq n+1} \operatorname{tr}[(H_{n+1} - HI)H_\beta^2] \\ &+ nH^2 \sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2 - \sum_{\beta \neq n+1} [\operatorname{tr}(H_{n+1} - HI)H_\beta]^2, \end{aligned} \tag{2.15}$$

where I is the unit $(n \times n)$-matrix. Fix a vector e_β and let $\{e_i\}$ be a local orthonormal frame such that the matrix H_β takes the diagonal form and such that

$$h_{ij}^\beta = 0, \qquad \text{for } i \neq j.$$

Then we get

$$\begin{aligned} &nH \operatorname{tr}[(H_{n+1} - HI)H_\beta^2] - [\operatorname{tr}(H_{n+1} - HI)H_\beta]^2 \\ &\qquad = nH\Big[\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\beta)^2\Big] - \Big[\sum_i (h_{ii}^{n+1} - H)h_{ii}^\beta\Big]^2. \end{aligned} \tag{2.16}$$

Using the Schwarz inequality and Proposition 1.6, we see from (2.16)

$$\begin{aligned} nH &\operatorname{tr}[(H_{n+1} - HI)H_\beta^2] - [\operatorname{tr}(H_{n+1} - HI)H_\beta]^2 \\ &\geq -(n-2)n^{1/2}(n-1)^{-1/2}H\Big[\sum_i (h_{ii}^{n+1} - H)^2\Big]^{1/2}\Big[\sum_i (h_{ii}^\beta)^2\Big] \\ &\quad - \Big[\sum_i (h_{ii}^{n+1} - H)^2\Big]\Big[\sum_i (h_{ii}^\beta)^2\Big] \\ &\geq -(n-2)n^{1/2}(n-1)^{-1/2}H(\operatorname{tr} H_{n+1}^2 - nH^2)^{1/2} \operatorname{tr} H_\beta^2 \\ &\quad - (\operatorname{tr} H_{n+1}^2 - nH^2) \operatorname{tr} H_\beta^2. \end{aligned}$$

This together with (2.15) implies the desired inequality. ∎

LEMMA 2.6.

$$Y \geq na \sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2 - \frac{2}{3}(p-2)(n-1)^{1/2}(b-a) \sum_{\beta \neq n+1} \operatorname{tr} H_\beta^2.$$

Proof. For a fixed index β we choose an orthonormal frame $\{e_i\}$ such that the matrix H_β takes the diagonal form. We then conclude the proof by

$$\begin{aligned}
&\sum_{i,j,k,m} (h_{ij}^\beta h_{jm}^\beta K_{mkik} + h_{ij}^\beta h_{mk}^\beta K_{mijk}) + \sum_{i,j,k,\alpha \neq n+1} h_{ik}^\alpha h_{ij}^\beta K_{\alpha\beta jk} \\
&\qquad = \sum_{i,k} (h_{ii}^\beta)^2 K_{ikik} + \sum_{i,k} h_{kk}^\beta h_{ii}^\beta K_{kiik} + \sum_{i,k,\alpha \neq n+1} h_{ik}^\alpha \lambda_i^\beta K_{\alpha\beta ik} \\
&\qquad \geq \frac{1}{2} \sum_{i,k} (h_{ii}^\beta - h_{kk}^\beta)^2 K_{ikik} - \sum_{i \neq k, \alpha \neq \beta, n+1} \frac{2}{3}(b-a)|h_{ik}^\alpha \lambda_i^\beta| \\
&\qquad \geq \frac{1}{2} a \sum_{i,k} (h_{ii}^\beta - h_{kk}^\beta)^2 \\
&\qquad\quad - \sum_{i \neq k, \alpha \neq \beta, n+1} \frac{1}{3}(b-a)[(n-1)^{1/2}(h_{ik}^\alpha)^2 + (n-1)^{-1/2}(\lambda_i^\beta)^2] \\
&\qquad \geq na \cdot \operatorname{tr} H_\beta^2 - \frac{1}{3}(n-1)^{1/2}(b-a) \sum_{\alpha \neq \beta, n+1} \operatorname{tr} H_\alpha^2 \\
&\qquad\quad - \frac{1}{3}(n-1)^{1/2}(p-2)(b-a) \operatorname{tr} H_\beta^2.
\end{aligned}$$

∎

LEMMA 2.7.

$$\int_M Z \, dM \geq -\frac{1}{72}(p-1)n(n-1)(26n-9) \int_M (b-a)^2 \, dM.$$

Proof. A differentiable 1-form θ is defined as follows

$$\theta := \sum_{i,j,k,\beta \neq n+1} (h_{ik}^\beta K_{\beta jij} + h_{ij}^\beta K_{\beta ijk})\omega_k.$$

We then have

$$
\begin{aligned}
(2.17)\qquad & -\sum_{i,j,k,\beta\neq n+1}(h^{\beta}_{ij}K_{\beta kikj}+h^{\beta}_{ij}K_{\beta ijkk})\\
&\qquad = \sum_{i,j,k,\beta\neq n+1}(h^{\beta}_{ijj}K_{\beta kik}+h^{\beta}_{ijk}K_{\beta ijk})-\operatorname{div}\theta.
\end{aligned}
$$

From $\sum_{i,j,k}(h^{\beta}_{ijk}+\frac{1}{2}K_{\beta ijk})^2\geq 0$ we have

$$
\begin{aligned}
&\sum_{i,j,k,\beta\neq n+1}[(h^{\beta}_{ijk})^2+h^{\beta}_{ijk}K_{\beta ijk}]\\
&\qquad \geq -\frac{1}{4}\sum_{i,j,k,\beta\neq n+1}K^2_{\beta ijk}\\
(2.18)\qquad &\qquad \geq -\frac{1}{4}\sum_{i\neq j\neq k\neq i,\beta\neq n+1}K^2_{\beta ijk}-\frac{1}{2}\sum_{i\neq j,\beta\neq n+1}K^2_{\beta iji}\\
&\qquad \geq -\frac{1}{72}(p-1)n(n-1)(8n-7)(b-a)^2.
\end{aligned}
$$

Since M admits parallel mean curvature normal field, we have

$$
(2.19)\qquad \sum_i h^{\beta}_{iij}=0,\qquad \text{for all } j.
$$

From (1.6), (2.19) and Proposition 1.5 we get

$$
\begin{aligned}
(2.20)\qquad \sum_{i,j,k,\beta\neq n+1}h^{\beta}_{ijj}K_{\beta kik} &= -\sum_{i,\beta\neq n+1}\Big(\sum_j K_{\beta jij}\Big)^2\\
&\geq -\frac{1}{4}(p-1)n(n-1)^2(b-a)^2.
\end{aligned}
$$

From (1.4) and Proposition 1.2,

$$
\begin{aligned}
&\sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}H_{\beta})^2-\sum_{\beta\neq n+1}\operatorname{tr}(H^2_{n+1}H^2_{\beta})\\
&\quad =\frac{1}{2}\sum_{i,j,k,\beta\neq n+1}(h^{n+1}_{ij}h^{\beta}_{ik}+h^{n+1}_{ik}h^{\beta}_{ij})K_{n+1\beta jk}-\frac{1}{2}\sum_{j,k,\beta\neq n+1}K^2_{n+1\beta jk}\\
&\quad \geq -\frac{2}{9}(p-1)n(n-1)(b-a)^2.
\end{aligned}
$$

Combining (2.13), (2.17), (2.18), (2.20) and the above inequality, we get

$$Z \geq -\frac{1}{72}(p-1)n(n-1)(26n-9)(b-a)^2 - \operatorname{div}\theta \tag{2.21}$$

and hence,

$$\int_M Z\,dM \geq -\frac{1}{72}(p-1)n(n-1)(26n-9)\int_M (b-a)^2\,dM.$$

□

We now set

$$E(n,p) := \frac{2}{3}(p-2)(n-1)^{1/2}, \quad F(n,p) := \frac{1}{72}(p-1)n(n-1)(26n-9).$$

THEOREM 2.8. *If $p \neq 1$, then*

$$\begin{aligned}\int_M \Big\{ S_I[na + 2nH^2 - S - \frac{1}{2}\operatorname{sgn}(p-2)S_I \\ -(n-2)n^{1/2}(n-1)^{-1/2}H(S_H - nH^2)^{1/2} \\ -E(n,p)(b-a)] - F(n,p)(b-a)^2\Big\}\,dM \leq 0.\end{aligned}$$

Proof. We see from (2.10), (2.14), Lemmas 2.5 and 2.6,

$$\begin{aligned}\frac{1}{2}\Delta S_I \geq S_I\lfloor na + 2nH^2 - S - \frac{1}{2}\operatorname{sgn}(p-2)S_I \\ - (n-2)n^{1/2}(n-1)^{-1/2}H(S_H - nH^2)^{1/2} \\ - \frac{2}{3}(p-2)(n-1)^{1/2}(b-a)] + Z.\end{aligned}$$

By the Green formula and Lemma 2.7,

$$\begin{aligned}\int_M \Big\{ S_I[na + 2nH^2 - S - \frac{1}{2}\operatorname{sgn}(p-2)S_I \\ - (n-2)n^{1/2}(n-1)^{-1/2}H(S_H - nH^2)^{1/2} \\ - E(n,p)(b-a)] - F(n,p)(b-a)^2\Big\}\,dM \leq 0.\end{aligned} \tag{2.22}$$

□

§3. Rigidity theorems for compact submanifolds

The compact case for Main Theorem is discussed here. By setting

$$\beta_1(n) := \frac{1}{6\sqrt{2}}\sqrt{n(n-1)(26n-25)},$$
$$\gamma_1(n) := n + \frac{1}{6\sqrt{2}}\sqrt{n(n-1)(26n-25)},$$

we first prove the following.

THEOREM 3.1. *Let $p = 1$. There exists a number $\delta_1(n)$ with $0 < \delta_1(n) < 1$ such that if the sectional curvature of N^{n+1} and the squared norm of the second fundamental form of M satisfy*

$$\delta_1(n) \leq K_N \leq 1, \quad nH^2 + \beta_1(n)(1-c) \leq S \leq \alpha(n,H) - \gamma_1(n)(1-c),$$

where $c := \inf K_N$, then N is isometric to $\mathbf{S}^{n+1}(1)$. Moreover M is congruent to either $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$ or the isoparametric hypersurface $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$. Here λ and $\alpha(n,H)$ are defined in Theorem 0.1.

Proof. Since $c \leq a(x) \leq b(x) \leq 1$, (2.8) implies that

$$\int_M \Big\{(S-nH^2)\Big[nc + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2}\Big] - D(n,p)(1-c)^2\Big\} \leq 0. \tag{3.1}$$

Setting $\delta_1(n) := 1 - 2n(n-1)[(n^2-2n+2)(\beta_1(n)+\gamma_1(n))]^{-1}$, we have

$$\begin{aligned} 1 - c \leq 1 - \delta_1(n) &= 2n(n-1)[(n^2-2n+2)(\beta_1(n)+\gamma_1(n))]^{-1} \\ &\leq \min_{H\geq 0}[\alpha(n,H) - nH^2][\beta_1(n)+\gamma_1(n)]^{-1} \\ &\leq [\alpha(n,H) - nH^2]\,[\beta_1(n)+\gamma_1(n)]^{-1}. \end{aligned}$$

Thus we get

$$nH^2 + \beta_1(n)(1-c) \leq \alpha(n,H) - \gamma_1(n)(1-c).$$

From assumption

$$nH^2 + \beta_1(n)(1-c) \leq S \leq \alpha(n,H) - \gamma_1(n)(1-c), \tag{3.2}$$

we see that the first term on the left hand side of (3.3) is not less than

$$\begin{aligned}\beta_1(n)(1-c)[n+2nH^2-\alpha(n,H)\\ -\frac{n(n-2)}{\sqrt{n(n-1)}}H(\alpha(n,H)-nH^2)^{1/2}+\alpha(n,H)-S+nc-n]\\ &=\beta_1(n)(1-c)[\alpha(n,H)-S+n(c-1)]\\ &\geq\beta_1(n)(1-c)[\gamma_1(n)(1-c)-n(1-c)]\\ &=D(n,p)(1-c)^2.\end{aligned}$$

Thus we have

$$(S-nH^2)\Big[nc+2nH^2-S-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2}\Big]-D(n,p)(1-c)^2\geq 0. \tag{3.3}$$

From (3.1) and (3.3) we observe that the left hand side of (3.3) is identically zero. This together with (2.8) and $c\leq a\leq b\leq 1$ implies $a\equiv c$ and $b\equiv 1$. Since equalities hold on (2.5) and (2.6), we have

$$\sum_{i,j,k}h_{ikk}^{n+1}K_{n+1jij}=-\frac{1}{4}n(n-1)^2(b-a)^2=-\frac{1}{4}n(n-1)^2(1-c)^2,$$

$$0=h_{ikk}^{n+1}+\frac{1}{2}K_{n+1ikk}=h_{ikk}^{n+1}.$$

This implies that $1-c=0$. Therefore all the equalities hold in (2.5), (2.6), (3.1) and (3.3). Since the left hand side of (3.3) is identically zero and $c=1$, we see that N is isometric to $\mathbf{S}^{n+1}(1)$, and

$$S=nH^2\qquad\text{or}\qquad S=\alpha(n,H).$$

The above relations imply that $S\leq C(n,p,H)$. From Theorem 0.1 we see that M is the small sphere $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$ or the isoparametric hypersurface $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}})\times\mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$. □

The following lemma is needed for the discussion of the case of higher codimensions.

LEMMA 3.2. *Let M^n be a closed and oriented submanifold in N^{n+p}. If*

$$S\leq nd+\frac{n^3}{2(n-1)}H^2-\frac{n(n-2)}{\sqrt{2(n-1)}}\sqrt{n^2H^4+4(n-1)H^2d}-\alpha_1(d-c),$$

then either $d = c$ *or*

$$\int_M (S_H - nH^2)\, dM \le \alpha_2 \int_M (b-a)\, dM.$$

Here α_1, α_2 *are positive constants with* $\alpha_2 = D(n,p)(\alpha_1 - n)^{-1}$ *and* $d = \sup_N b$ *as defined in* §1.

Proof. Let

$$\alpha(n,H,d) := nd + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2 d}.$$

From assumption, the left hand side of (3.4) is not less than

$$\begin{aligned} nd + 2nH^2 - \alpha(n,H,d) - \frac{n(n-2)}{\sqrt{n(n-1)}}H(\alpha(n,H,d) - nH^2) \\ + \alpha(n,H,d) - S - \alpha_1(d-c) = a(n,H,d) - S - \alpha_1(d-c) \ge 0. \end{aligned}$$

Then we see

$$nd - \alpha_1(d-c) + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S - nH^2)^{1/2} \ge 0. \tag{3.4}$$

From Theorem 2.4,

$$\begin{aligned} \int_M \{(S_H - nH^2)[nc + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2}] \\ -D(n,p)(d-c)(b-a)\}\, dM \le 0. \end{aligned} \tag{3.5}$$

From (3.4),

$$nc + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2} \ge \alpha_2^{-1} D(n,p)(d-c),$$

where $\alpha_2 = D(n,p)(\alpha_1 - n)^{-1}$. Substituting the above inequality into (3.5) gives

$$\int_M \{(S_H - nH^2)[\alpha_2^{-1}D(n,p)(d-c)] - D(n,p)(d-c)(b-a)\}\, dM \le 0$$

Therefore we have either $d = c$ or

$$\int_M (S_H - nH^2)\, dM \le \alpha_2 \int_M (b-a)\, dM.$$

$\square$

We continue the proof of our Main Theorem.

If $p = 2$, then (2.8) and (2.22) imply that

$$\begin{aligned}(3.6)\quad \int_M \Big\{(S - nH^2)\Big[na + 2nH^2 - S \\ - (n-2)n^{1/2}(n-1)^{-1/2}H(S_H - nH^2)^{1/2}\Big] \\ - (D(n,p) + F(n,p))(b-a)^2\Big\}\, dM \leq 0.\end{aligned}$$

This implies

$$\begin{aligned}(3.7)\quad \int_M \Big\{(S - nH^2)\Big[na + 2nH^2 - S \\ - (n-2)n^{1/2}(n-1)^{-1/2}H(S - nH^2)^{1/2}\Big] - G(n)(b-a)^2\Big\}\, dM \leq 0,\end{aligned}$$

where we set

$$G(n) := \frac{1}{36}n(n-1)(26n-9).$$

By using the similar method as developed in the proof of Theorem 3.1, we obtain the following

THEOREM 3.3. *Let $p = 2$. There exists a number $\delta_2(n)$ with $0 < \delta_2(n) < 1$ such that if $\delta_2(n) \leq K_N \leq 1$ and if*

$$nH^2 + \beta_2(n)(1-c) \leq S \leq \alpha(n, H) - \gamma_2(n)(1-c),$$

where $c := \inf_N K_N$, then N^{n+2} is isometric to $\mathbf{S}^{n+2}(1)$. Moreover M is congruent to one of the following:

(1) $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$.

(2) *the isoparametric hypersurface $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ in $\mathbf{S}^{n+1}(1)$.*

(3) *the Clifford torus $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2)$ in $\mathbf{S}^3(r)$ with constant mean curvature H_0, where r_1, $r_2 = [2(1 + H^2) \pm 2H_0(1 + H^2)^{1/2}]^{-1/2}$, $r = (1 + H^2 - H_0^2)^{-1/2}$, and $0 \leq H_0 \leq H$.*

Here the constants are given by

$$\beta_2(n) := \frac{1}{6\sqrt{2}}\sqrt{n(n-1)(26n-9)},$$

$$\gamma_2(n) := n + \frac{1}{6\sqrt{2}}\sqrt{n(n-1)(26n-9)},$$

$$\delta_2(n) := 1 - 2n(n-1)[(n^2 - 2n + 2)(\beta_2(n) + \gamma_2(n))]^{-1}.$$

We now discuss the case where $p \geq 3$.

THEOREM 3.4. *Let $p \geq 3$. There exists a number $\delta_3(n,p) \in (0,1)$ such that if $\delta_3(n,p) \leq K_N \leq 1$, and if*

$$\begin{aligned}&nH^2 + \beta_3(n,p)(1-c) + \beta_4(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}\\ &\quad \leq S \leq C(n,p,H) - \gamma_3(n,p)(1-c) - \gamma_4(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4},\end{aligned}$$

then N is isometric to $\mathbf{S}^{n+p}(1)$ and M is congruent to one of the following:

(1) $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$.

(2) *the isoparametric hypersurface* $\mathbf{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbf{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ *in* $\mathbf{S}^n(1)$.

(3) *the Clifford torus $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2)$ in $\mathbf{S}^3(r)$ with constant mean curvature H_0, where r_1, $r_2 = [2(1+H^2) \pm 2H_0(1+H^2)^{1/2}]^{-1/2}$, $r = (1+H^2-H_0^2)^{-1/2}$, and $0 \leq H_0 \leq H$.*

(4) *the Veronese surface in* $\mathbf{S}^4(\frac{1}{\sqrt{1+H^2}})$.

Here the constants β_3, β_4, γ_3, γ_4 and δ_3 are given later in the remark.

Proof. We argue the proof by deriving a contradiction. Suppose that $c \neq 1$. We then have from (2.22)

$$\begin{aligned}(3.8)\quad \int_M \Big\{S_I\Big[na + \frac{5}{2}nH^2 - \frac{3}{2}S - (n-2)n^{1/2}(n-1)^{-1/2}H(S_H - nH^2)^{1/2}\\ - E(n,p)(1-c)\Big] - F(n,p)(1-c)^2\Big\}\, dM \leq 0.\end{aligned}$$

In fact, since $S = S_H + S_I \geq nH^2 + S_I$ we observe

$$\frac{5}{2}nH^2 - \frac{3}{2}S \leq 2nH^2 - S - \frac{1}{2}S_I.$$

This together with $0 \leq b - a \leq 1 - c$ implies that the left hand side of (3.8) is not greater than the left hand side of (2.22) which is nonpositive.

Note that the assumption implies

$$S \leq C(n,p,H) - \gamma_3(n,p)(1-c) \leq \alpha(n,H) - \alpha_1(n,p)(1-c).$$

This shows that the assumption of Lemma 3.2 is fulfilled, where $d = 1$. By the Schwarz inequality and Lemma 3.2,

$$\begin{aligned}\int_M & H S_I (S_H - nH^2)^{1/2}\, dM \\ & \le H \max S_I \cdot \mathrm{vol}(M)^{1/2} \Big[\int_M (S_H - nH^2)\Big]^{1/2} \\ & \le \frac{2}{3} n \alpha_2(n,p)^{1/2} H(1+H^2)(1-c)^{1/2}\,\mathrm{vol}(M).\end{aligned} \tag{3.9}$$

Here the second inequality is obtained as follows. By Lemma 3.2,

$$\int_M (S_H - nH^2)\, dM \le \alpha_2 (1-c)\,\mathrm{vol}(M).$$

This together with $S_I \le S - nH^2 \le C(n,p,H) - nH^2 \le \frac{2}{3} n(1+H^2)$ implies

$$\begin{aligned} H \max S_I\, \mathrm{vol}(M)^{1/2} & \Big[\int_M (S_H - nH^2)\, dM\Big]^{1/2} \\ & \le \frac{2}{3} n \alpha_2(n,p)^{1/2} H(1+H^2)(1-c)^{1/2}\,\mathrm{vol}(M).\end{aligned}$$

Combining (3.8) and (3.9), we have

$$\begin{aligned}\int_M \Big\{ S_I & \Big[nc + \frac{5}{2} nH^2 - \frac{3}{2} S - E(n,p)(1-c)\Big] \\ & - E_1(n,p,H)(1-c)^{1/2} - F(n,p)(1-c)^2 \Big\}\, dM \le 0,\end{aligned} \tag{3.10}$$

where we set

$$\begin{aligned} & E_1(n,p,H) \\ & \quad := \frac{2}{3} n^{3/2}(n-2)(n-1)^{-1/2} D(n,p)^{1/2} (\alpha_1(n,p) - n)^{-1/2}(1+H^2)H.\end{aligned}$$

Since $C(n,p,H) \le \frac{1}{3}(2n + 5nH^2)$ and $\gamma_3(n,p) \ge \frac{2}{3}[n + E(n,p) + F(n,p)^{1/2}]$ and $\gamma_4(n,p)[(1+H^2)H]^{1/2} = \frac{2}{3} E_1(n,p,H)^{1/2}$, we obtain

$$\begin{aligned} S \le \frac{1}{3}(2n + 5nH^2) - \frac{2}{3}(n + E(n,p) + F(n,p)^{1/2})(1-c) \\ - \frac{2}{3} E_1(n,p,H)^{1/2} (1-c)^{1/4}.\end{aligned}$$

Substituting the above into (3.10), we get

$$\int_M S_I\, dM \le [F(n,p)^{1/2}(1-c) + E_1(n,p,H)^{1/2}(1-c)^{1/4}]\,\mathrm{vol}(M), \tag{3.11}$$

and this together with Lemma 3.2 yields

$$\begin{aligned}\int_M (S - nH^2)\, dM \le {} & [(\alpha_2(n,p) + F(n,p)^{1/2})(1-c) \\ & + E_1(n,p,H)^{1/2}(1-c)^{1/4}]\,\mathrm{vol}(M).\end{aligned} \tag{3.12}$$

From assumption follows

$$S - nH^2 \ge \beta_3(n,p)(1-c) + \beta_4(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}.$$

This and (3.12) imply that

$$S - nH^2 \equiv [\alpha_2(n,p) + F(n,p)^{1/2}](1-c) + E_1(n,p,H)^{1/2}(1-c)^{1/4}.$$

Therefore we see that all the inequalities in (2.18), (2.20), (2.22), (3.8) and (3.12) are actually equalities. From (2.22), (3.8) and (3.12) we get $1 - c = b - a$. Because $c \le a \le b \le 1$, we have $a = c$, $b = 1$. It follows from (2.18) and (2.20) that

$$0 = h^\beta_{ijj} + \frac{1}{2}K_{\beta ijj} = h^\beta_{ijj}, \quad \sum h^\beta_{ijj}K_{\beta kik} = -\frac{1}{4}(p-1)n(n-1)^2(1-c)^2.$$

This implies $1 - c = 0$, contradicting to what is supposed at the beginning of the proof. Since N is assumed to be complete and simply connected, N is isometric to $\mathbf{S}^{n+p}(1)$. Moreover from (3.5) and (3.10),

$$S = nH^2 \qquad \text{or} \qquad S = C(n,p,H).$$

From Theorem 0.1 we see that M is one of the (1), (2), (3) and (4). This proves Theorem 3.4. □

Remark 3.5. To make sure of the assumption in Theorem 3.4 we set

$$\delta_3(n,p) = \begin{cases} 1 - \dfrac{4}{3}(\beta_3 + \gamma_3)^{-1}, & \text{for } n = 2 \\ 1 - \min\{16(\beta_3 + \gamma_3 + \dfrac{5}{4}\gamma_4)^{-4}, & \\ \qquad \dfrac{256}{14641}[n^{-1}(n-1)(\beta_4 + \gamma_4)]^{-4}\}, & \text{for } n \ge 3. \end{cases}$$

Then the following inequality makes sense.

$$nH^2 + \beta_3(n,p)(1-c) + \beta_4(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}$$
$$< C(n,p,H) - \gamma_3(n,p)(1-c) - \gamma_4(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}.$$

Here α_1, α_2, β_3, β_4, γ_3 and γ_4 are constants precisely given as follows.

$$\alpha_1(n,p) = n + \frac{1}{6\sqrt{2}}[n(n-1)(26n+16p-41)]^{1/2},$$
$$\alpha_2(n,p) = \frac{1}{6\sqrt{2}}[n(n-1)(26n+16p-41)]^{1/2},$$
$$\beta_3(n,p) = \alpha_2(n,p) + F(n,p)^{1/2},$$
$$\gamma_3(n,p) = \max\{\alpha_1(n,p), \frac{2}{3}(n + E(n,p) + F(n,p)^{1/2})\},$$
$$\beta_4(n,p) = 3^{-3/4}[2n^7(n-2)^4(n-1)^{-1}(26n+16p-41)]^{1/8},$$
$$\gamma_4(n,p) = 3^{-7/4}[512n^7(n-2)^4(n-1)^{-1}(26n+16p-41)]^{1/8}.$$

The constants in Main Theorem are given as follows.

$$\tau(n,p) = \begin{cases} \delta(n,p), & \text{for } H=0 \\ \delta_1(n), & \text{for } p=1 \text{ and } H\neq 0 \\ \delta_2(n), & \text{for } p=2 \text{ and } H\neq 0 \\ \delta_3(n,p), & \text{for } p\geq 3 \text{ and } H\neq 0, \end{cases}$$

$$A_1(n,p) = \begin{cases} \beta(n,p), & \text{for } H=0 \\ \beta_1(n), & \text{for } p=1 \text{ and } H\neq 0 \\ \beta_2(n), & \text{for } p=2 \text{ and } H\neq 0 \\ \beta_3(n,p), & \text{for } p\geq 3 \text{ and } H\neq 0, \end{cases}$$

$$B_1(n,p) = \begin{cases} \gamma(n,p), & \text{for } H=0 \\ \gamma_1(n), & \text{for } p=1 \text{ and } H\neq 0 \\ \gamma_2(n), & \text{for } p=2 \text{ and } H\neq 0 \\ \gamma_3(n,p), & \text{for } p\geq 3 \text{ and } H\neq 0, \end{cases}$$

$$A_2(n,p) = \begin{cases} \beta_4(n,p), & \text{for } n\geq 3,\ p\geq 3 \text{ and } H\neq 0 \\ 0, & \text{otherwise}, \end{cases}$$

$$B_2(n,p) = \begin{cases} \gamma_4(n,p), & \text{for } n\geq 3,\ p\geq 3 \text{ and } H\neq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Combining Theorems 0.2, 3.1, 3.3 and 3.4, we conclude the proof of Main Theorem for compact case.

§4. The complete case

The generalized maximum principle due to Omori [O] and Yau [Y2] is the useful tool to generalize rigidity theorems such as the Chern-do Carmo-Kobayashi theorem to complete cases where compactness is not assumed (see [H], [P]). However it does not apply to our case because the divergence of the two 1-forms in (2.7) and (2.21) require pointwise estimates. We shall employ the following useful Lemma 4.1 for the discussion of the rigidity of complete submanifolds with parallel mean curvature normal fields where compactness is not assumed. We emphasize that all the results obtained here include *the minimal case.*

LEMMA 4.1. (see [X-4]) *Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p} with $K_N \geq c$. Here c is a constant satisfying $c+H^2 > 0$. If $S \leq \alpha(n, H, c)$, then the Ricci curvature satisfies*

$$\mathrm{Ric}_M \geq \frac{n-1}{n}\left[nc + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{\frac{1}{2}}\right].$$

Moreover, if $\sup_M(S - \alpha(n, H, c)) < 0$, then M is compact, where

$$\alpha(n, H, c) := nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2}.$$

Thus from the rigidity theorems in previous section and Lemma 4.1, we obtain the following

THEOREM 4.2. *For given positive integers $n \geq 2$, p and a nonnegative constant H there exists a number $\tau(n,p)$ with $0 < \tau(n,p) < 1$ such that if M^n is an oriented complete submanifold in a complete simply connected Riemannian $(n+p)$-manifold N^{n+p} with $\tau(n,p) \leq K_N \leq 1$, and if*

$$\begin{aligned} &nH^2 + A_1(n,p)(1-c) + A_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4} \\ &\quad \leq S \leq C(n,p,H) - B_1(n,p)(1-c) - B_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}, \end{aligned}$$

then N is isometric to $\mathbf{S}^{n+p}(1)$. Moreover if

$$\sup_M S < \alpha(n, H),$$

then M is congruent to either $\mathbf{S}^n(\frac{1}{\sqrt{1+H^2}})$ or the Veronese surface in $\mathbf{S}^4(\frac{1}{\sqrt{1+H^2}})$.

We conclude the proof of Main Theorem by the combination of Theorems 3.1, 3.3, 3.4 and 4.2.

In due course of the proof of Main Theorem the completeness and simple connectedness of N is used only to guarantee that N is isometric to the round sphere. We can derive $K_N \equiv 1$ without these assumptions. Thus we obtain the following Theorem 4.3 by using the orientable double covering.

THEOREM 4.3. *For given positive integers $n \geq 2$, p and a nonnegative constant H there exists a number $\tau(n,p)$ with $0 < \tau(n,p) < 1$ such that if M^n is a complete submanifold with parallel mean curvature normal field in a Riemannian $(n+p)$-manifold N^{n+p} with $\tau(n,p) \leq K_N \leq 1$, and if*

$$\begin{aligned} &nH^2 + A_1(n,p)(1-c) + A_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4} \\ &\quad \leq S \leq C(n,p,H) - B_1(n,p)(1-c) - B_2(n,p)[(1+H^2)H]^{1/2}(1-c)^{1/4}, \end{aligned}$$

then $K_N \equiv 1$. Moreover,

(1) *if $\sup_M S < \alpha(n,H)$, then $S = nH^2$ and M is totally umbilic, or $S = \frac{2}{3}(2+5H^2)$*

(2) *if M is compact, then $S = nH^2$ and M is totally umbilic, or $S = C(n,p,H)$.*

REFERENCES

[AdC] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. A. M. S., **120** (1994), 1223–1229.

[CDK] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in Functional Analysis and Related Fields (F. E. Browder, ed.), Springer-Verlag, New York (1970).

[Ga] H. Gauchman, *Minimal submanifolds of a sphere with bounded second fundamental form*, Trans. A. M. S., **298** (1986), 779–791.

[G] S. Goldberg, Curvature and Homology, Academic Press, London, 1962.

[H] T. Hasanis, *Characterization of totally umbilical hypersurfaces*, Proc. Amer. Math. Soc., **81** (1981), 447–450.

[L1] B. Lawson, *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math., **89** (1969), 187–197.

[L2] ———, *Rigidity theorems in rank one symmetric space*, J. Differential Geometry, **4** (1970), 349–357.

[LL] A. M. Li and J. M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Archiv der Math., **58** (1992), 582–594.

[O] H. Omori, *Isometric immersion of Riemannian manifold*, J. Math. Soc. Japan, **19** (1967), 205–214.

[P] Y. L. Pan, *Pinching theorems of Simon's type for complete minimal submanifolds in the sphere*, Proc. Amer. Math. Soc., **93** (1985), 710–712.

[Sh] Y. B. Shen, *On intrinsic rigidity for minimal submanifolds in a sphere*, Sci. Sinica, Ser. A, **32** (1989), 769–781.

[Si] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math., **88, (2)** (1968), 62–105.

[X1] H. W. Xu, *A rigidity theorem for submanifolds with parallel mean curvature in a sphere*, Archiv der Math., **61** (1993), 489–496.

[X2] ———, *On closed minimal submanifolds in pinched Riemannian manifolds*, Trans. A. M. S., **347** (1995), 1743–1751.

[X3] ———, *A pinching constant of Simons' type and isometric immersion*, Chinese Ann. Math., Ser. A, **12** (1991), 261–269.

[X4] ———, *Lower bounds for eigenvalues of submanifolds in a Riemannian manifold* (1992), no. Research Report, Zhejiang University.

[Y1] S. T. Yau, *Submanifolds with constant mean curvature* I, II, Amer. J. Math., **96, 97** (1974, 1975), 346–366, 76–100.

[Y2] ———, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math., **28** (1975), 201–228.

Katsuhiro Shiohama
Department of Mathematics, Faculty of Science and Engineering
Saga University
Saga, 840–8502, Japan

Hongwei Xu
Institute of mathematics
Zhejiang University
Hangzhou, 310027-China