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ASYMPTOTIC EXPANSIONS OF RECURSION COEFFICIENTS OF ORTHOGONAL POLYNOMIALS WITH TRUNCATED EXPONENTIAL WEIGHTS

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Abstract. Let $\beta > 0$ and $W_\beta(x) = \exp(-|x|^\beta)$, $x \in \mathbb{R}$. For $c > 0$, define $W_{\beta,cn}(x) = W_\beta(x)$ if $|x| \le c^{1/\beta} a_{2n}$ and $W_{\beta,cn}(x) = 0$ if $|x| > c^{1/\beta} a_{2n}$, where a_{2n} denotes Mhaskar-Rahmanov-Saff number for W_β. Let $\gamma_n(W_{\beta,cn})$ be the leading coefficient of the nth orthonormal polynomial corresponding to $W_{\beta,cn}$ and write $\alpha_n(W_{\beta,cn}) = \gamma_{n-1}(W_{\beta,cn})/\gamma_n(W_{\beta,cn})$. It is shown that if $c > 1$ and β is a positive even integer then $\alpha_n(W_{\beta,cn})/n^{1/\beta}$ has an asymptotic expansion. Also when $0 < c < 1$, asymptotic expansions of recursion coefficients of the truncated Hermite weights are given.

§1. Introduction

Let $\beta > 0$. Let $W_\beta(x) = \exp(-|x|^\beta)$, $x \in \mathbb{R}$, and let $\gamma_n = \gamma_n(W_\beta) > 0$ denote the leading coefficient of nth orthonormal polynomial p_n corresponding to W_β. The leading coefficients γ_n have the following minimum property. Denoting by $\mathbb{P}_n$ the set of all polynomials of degree at most n we have

$$\gamma_n^{-2}(W_\beta) = \min_{q \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} [x^n + q(x)]^2 W_\beta(x)\, dx \, . \tag{1.1}$$

Since W_β is even, the recursion formula of the orthonormal polynomials has the form

$$x p_{n-1}(x) = \alpha_n p_n(x) + \alpha_{n-1} p_{n-2}(x) \, , \tag{1.2}$$

$n = 1, 2, 3, \cdots$, where $\alpha_0 = 0$ and $\alpha_n = \gamma_{n-1}/\gamma_n$, $n \ge 1$. Associated with the weight function W_β, there are Mhaskar-Rahmanov-Saff numbers $a_n = a_n(W_\beta)$, which is a positive solution of the equation

$$n = (2/\pi) \int_0^1 a_n t Q'(a_n t)(1 - t^2)^{-1/2}\, dt \, ,$$

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$n = 1, 2, 3, \cdots$, where $Q(x) = |x|^{\beta}$ (cf. [4]). Explicitly,

$$a_n = a_n(W_\beta) = (n/\lambda_\beta)^{1/\beta}\,, \quad n = 1, 2, 3, \cdots, \tag{1.3}$$

where

$$\lambda_\beta = \frac{2^{2-\beta}\Gamma(\beta)}{\{\Gamma(\beta/2)\}^2}\,.$$

Using Mhaskar-Rahmanov-Saff numbers $a_n = a_n(W_\beta)$, we define the truncated exponential weights $W_{\beta,cn}$, $c > 0$, $n = 1, 2, 3, \cdots$, by

$$W_{\beta,cn}(x) = \begin{cases} W_\beta(x) & \text{if} \quad |x| \le c^{1/\beta}\, a_{2n}\,, \\ 0 & \text{if} \quad |x| > c^{1/\beta}\, a_{2n}\,. \end{cases} \tag{1.4}$$

Our purpose in this paper is to prove the followings.

Theorem 1.1. *Let $c > 1$ and β be a positive real number. Then*

$$\alpha_n(W_{\beta,cn})/\alpha_n(W_\beta) \sim 1 \quad as \quad n \to \infty\,. \tag{1.5}$$

In other words,

$$\alpha_n(W_{\beta,cn})/\alpha_n(W_\beta) = 1 + o(n^{-k})\,,$$

for every integer $k \ge 1$ as $n \to \infty$.

Combining a result of Máté, Nevai and Zaslavsky [6, Theorem 1, p. 496] and Theorem 1.1 we have

Theorem 1.2. *Let $c > 1$ and β be a positive even integer. Then $\alpha_n(W_{\beta,cn})/n^{1/\beta}$ has an asymptotic expansion*

$$\alpha_n(W_{\beta,cn})/n^{1/\beta} \sim \sum_{k=0}^{\infty} c_{2k}\, n^{-2k} \quad as \quad n \to \infty\,, \tag{1.6}$$

where the constants c_{2k}'s are independent on c.

Theorems 1.1 and 1.2 depend on infinite-finite range inequality (2.1). When $0 < c < 1$, (2.1) is not true any more. Using the recurrence equation, we obtain following result for the truncated Hermite weights.

THEOREM 1.3. *If* $0 < c < 1$, *then* $\alpha_n^2(W_{2,cn})/n$ *has an asymptotic expansion*

$$\alpha_n^2(W_{2,cn})/n \sim \sum_{k=0}^{\infty} d_{2k}\, n^{-2k} \quad as \quad n \to \infty \,, \tag{1.7}$$

where $d_0 = c/2$, $d_2 = \frac{c}{8(1-c)^2}$, $d_4 = \frac{17c^2+c}{32(1-c)^5}$, *and* $d_6 = \frac{1126c^3+196c^2+c}{128(1-c)^8}$.

Of course if $c > 1$, then

$$\alpha_n^2(W_{2,cn})/n \sim 1/2 \quad as \quad n \to \infty \,,$$

as a consequence of Theorem 1.1 and $\alpha_n^2(W_2) = n/2$.

§2. Proof of Theorems 1.1 and 1.2

To prove Theorem 1.1 we need following infinite-finite range inequality, which is a special case of [3, formula (3. 19) in Lemma 3.3, p. 32].

LEMMA 2.1. *Let* $\beta > 0$ *and* $c > 1$. *Then there exist a positive constant* C_0 *and a positive integer* n_0 *such that, for* $n \geq n_0$, *and for* $P \in \mathbb{P}_n$,

$$\int_{-\infty}^{\infty} P^2(x)\, W_\beta(x)\, dx \leq (1 + e^{-C_0 n}) \int_{-\infty}^{\infty} P^2(x)\, W_{\beta,cn}(x)\, dx \,. \tag{2.1}$$

Proof of Theorem 1.1. Since $W_\beta(x) \geq W_{\beta,cn}(x)$, $\gamma_n(W_\beta) \leq \gamma_n(W_{\beta,cn})$, hence, in view of (2.1) and (1.1), we have

$$1 \leq \gamma_n^2(W_{\beta,cn})/\gamma_n^2(W_\beta) \leq 1 + e^{-C_0 n} \quad \text{for all} \quad n \geq n_0 \,,$$

and

$$1 \leq \gamma_{n-1}^2(W_{\beta,cn})/\gamma_{n-1}^2(W_\beta) \leq 1 + e^{-C_0 n} \quad \text{for all} \quad n \geq n_0 \,,$$

which implies Theorem 1.1.

Proof of Theorem 1.2. Máté, Nevai and Zaslavsky [6, Theorem 1, p.496] showed that, if β is a positive even integer, $\alpha_n(W_\beta)/n^{1/\beta}$ has an asymptotic expansion

$$\alpha_n(W_\beta)/n^{1/\beta} \sim \sum_{k=0}^{\infty} c_{2k} n^{-2k} \quad as \quad n \to \infty \,. \tag{2.2}$$

Thus Theorem 1.2 follows from (2.2) and Theorem 1.1.

§3. Proof of Theorem 1.3

Instead of dealing with the weights $W_{2,cn}$, we consider the weights w_m, $m > 0$, defined by

$$w_m(x) = \begin{cases} \exp(-2mx^2) & \text{if} \quad |x| \le 1\,, \\ 0 & \text{if} \quad |x| > 1\,. \end{cases} \tag{3.1}$$

Let $p_n(x) = \gamma_n x^n + \cdots$, $\gamma_n > 0$, denote the orthonormal polynomials corresponding to w_m. Since w_m is even, the recursion formula becomes

$$xp_{n-1}(x) = \alpha_n p_n(x) + \alpha_{n-1} p_{n-2}(x)\ , \tag{3.2}$$

$n = 1, 2, 3, \cdots$, where $\alpha_0 = 0$ and $\alpha_n = \gamma_{n-1}/\gamma_n$, $n \ge 1$, and p_n is even if n is even and odd if n is odd. Observe that

$$\frac{n}{\alpha_n} = \int_{-1}^{1} p_n'(x) p_{n-1}(x) w_m(x)\, dx\,,\ n \ge 1.$$

Integrating by parts the right hand side of the above equation, we obtain

$$\frac{n}{\alpha_n} - 4m\alpha_n\ = 2p_n(1)p_{n-1}(1)w_m(1)\,.$$

Squaring both sides, we have

$$\frac{n^2}{\alpha_n^2} - 8nm + 16m^2\alpha_n^2\ = 4p_n^2(1)p_{n-1}^2(1)w_m^2(1)\,. \tag{3.3}$$

We remark[1] that (3.3) can be obtained by equating leading terms in

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x),$$

where

$$\begin{aligned} A_n(x) = {} & \frac{\alpha_n w_m(1)p_n^2(1)}{1-x} + \frac{\alpha_n w_m(-1)p_n^2(-1)}{x+1} \\ & +\alpha_n \int_{-1}^{1} \frac{v'(x)-v'(y)}{x-y} p_n^2(y) w_m(y)dy, \\ B_n(x) = {} & \frac{\alpha_n w_m(1)p_n(1)p_{n-1}(1)}{1-x} + \frac{\alpha_n w_m(-1)p_n(-1)p_{n-1}(-1)}{x+1} \\ & +\alpha_n \int_{-1}^{1} \frac{v'(x)-v'(y)}{x-y} p_n(y)p_{n-1}(y) w_m(y)dy, \end{aligned}$$

[1] We thank the referee for this remark.

and $v(x) = 2mx^2$ (cf. [1], [2]).

Also note that

$$(3.4)\quad 1 + 2n = \int_{-1}^{1} [xp_n^2(x)]' w_m(x)\, dx = 2p_n^2(1) w_m(1) + 4m(\alpha_{n+1}^2 + \alpha_n^2).$$

From (3.3) and (3.4), we obtain the recurrence equation

$$(3.5)\quad \frac{1}{4} = \alpha_n^2 \left[1 + \frac{1}{2n} - \frac{2m(\alpha_{n+1}^2 + \alpha_n^2)}{n}\right] \left[1 - \frac{1}{2n} - \frac{2m(\alpha_n^2 + \alpha_{n-1}^2)}{n}\right] + \frac{2m\alpha_n^2}{n} - \frac{4m^2\alpha_n^4}{n^2},$$

$n = 1, 2, 3, \cdots,$ where $\alpha_0 = 0$ and $\alpha_1 = \gamma_0/\gamma_1$. Since

$$0 < \alpha_n = \int_{-1}^{1} x\, p_n(x)\, p_{n-1}(x)\, w_m(x)\, dx \le \int_{-1}^{1} |p_n(x)\, p_{n-1}(x)|\, w_m(x)\, dx$$
$$\le \left(\int_{-1}^{1} p_n^2(x)\, w_m(x)\, dx\right)^{1/2} \left(\int_{-1}^{1} p_{n-1}^2(x)\, w_m(x)\, dx\right)^{1/2} = 1,$$

$$(3.6)\qquad \lim_{n\to\infty} \alpha_n^2(w_m) = 1/4$$

follows from (3.5). Now we show that $\alpha_n^2(w_m)$ has an asymptotic expansion.

LEMMA 3.1. *Let $m > 0$. Then $\alpha_n^2(w_m)$ has an asymptotic expansion*

$$(3.7)\qquad \alpha_n^2 \sim \sum_{k=0}^{\infty} r_k\, n^{-k} \quad \text{as} \quad n \to \infty,$$

that is,

$$(3.8)\qquad \alpha_n^2 = \sum_{k=0}^{j} r_k\, n^{-k} + o(n^{-j}),$$

for every integer $j \ge 0$ as $n \to \infty$, where $r_0 = 1/4$.

Proof of Lemma 3.1. Let

$$(3.9)\quad H(x,y,z,w) = [y + (1/4)][1 - (1/2)(2m-1)w - 2mw(y+z)] \cdot [1 - (1/2)(1+2m)w - 2mw(x+y)] + 2myw - 4m^2y^2w^2 - (1/4).$$

Then

$$\frac{\partial H(0,0,0,0)}{\partial x} = \frac{\partial H(0,0,0,0)}{\partial z} = 0\,, \quad \frac{\partial H(0,0,0,0)}{\partial y} = 1\,. \tag{3.10}$$

Putting

$$y_n = \alpha_n^2 - (1/4)\,, \tag{3.11}$$

we have

$$H(y_{n-1}, y_n, y_{n+1}, 1/n) = 0 \qquad \text{by (3.5)}. \tag{3.12}$$

By (3.10), (3.11), (3.6), and (3.12), the conditions of the theorem of Máté and Nevai [5, Theorem in p. 423] are satisfied, hence,

$$\alpha_n^2 \sim \sum_{k=0}^{\infty} r_k\, n^{-k} \quad \text{as} \quad n \to \infty\,,$$

where $r_0 = 1/4$ by (3.6). □

Once we know (3.8) is true, we expand α_{n-1}^2 and α_{n+1}^2 in terms of n^{-k} using the binomial theorem, and then, putting them into (3.5), we find $r_1 = 0$, $r_2 = 1/16$, and $r_3 = m/8$.

Rewriting (3.7) as

$$\alpha_n^2(w_m) \sim 1/4 + \sum_{k=0}^{\infty} e_k\, n^{-(k+2)} \quad \text{as} \quad n \to \infty\,, \tag{3.13}$$

where $e_0 = 1/16$ and $e_1 = m/8$, next we show

LEMMA 3.2. *The coefficient e_k in (3.13) is a polynomial in m of degree k with the leading coefficient $(k+1)/16$, and e_k is even, if k is even, and odd, if k is odd.*

Proof of Lemma 3.2. When $k = 0, 1$, the lemma is true according to (3.13). Using induction on k, we may assume that the lemma is true for $e_0, e_1, e_2, \cdots$, and e_{k-1}. Let

$$\alpha_n^2 = 1/4 + \sum_{j=0}^{k} e_j\, n^{-(j+2)} + o(n^{-k-2})\,. \tag{3.14}$$

Writing $(n \pm 1)^{-j} = n^{-j}(1 \pm 1/n)^{-j}$, and then, using the binomial expansion for $(1 \pm 1/n)^{-j}$, we obtain

$$\alpha_{n-1}^2 = 1/4 + \sum_{j=0}^{k}\Big[\sum_{i=0}^{j}\binom{j+1}{i+1} e_i\Big] n^{-(j+2)} + o(n^{-k-2}), \tag{3.15}$$

and

$$\alpha_{n+1}^2 = 1/4 + \sum_{j=0}^{k}\Big[\sum_{i=0}^{j}(-1)^{j-i}\binom{j+1}{i+1} e_i\Big] n^{-(j+2)} + o(n^{-k-2}). \tag{3.16}$$

Now, expanding (3.5) yields

$$\begin{aligned}(3.17)\ 0 = \alpha_n^2\Big[1 + \frac{2m}{n} - \frac{1}{4n^2}\Big] - \alpha_n^4\Big[\frac{4m}{n} + \frac{4m^2}{n^2}\Big] + \frac{4m^2\alpha_n^6}{n^2} \\ + \frac{m\alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)}{n^2} \\ + (\alpha_{n+1}^2 + \alpha_{n-1}^2)\Big[-\frac{2m\alpha_n^2}{n} + \frac{4m^2\alpha_n^4}{n^2}\Big] + \frac{4m^2\alpha_n^2\alpha_{n+1}^2\alpha_{n-1}^2}{n^2} - \frac{1}{4}.\end{aligned}$$

Substituting (3.14), (3.15), and (3.16) into (3.17) yields

$$\sum_{j=0}^{k} C_j\, n^{-(j+2)} + o(n^{-k-2}) = 0,$$

which implies $C_j = 0$, for $j = 0, 1, 2, \cdots, k$. In particular from $C_k = 0$, we can express e_k in terms of $e_0, e_1, e_2, \cdots, e_{k-1}$ as follows. Setting

$$s_j = \sum_{i=0}^{j}\binom{j+1}{i+1} e_i, \quad \text{and} \quad t_j = \sum_{i=0}^{j}(-1)^{j-i}\binom{j+1}{i+1} e_i,$$

for $j = 0, 1, 2, \cdots, k-1$, and let

$$f_j = s_j + t_j, \quad g_j = t_j - s_j, \quad \text{and} \quad h_j = \frac{s_j + t_j}{4} + \sum_{i=0}^{j-2} s_i\, t_{j-2-i},$$

for $j = 0, 1, 2, \cdots, k-1$. Since t_j is alternating, it follows by the induction hypothesis that

(3.18) f_j is a polynomial in m of degree j, and is odd, if j is odd,
and even, if j is even,

for $j = 0, 1, 2, \cdots, k-1$,

(3.19) $g_0 = 0$ and

$$g_j \text{ is a polynomial in } m \text{ of degree } j-1, \text{ and is odd, if } j \text{ is even, and even, if } j \text{ is odd,}$$

for $j = 1, 2, 3, \cdots, k-1$, and

(3.20) h_j is a polynomial in m of degree j, and is odd, if j is odd, and even, if j is even,

for $j = 0, 1, 2, \cdots, k-1$.

Now we have

$$\begin{aligned}(3.21)\quad e_k = {} & \frac{e_{k-2}}{4} + m\Big[e_{k-1} + \frac{f_{k-1}}{2} - \frac{g_{k-2}}{4} + 4\sum_{j=0}^{k-3} e_j\, e_{k-3-j} \\ & + 2\sum_{j=0}^{k-3} e_j\, f_{k-3-j} - \sum_{j=0}^{k-4} e_j\, g_{k-4-j}\Big] \\ & - m^2\Big[\frac{f_{k-2}}{4} + h_{k-2} + 2e_0\,(e_{k-4} + f_{k-4}) \\ & + 2e_1\,(e_{k-5} + f_{k-5}) - \sum_{j=0}^{k-4} e_j\, e_{k-4-j} + 4\sum_{j=0}^{k-4} e_j\, h_{k-4-j} \\ & + \sum_{i=0}^{k-6}(e_i + f_i)\Big(2e_{k-4-i} + 4\sum_{j=0}^{k-6-i} e_j\, e_{k-6-i-j}\Big)\Big].\end{aligned}$$

Since the coefficient of m^j in f_j is $(j+1)/8$ and the coefficient of m^j in h_j is $(j+1)/32$, $j = 0, 1, 2, \cdots, k-1$, Lemma 3.2 follows from (3.18), (3.19), (3.20), and (3.21) and the induction hypothesis. □

By Lemma 3.2 we write

$$\begin{aligned}(3.22)\quad e_k = e_k(m) = {} & b_{k,0}\, m^k + b_{k,1}\, m^{k-2} \\ & + b_{k,2}\, m^{k-4} + \cdots + b_{k,[k/2]}\, m^{k-2[k/2]},\end{aligned}$$

where $[\cdot]$ denotes the integer part. Note that $b_{2p,p}$ is the constant term in e_{2p} and $b_{2p,p} = b_{2p-2,p-1}/4$ by (3.21). Since $b_{0,0} = 1/16$,

$$(3.23)\qquad b_{2p,p} = \frac{1}{4^{p+2}}, \quad p = 0, 1, 2, \cdots .$$

We need more information about $b_{2p+k,p}$ as $k \to \infty$, that is, we want that for each $p = 0, 1, 2, \cdots$, $b_{2p+k,p}$ can not grow too fast as $k \to \infty$ so that $\sum_{k=0}^{\infty} b_{2p+k,p}\, c^k$ converges provided $0 < c < 1$. In fact we show that for each $p = 0, 1, 2, \cdots$, $b_{2p+k,p}$ is a polynomial in k whose degree depends only on p. For $p = 0$,

$$b_{k,0} = (k+1)/16\ , \quad k = 0, 1, 2, \cdots , \tag{3.24}$$

by Lemma 3.2. When $p = 1$, we find

$$\begin{aligned} b_{2+k,1} = \Big(\frac{3}{256}\Big)&(k+4)(k+3)(k+2)(k+1) \\ &-\Big(\frac{17}{384}\Big)(k+3)(k+2)(k+1)\,, \end{aligned} \tag{3.25}$$

for $k = 0, 1, 2, \cdots$, as follows. Note that $b_{2+k,1}$ is the coefficient of m^k in e_{2+k}. Comparing the coefficient of m^k in both sides of (3.21) with $2 + k$ replacing k, we obtain

$$\begin{aligned} &b_{2+k,1} - 2b_{2+(k-1),1} + b_{2+(k-2),1} \\ &= \frac{b_{k,0}}{4} + \frac{(k+1)(k+2)b_{k-1,0}}{2} + \frac{(k+1)b_{k-1,0}}{2} \\ &\quad - \frac{k(k+1)b_{k-2,0}}{2} - 6(b_{0,0}b_{k-2,0} + b_{1,0}b_{k-3,0}) + 8\sum_{j=0}^{k-1} b_{j,0}b_{k-1-j,0} \\ &\quad -2\sum_{j=0}^{k-2} b_{j,0}b_{k-2-j,0} - 6\sum_{j=0}^{k-4} b_{j,0}b_{k-2-j,0}\ . \end{aligned} \tag{3.26}$$

Substituting (3.24) into (3.26), it follows that

$$b_{2+k,1} - 2b_{2+(k-1),1} + b_{2+(k-2),1} = \frac{9k^2}{64} + \frac{5k}{32} + \frac{1}{64}\ ,$$

for $k = 1, 2, \cdots$. Solving the above difference equation with $b_{2,1} = 1/64$ by (3.23) and $b_{1,1} = 0$ yields (3.25). When $p = 2$, similarly as done above, we have

$$\begin{aligned} b_{4+k,2} = &\frac{21(k+7)(k+6)(k+5)\cdots(k+1)}{20480} \\ &-\frac{17(k+6)(k+5)\cdots(k+1)}{1280} \\ &+\frac{563(k+5)(k+4)(k+3)(k+2)(k+1)}{15360}\ , \end{aligned} \tag{3.27}$$

for $k = 0, 1, 2, \cdots$. Now using induction on p we assume that $b_{2i+k,i}$ is a polynomial in k for $i = 0, 1, 2, \cdots, p-1$. Replacing k by $2p+k$ in (3.21) and then comparing the coefficient of m^k in both sides we find that $b_{2p+k,p} - 2b_{2p+(k-1),p} + b_{2p+(k-2),p}$ can be expressed in terms of k and $b_{2i+j,i}$, $i = 0, 1, 2, \cdots, p-1$, $j = 0, 1, 2, \cdots, k$. Using the induction hypothesis yields

$$b_{2p+k,p} - 2b_{2p+(k-1),p} + b_{2p+(k-2),p} \;=\; \text{a polynomial in } k\,.$$

Then the following lemma follows by solving the above difference equation.

Lemma 3.3. *For each $p = 0, 1, 2, \cdots$, $b_{2p+k,p}$ is a polynomial in k whose degree depends only on p.*

Recall the definitions of w_{cn} in (3.1). Now we prove

Lemma 3.4. *Let $0 < c < 1$. Then $\alpha_n^2(w_{cn})$ has an asymptotic expansion*

$$\alpha_n^2(w_{cn}) \sim \frac{1}{4} + \sum_{p=0}^{\infty} \delta_{2p}\, n^{-2p-2} \qquad as \qquad n \to \infty\,, \tag{3.28}$$

where $\delta_0 = \frac{1}{16(1-c)^2}$, $\delta_2 = \frac{17c+1}{64(1-c)^5}$, and $\delta_4 = \frac{1126c^2+196c+1}{256(1-c)^8}$.

Proof of Lemma 3.4. From Lemma 3.1 and (3.13)

$$\alpha_n^2(w_m) \sim 1/4 + \sum_{k=0}^{\infty} \frac{e_k(m)}{n^{k+2}} \quad \text{as} \quad n \to \infty\,,$$

for each fixed $m > 0$. If we replace m by cn, then, with the notations in (3.22),

$$\frac{e_k(cn)}{n^{k+2}} = \sum_{p=0}^{[k/2]} \frac{b_{k,p}\, c^{k-2p}}{n^{2p+2}}\,.$$

Since $\sum_{k=0}^{\infty} b_{2p+k,p}\, c^k$ converges for each $p = 0, 1, 2, \cdots$, by Lemma 3.3, we have

$$\alpha_n^2(w_{cn}) \sim \frac{1}{4} + \sum_{p=0}^{\infty} \delta_{2p}\, n^{-2p-2} \qquad \text{as} \qquad n \to \infty\,,$$

where $\delta_{2p} = \sum_{k=0}^{\infty} b_{2p+k,p}\, c^k$. From (3.24), (3.25), and (3.27), we obtain δ_0, δ_2, and δ_4, completing the proof of Lemma 3.4. $\square$

Proof of Theorem 1.3. Since

$$\alpha_n^2(w_{cn}) = \frac{\alpha_n^2(W_{2,cn})}{2cn} ,$$

Theorem 1.3 follows from Lemma 3.4. ∎

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