K. Shimomura
Nagoya Math. J.
Vol. 158 (2000), 133–166

THE DETERMINATION OF CALORIC MORPHISMS ON EUCLIDEAN DOMAINS

KATSUNORI SHIMOMURA

Dedicated to Professor Masayuki Itô in honour of his sixtieth birthday

Abstract. Let D be a domain in $\mathbb{R}^{m+1}$ and E be a domain in $\mathbb{R}^{n+1}$. A pair of a smooth mapping $f : D \to E$ and a smooth positive function φ on D is called a caloric morphism if $\varphi \cdot u \circ f$ is a solution of the heat equation in D whenever u is a solution of the heat equation in E. We give the characterization of caloric morphisms, and then give the determination of caloric morphisms. In the case of $m < n$, there are no caloric morphisms. In the case of $m = n$, caloric morphisms are generated by the dilation, the rotation, the translation and the Appell transformation. In the case of $m > n$, under some assumption on f, every caloric morphism is obtained by composing a projection with a direct sum of caloric morphisms of $\mathbb{R}^{n+1}$.

§1. Introduction

For a non-negative integer k, $\mathbb{R}^{k+1}$ denotes the $k+1$-dimensional Euclidean space. The coordinates in $\mathbb{R}^{k+1}$ is denoted by (t, x) or (x_0, x) where $x = (x_1, \ldots, x_k)$.

We shall use the following notation:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right), \quad \Delta = \sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2}, \quad H = \frac{\partial}{\partial t} - \Delta.$$

A C^2-function h is said to be caloric if h satisfies the heat equation

$$Hh = 0.$$

Since the heat operator H is hypoelliptic (see, e.g. [9]), every caloric function is infinitely differentiable.

Received November 12, 1998.
2000 *Mathematics Subject Classification*: 31B99, 35K99, 35A30

Let m, n be positive integers and D a domain in $\mathbb{R}^{m+1}$. We denote by $(t,x) = (t, x_1, \dots, x_m)$, $(\tau, y) = (\tau, y_1, \dots, y_n)$ the points of $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ respectively. We consider a mapping $f(t,x) = (f_0(t,x), f_1(t,x), \dots, f_n(t,x)) : D \to \mathbb{R}^{n+1}$ and a weight function φ which preserve solutions of the heat equation in the following sense. *A pair* (f, φ) *of* C^2*-mapping* $f : D \to \mathbb{R}^{n+1}$ *and a positive* C^2*-function* φ *on* D *is said to be a caloric morphism if* $f(D)$ *is a domain in* $\mathbb{R}^{n+1}$ *and if for every caloric function* u *on* $f(D)$, $\varphi(t,x)(u \circ f)(t,x)$ *is also a caloric function on* D.

In the case of $m = n$, the following three typical caloric morphisms are known.

The Appell transformation

Let $D = (0, \infty) \times \mathbb{R}^n$ (resp. $= (-\infty, 0) \times \mathbb{R}^n$). Put

$$f(t,x) = \Big(-\frac{1}{t}, \frac{x}{t} \Big), \quad \varphi(t,x) = \frac{1}{\sqrt{4\pi|t|}^{\,n}} e^{-|x|^2/4t}.$$

Then $f(D) = (-\infty, 0) \times \mathbb{R}^n$ (resp. $= (0, \infty) \times \mathbb{R}^n$) and (f, φ) is a caloric morphism.

The dilation and the rotation in x

Let $\lambda > 0$ and U be an (n,n)-orthogonal matrix. Put

$$f(t,x) = (\lambda^2 t, \lambda U x), \quad \varphi(t,x) = 1.$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n+1}$.

The translation

Let $a \in \mathbb{R}$ and b, $c \in \mathbb{R}^n$. Put

$$f(t,x) = (t + a, x + tb + c), \quad \varphi(t,x) = e^{\frac{1}{4}|b|^2 t + \frac{1}{2} b \cdot x}.$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n+1}$.

We give two simple examples in the case of $m > n$.

Example 1. The symmetrization in $\mathbb{R}^m$ with respect to a subspace with codimension 2.

Let $m \geqq 4$, $n = m - 2$ and $D = \{(t,x) \,;\, t > 0, |x'| > 0\}$ (resp. $D = \{(t,x); t < 0, |x'| > 0\}$), where $x' = (x_1, x_2, x_3, 0, \dots, 0)$ for $x = (x_1, \dots, x_m)$. Put

$$\begin{cases} f_0(t) = -t^{-1}, \\ f_1(t,x) = t^{-1}|x'|, \\ f_j(t,x) = t^{-1} x_{j+2}, \qquad 2 \leqq j \leqq n, \end{cases}$$

$$\varphi(t,x) = |x'|^{-1}|t|^{-(m-2)/2}\exp\Big(-\frac{|x|^2}{4t}\Big).$$

Then $f(D) = \{(\tau, y); \tau < 0,\ y_1 > 0\}$ (resp. $f(D) = \{(\tau, y); \tau > 0,\ y_1 < 0\}$) and (f, φ) is a caloric morphism.

Example 2. The projection in x.

Let h be an arbitrary positive caloric function on $\mathbb{R}^{m-n+1}$. Put

$$f(t, x_1, \dots, x_m) = (t, x_1, \dots, x_n), \quad \varphi(t,x) = h(t, x_{n+1}, \dots, x_m).$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{m+1}$ onto $\mathbb{R}^{n+1}$.

In the case of $m = n$, Leutwiler [7] proved that every caloric morphism has the following form:

$$f(t,x) = \Big(\frac{\alpha t+\beta}{\gamma t+\delta}, \frac{Rx + tv + w}{\gamma t+\delta}\Big),$$
$$\varphi(t,x) = \begin{cases} \frac{C}{|\gamma t+\delta|^{n/2}}\exp\Big[-\frac{|\gamma Rx+\gamma w-\delta v|^2}{\gamma|\gamma t+\delta|}\Big], & \gamma \neq 0, \\ C\exp\Big[\frac{|v|^2}{4}t + \frac{1}{2}v\cdot Rx\Big], & \gamma = 0, \end{cases} \tag{0}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha\delta - \beta\gamma = 1$, $v, w \in \mathbb{R}^n$, R is an n-dimensional orthogonal matrix, $C > 0$ and $\cdot$ denotes the inner product of $\mathbb{R}^n$. It is a composition of the above three morphisms: the Appell transformation, the dilation, the translation.

The aim of this paper is to extend this to the case of $m \neq n$.

We first give a general characterization of caloric morphisms, which is essentially obtained by Leutwiler. As its corollary, there are no caloric morphism if $m < n$. Also by virtue of the characterization, we obtain a new systematic way to construct a caloric morphism by a “direct sum” of caloric morphisms in the case of $m > n$. It is remarkable that the direct sum gives caloric morphisms of new type such that f_0 is a sum of fractional linear functions. Note that in the case of $m = n$, f_0 is just a fractional linear function.

Our main result is the determination of caloric morphisms (f, φ) in the case of $m > n$ under the assumption that each f_i, $1 \leqq i \leqq n$ is a polynomial in x for every t and that f_0 is real analytic. Under the assumption, we can give an explicit form of caloric morphisms (Theorem 7 below). Although it seems to be complicated, it turns out to be a direct sum of the caloric morphisms of form (0) composed with a projection, as is shown in Corollary 10.

§2. Characterization of caloric morphisms

Definition 1. A pair (f, φ) of C^2-mapping $f : D \to \mathbb{R}^{n+1}$ and a positive C^2-function on D is said to be a caloric morphism, if $f(D)$ is a domain and if for every caloric function u on $f(D)$, $\varphi(t,x)(u \circ f)(t,x)$ is also a caloric function on D.

Remark 1. Using derivatives in the sense of distribution, we may assume f and φ to be continuous rather than of C^2. For the sake of simplicity, we assume here that f and φ are of C^2.

Theorem 1. *Let $f = (f_0, f_1, \dots, f_n) : D \to \mathbb{R}^{n+1}$ be a C^2-mapping such that $f(D)$ is a domain and let φ be a positive C^2-function on D. Then the following statements are equivalent*:

(i) (f, φ) *is a caloric morphism.*

(ii) *For every polynomial $P(\tau, y)$ which is caloric and of degree $\leqq 4$,*

$$\varphi(t,x)(P \circ f)(t,x)$$

is caloric on D.

(iii) f *and* φ *satisfy the following equations*:

$$H\varphi = 0, \tag{1}$$

$$\varphi H f_i = 2\nabla\varphi \cdot \nabla f_i, \quad 1 \leqq i \leqq n, \tag{2}$$

$$\nabla f_0 = 0, \tag{3}$$

$$\nabla f_i(t,x) \cdot \nabla f_j(t,x) = \delta_{ij} \frac{df_0}{dt}(t), \quad 1 \leqq i,j \leqq n, \tag{4}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^m$.

(iv) *There exists a continuous function $\lambda(t) \geqq 0$ on D such that*

$$H\{\varphi(u \circ f)\}(t,x) = \lambda(t)^2 \varphi(t,x)(Hu \circ f)(t,x) \tag{5}$$

holds for every C^2 function u on $f(D)$ where H in the right hand side means the heat operator on $\mathbb{R}^{n+1}$.

Remark 2. By (3), f_0 depends only on t. And (4) shows that $df_0/dt \geqq 0$ and $|\nabla f_j(t,x)|^2$ is independent of x, where $|\cdot|$ denotes the norm of $\mathbb{R}^m$.

Proof.
(i)⇒(ii) is trivial.
(ii)⇒(iii): By the chain rule,

$$(6)\qquad H\{\varphi(P\circ f)\} = H\varphi(P\circ f) + \sum_{i=0}^{n}(\varphi H f_i - 2\nabla\varphi\cdot\nabla f_i)\frac{\partial P}{\partial y_i}\circ f - \varphi\sum_{i,j=0}^{n}(\nabla f_i\cdot\nabla f_j)\frac{\partial^2 P}{\partial y_i\partial y_j}\circ f.$$

Let $P = 1$. Then we have $H\varphi = 0$. Let $P(y_0, y) = y_i$, $1 \leqq i \leqq n$ in the equation (6). Then we obtain

$$\varphi H f_i = 2\nabla\varphi\cdot\nabla f_i, \quad 1 \leqq i \leqq n.$$

Take a point $p \in D$ and put $q = f(p)$. Let $P(y_0, y) = (y_i - q_i)(y_j - q_j)$, $1 \leqq i, j \leqq n$, $i \neq j$ in the equation (6). Since $(\partial^2 P/\partial y_i \partial y_j)(q) = 1$ and the other derivatives of P vanish at q, we have

$$\nabla f_i(p)\cdot\nabla f_j(p) = 0, \quad 1 \leqq i, j \leqq n,\ i \neq j.$$

Since p is arbitrary,

$$(7)\qquad \nabla f_i\cdot\nabla f_j = 0, \quad 1 \leqq i, j \leqq n,\ i \neq j,$$

in D. Let $P(y_0, y) = (y_0 - q_0)^2 + (y_0 - q_0)(y_i - q_i)^2 + \frac{1}{12}(y_i - q_i)^4$, $1 \leqq i \leqq n$. Since $(\partial^2 P/\partial y_0^2)(q) = 1$ and the other derivatives of order $\leqq 2$ vanish at q, we have

$$(8)\qquad |\nabla f_0(p)|^2 = 0, \text{ and thus } \nabla f_0(p) = 0.$$

Since p is arbitrary, (3) holds. Finally, let $P(y_0, y) = y_0 - q_0 + \frac{1}{2}(y_i - q_i)^2$, $1 \leqq i \leqq n$. Since $(\partial P/\partial y_0)(q) = (\partial^2 P/\partial y_i^2)(q) = 1$ and the other derivatives vanish at q, we have

$$(9)\qquad \varphi(p) H f_0(p) = \varphi(p)|\nabla f_i(p)|^2, \quad 1 \leqq i \leqq n.$$

Combining (7), (8) and (9), we obtain (4).

(iii)⇒(iv): Let u be of C^2 in $f(D)$. By the chain rule

$$(10)\qquad H\{\varphi(u\circ f)\} = H\varphi(u\circ f) + \sum_{i=0}^{n}(\varphi H f_i - 2\nabla\varphi\cdot\nabla f_i)\frac{\partial u}{\partial y_i}\circ f - \varphi\sum_{i,j=0}^{n}(\nabla f_i\cdot\nabla f_j)\frac{\partial^2 u}{\partial y_i\partial y_j}\circ f.$$

Substituting (1)–(4) into (10), we have

$$H\{\varphi(u\circ f)\} = \varphi H f_0 \frac{\partial u}{\partial y_0}\circ f - \varphi\sum_{i=1}^{n}|\nabla f_i|^2\frac{\partial^2 u}{\partial y_i^2}\circ f = \varphi\frac{df_0}{dt}Hu\circ f.$$

Putting $\lambda(t) = (df_0/dt(t))^{1/2}$, we obtain

$$H\{\varphi(u\circ f)\}(t,x) = \lambda(t)^2\varphi(t,x)(Hu\circ f)(t,x).$$

Note that $\lambda(t) = |\nabla f_i(t,x)|$ by (4).

(iv)$\Rightarrow$(i) is evident. ∎

Corollary 2. *For every caloric morphism (f,φ), f and φ are of C^∞.*

Proof. By (2), φf_i is caloric ($1 \leqq i \leqq n$), so φf_i is of C^∞. Since $\varphi > 0$ and φ is caloric, f_i is of C^∞, $1 \leqq i \leqq n$. f_0 is of C^∞ by (4). Thus f is a C^∞-mapping. ∎

Corollary 3. *Let (f,φ) be a caloric morphism from D to $\mathbb{R}^{n+1}$. Then for any C^2-function u on $f(D)$, we have the following implications*:

$$Hu \geqq 0 \Longrightarrow H\{\varphi(u\circ f)\} \geqq 0,$$
$$Hu \leqq 0 \Longrightarrow H\{\varphi(u\circ f)\} \leqq 0.$$

They immediately follow from (5).

Corollary 4. (i) *Let $(f,\varphi) = ((f_0,\dots,f_n),\varphi)$ be a caloric morphism from $D\subset\mathbb{R}^{m+1}$ to $\mathbb{R}^{n+1}$. Then $f_0'(t) > 0$ on D.*

(ii) *If $n > m$, there are no caloric morphisms.*

Proof. (i) Suppose that $f_0'(t_0) = 0$ for some $(t_0,x_0)\in D$. Let $I\subset\mathbb{R}$ be the connected component of $\{t; f_0'(t) = 0\}$ such that $t_0\in I$. Since f_0 is a non-decreasing function, $f_0(t)\neq f_0(t_0)$ for all $t\notin I$. So we have

$$f(\{(t,x)\in D; t\in I\}) = f(D)\cap\{(\tau,y)\in\mathbb{R}^{n+1}; \tau = f_0(t_0)\}.$$

Then by (4)

$$\nabla f_i(t,x) = 0,\quad (t,x)\in D,\ t\in I,\ 1\leqq i\leqq n.$$

This and (2) imply

$$\frac{\partial f_i}{\partial t}(t,x) = 2\nabla \log \varphi \cdot \nabla f_i = 0, \quad (t,x) \in D,\, t \in I,\, 1 \leqq i \leqq n.$$

Therefore the set $f(\{(t,x) \in D; t \in I\})$ consists of one point. Thus the set $f(D) \cap \{(\tau, y); \tau = f_0(t_0)\}$ consists of one point. It is contrary to the condition that $f(D)$ is a domain. Therefore $f_0'(t) > 0$ for all t.

(ii) Let $m < n$. By virtue of (4), $\nabla f_1, \ldots, \nabla f_n$ are n orthogonal vectors in $\mathbb{R}^m$ with same length. Since $n > m$, we have $\nabla f_1 = \cdots = \nabla f_n = 0$ in D. Then (4) gives $f_0' = 0$ in D. This contradicts to (i). ∎

Let m, n, k be positive integers and let D, E be domains in $\mathbb{R}^{m+1}$, in $\mathbb{R}^{n+1}$, respectively. If $(f, \varphi) : E \to \mathbb{R}^{k+1}$ and $(g, \psi) : D \to \mathbb{R}^{n+1}$ are caloric morphisms such that $g(D) \subset E$, then we can make a caloric morphism $(F, \Phi) : D \to \mathbb{R}^{k+1}$ from (f, φ) and (g, ψ) by the composition $(F, \Phi) = (f \circ g, (\varphi \circ g)\psi)$.

The next proposition provides a manner for the construction of new caloric morphisms.

Proposition 5. *Let $l, m_1, \ldots, m_l, n$ be positive integers and I be an open interval. For each $j = 1, \ldots, l$, suppose that D_j is a domain in $\mathbb{R}^{m_j}$ and that $(g_j, \varphi_j) = ((g_{j0}, g_{j1}, \ldots, g_{jn}), \varphi_j)$ is a caloric morphism : $I \times D_j \subset \mathbb{R}^{m_j+1} \to \mathbb{R}^{n+1}$. Put*

$$\begin{aligned}
f_0(t) &= g_{10}(t) + \cdots + g_{l0}(t),\\
f_i(t, x_1, \ldots, x_{m_1+\cdots+m_l}) &= g_{1i}(t, x_1, \ldots, x_{m_1})\\
&\quad + g_{2i}(t, x_{m_1+1}, \ldots, x_{m_1+m_2}) + \cdots\\
&\quad + g_{li}(t, x_{m_1+\cdots+m_{l-1}+1}, \ldots, x_{m_1+\cdots+m_l}), \quad 1 \leqq i \leqq n,\\
\varphi(t, x_1, \ldots, x_{m_1+\cdots+m_l}) &= \varphi_1(t, x_1, \ldots, x_{m_1})\varphi_2(t, x_{m_1+1}, \ldots, x_{m_1+m_2}) \cdots\\
&\quad \varphi_l(t, x_{m_1+\cdots+m_{l-1}+1}, \ldots, x_{m_1+\cdots+m_l}).
\end{aligned}$$

Then $(f, \varphi) : I \times D_1 \times \cdots \times D_l \subset \mathbb{R}^{m_1+\cdots+m_l+1} \to \mathbb{R}^{n+1}$ is a caloric morphism.

We call the above caloric morphism (f, φ) the direct sum of $(g_1, \varphi_1), \ldots, (g_l, \varphi_l)$.

Proof. For each j, we denote by H_j, ∇_j and Δ_j the heat operator, the gradient and the Laplacian in $\mathbb{R}^{m_j+1}$. The heat operator, the gradient and the Laplacian in $\mathbb{R}^{m_1+\cdots+m_l+1}$ are denoted by H, ∇ and Δ. Since (g_j,φ_j) is a caloric morphism, (1), (2) and (4) show

$$H_j\varphi_j = 0, \quad \varphi_j H_j g_{ji} = 2\nabla_j\varphi_j\cdot\nabla_j g_{ji}, \quad \nabla_j g_{ji}\cdot\nabla_j g_{jk} = \delta_{ik}\frac{dg_{j0}}{dt}, \qquad 1\leqq i,k\leqq n,\ 1\leqq j\leqq l.$$

Using

$$\begin{aligned}\nabla f_i &= (\nabla_1 g_{1i}, \nabla_2 g_{2i},\ldots,\nabla_l g_{li}),\\ \nabla\varphi &= \varphi\Big(\frac{\nabla_1\varphi_1}{\varphi_1},\frac{\nabla_2\varphi_2}{\varphi_2},\ldots,\frac{\nabla_l\varphi_l}{\varphi_l}\Big),\\ Hf_i &= H_1g_{1i}+H_2g_{2i}+\cdots+H_lg_{li},\end{aligned}$$

we have

$$\begin{aligned}2\nabla\varphi\cdot\nabla f_i &= \varphi\Big(\frac{2\nabla_1\varphi_1\cdot\nabla_1 g_{1i}}{\varphi_1},\frac{2\nabla_2\varphi_2\cdot\nabla_2 g_{2i}}{\varphi_2},\ldots,\frac{2\nabla_l\varphi_l\cdot\nabla_l g_{li}}{\varphi_l}\Big)\\ &= \varphi(H_1g_{1i}+H_2g_{2i}+\cdots+H_lg_{li})\\ &= \varphi Hf_i, \quad 1\leqq i\leqq n,\end{aligned}$$

and

$$\begin{aligned}\nabla f_i\cdot\nabla f_k &= \nabla_1 g_{1i}\cdot\nabla_1 g_{1k}+\nabla_2 g_{2i}\cdot\nabla_2 g_{2k}+\cdots+\nabla_l g_{li}\cdot\nabla_l g_{lk}\\ &= \delta_{ik}\Big(\frac{dg_{10}}{dt}+\frac{dg_{20}}{dt}+\cdots+\frac{dg_{l0}}{dt}\Big)\\ &= \delta_{ik}\frac{df_0}{dt}.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= \varphi\Big(\frac{1}{\varphi_1}\frac{\partial\varphi_1}{\partial t}+\frac{1}{\varphi_2}\frac{\partial\varphi_2}{\partial t}+\cdots+\frac{1}{\varphi_l}\frac{\partial\varphi_l}{\partial t}\Big),\\ \Delta\varphi &= \varphi\Big(\frac{\Delta_1\varphi_1}{\varphi_1}+\frac{\Delta_2\varphi_2}{\varphi_2}+\cdots+\frac{\Delta_l\varphi_l}{\varphi_l}\Big),\end{aligned}$$

we obtain $H\varphi = 0$. Thus (f,φ) is a caloric morphism. ∎

§3. Main result

In the case of $m = n$, the form of caloric morphism is explicitly determined by Leutwiler [7]. So hereafter, we assume $m > n$ in the rest of this paper.

In the sequel, we shall determine caloric morphisms (f, φ), $f = (f_0, f_1, \ldots, f_n)$ in the case that f_i, $1 \leqq i \leqq n$ is a polynomial of x for each t and that f_0 is real analytic.

Proposition 6. *Let (f, φ) be a caloric morphism and assume that f_i, $1 \leqq i \leqq n$ is a polynomial of x for each fixed t. Then*

$$f_i(t, x) = \sum_{j=1}^{m} a_{ij}(t) x_j + b_i(t), \quad 1 \leqq i \leqq n,$$

where a_{ij}, b_i, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ are C^∞-functions.

Proof. Let t be fixed. Suppose that $f_i(t, x)$ is a polynomial of degree $l \geqq 1$. Write $f_i(t, x) = h(t, x) + g(t, x)$, where h is a homogeneous polynomial of degree l and g is a polynomial of degree $\leqq l - 1$. Since $\nabla h \neq 0$, the degree of the polynomial $|\nabla f_i|^2 = |\nabla h|^2 + 2\nabla h \cdot \nabla g + |\nabla g|^2$ is equal to $2l - 2$. On the other hand, $|\nabla f_i|^2$ is of degree 0 by (4) of Theorem 1. Thus $\deg f_i \leqq 1$. $\square$

Remark 3. We cannot replace real analytic functions in the place of polynomials in the above proposition. In the above Example 1, f_1 is not a polynomial.

Main result of this paper is the following

Theorem 7. *Let $(f, \varphi) = ((f_0, f_1, \ldots, f_n), \varphi)$ be a caloric morphism defined on a domain $D \subset \mathbb{R}^{m+1}$. Assume that for each $1 \leqq i \leqq n$ and each t, $f_i(t, x)$ is a polynomial of x and that $f_0(t)$ is real analytic.*

Then there exist a positive integer $k \leqq m/n$ and an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \ldots, x_m)$ again with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $0 \leqq i \leqq n$ and γ_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$, and a positive caloric function $h = h(t, x_{kn+1}, \ldots, x_m)$ (in the case of $m = nk$, h is a positive constant) such that f and φ are of form (I) *or* (II).

(I)

$$
\begin{aligned}
f_0(t) &= \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\
f_i(t,x) &= \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i, \quad 1 \leqq i \leqq n, \\
\varphi(t,x) &= h \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{n/2}} \exp \sum_{i=1}^{n} \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)},
\end{aligned}
$$

(II)

$$
\begin{aligned}
f_0(t) &= \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\
f_i(t,x) &= \alpha_1(x_i + \gamma_{i1} t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i, \quad 1 \leqq i \leqq n, \\
\varphi(t,x) &= h \exp \sum_{i=1}^{n} \Big[\frac{\gamma_{i1}^2}{4} t + \frac{\gamma_{i1}}{2} x_i\Big] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{n/2}} \exp \sum_{i=1}^{n} \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)}.
\end{aligned}
$$

First we shall prove the assertion of the theorem in the case of $n = 1$ under the assumption that $\log \varphi$ is a polynomial of x of degree $\leqq 2$.

Lemma 8. *Let $(f, \varphi) = ((f_0, f_1), \varphi)$ be a caloric morphism from $D \subset \mathbb{R}^{m+1}$ to $\mathbb{R}^{1+1}$. Assume that f_1 and φ are of the following form:*

$$
\begin{aligned}
f_1(t,x) &= \sum_{j=1}^{m} a_j(t) x_j + b(t), \\
\varphi(t,x) &= \exp\Big(\frac{1}{4} x \cdot U(t) x + v(t) \cdot x + w(t)\Big),
\end{aligned}
$$

where $a_1, \dots, a_m$, b and w are C^∞-functions, v is a C^∞-vector and where U is a symmetric (m,m)-matrix of C^∞-functions.

Then there exist a positive integer $k \leqq m$ and an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \dots, x_m)$ again with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $i = 0, 1$ and γ_i, $1 \leqq i \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$, and a positive caloric function $h = h(t, x_{k+1}, \dots, x_m)$

(*in the case of* $m = k$, h *is a positive constant*) *such that* f *and* φ *are of form* (1) *or* (2).

(1)

$$
\begin{aligned}
f_0(t) &= \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\
f_1(t,x) &= \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1, \\
\varphi(t,x) &= h(t,x) \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)},
\end{aligned}
$$

if $U(t_0)$ *is invertible or* $a(t_0)$ *is orthogonal to the zero-eigenspace of* $U(t_0)$ *for some* t_0.

(2)

$$
\begin{aligned}
f_0(t) &= \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\
f_1(t,x) &= \alpha_1(x_1 + \gamma_1 t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1, \\
\varphi(t,x) &= h(t,x) \exp\left[\frac{\gamma_1^2}{4}t + \frac{\gamma_1}{2}x_1\right] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)},
\end{aligned}
$$

otherwise.

Proof of Lemma 8. We may assume $t_0 = 0$ by some translation of t. Since (f, φ) is a caloric morphism, f_1 and $\log\varphi$ satisfy the equations

$$
\begin{aligned}
&\frac{\partial \log\varphi}{\partial t} - \Delta \log\varphi - |\nabla \log\varphi|^2 = 0, \\
&H f_1 = 2\nabla \log\varphi \cdot \nabla f_1,
\end{aligned}
$$

by (1) and (2). Then we have the following differential equations

$$
\begin{aligned}
&U' = U^2, \quad v' = Uv, \quad w' = \frac{|v|^2}{4} + \frac{\operatorname{tr} U}{2}, \\
&a' = Ua, \quad b' = a \cdot v,
\end{aligned}
$$

where $a = (a_1, \ldots, a_m)$ and $\operatorname{tr} U$ denotes the trace of the matrix U.

Since $U(0)$ is real symmetric, we have the spectral decomposition $U(0) = \sum_{j=1}^{l} \lambda_j P_j$, where λ_j is a real eigenvalue of $U(0)$ with multiplicity n_j, and P_j is the orthogonal projection of $\mathbb{R}^m$ to the corresponding eigenspace. Since $U(t)$ is the solution of $U' = U^2$,

$$U(t) = \sum_{j=1}^{l} \frac{\lambda_j}{1 - \lambda_j t} P_j,$$

and so the solutions of $a' = Ua$, $v' = Uv$ are

$$a(t) = \sum_{j=1}^{l} \frac{1}{1 - \lambda_j t} P_j a_0, \quad v(t) = \sum_{j=1}^{l} \frac{1}{1 - \lambda_j t} P_j v_0,$$

where $a_0 = a(0)$ and $v_0 = v(0)$.

Let k be the cardinal of $\{P_j; P_j a_0 \neq 0\}$ (note that $a_0 \neq 0$ because of (4) and Corollary 4). We may assume $P_j a_0 \neq 0$, $1 \leqq j \leqq k$, $P_j a_0 = 0$, $k < j \leqq l$ and $\lambda_j \neq 0$, $1 < j \leqq k$, $k + 1 < j \leqq l$ by some rearrangement of $\lambda_1, \ldots, \lambda_l$, if necessary.

Assume that $U(0)$ is invertible. Then $\lambda_j \neq 0$ for all j and the solutions of $b' = a \cdot v$ and $w' = |v|^2/4 + \operatorname{tr} U/2$ are

$$\begin{aligned} b(t) &= \sum_{j=1}^{k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j (1 - \lambda_j t)} + \delta_1, \\ w(t) &= \sum_{j=1}^{l} \Big(\frac{|P_j v_0|^2}{4\lambda_j (1 - \lambda_j t)} - \frac{n_j}{2} \log(1 - \lambda_j t) \Big) + \delta_2 \end{aligned}$$

with some constants δ_1 and δ_2. By $f_0' = |\nabla f_1|^2$ we have

$$f_0(t) = \int |a(t)|^2 \, dt = \sum_{j=1}^{k} \frac{|P_j a_0|^2}{\lambda_j (1 - \lambda_j t)} + \delta_0$$

with some constant δ_0. Put

$$\alpha_j = \frac{|P_j a_0|}{|\lambda_j|} > 0, \quad e_j = \frac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|} \in \mathbb{R}^m, \quad \beta_j = \frac{1}{\lambda_j}, \quad 1 \leqq j \leqq k.$$

Note that $\beta_1, \ldots, \beta_k$ are mutually distinct. Adding $m - k$ eigenvectors of $U(0)$ to $\{e_1, \ldots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis

$\{e_1, \dots, e_m\}$ of $\mathbb{R}^m$. For $j > k$, we denote by λ_j the eigenvalue of $U(0)$ corresponding to e_j and put $\beta_j = \dfrac{1}{\lambda_j}$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \dots, e_m\}$, we write $x = (x_1, \dots, x_m)$ again for every $x \in \mathbb{R}^m$. Putting $\gamma_j = e_j \cdot \sum_{i=1}^{l} P_i v_0/\lambda_i$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j=1}^{m} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)},$$

where C is a positive constant.
Put

$$h(t,x) = C \prod_{k<j\leqq m} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}.$$

Then $h = h(t, x_{k+1}, \dots, x_m)$ is a positive caloric function and

$$\varphi(t,x) = h \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}.$$

Assume that $U(0)$ is not invertible. Then there are two cases: a_0 is not orthogonal to the zero-eigenspace of $U(0)$, or a_0 is orthogonal to the zero-eigenspace. They are equivalent to $\lambda_1 = 0$, or $\lambda_{k+1} = 0$, respectively.

If $\lambda_1 = 0$, then $b(t)$, $w(t)$ are given by

$$b(t) = P_1 a_0 \cdot P_1 v_0 t + \sum_{1<j\leqq k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j(1 - \lambda_j t)} + \delta_1,$$

$$w(t) = \frac{|P_1 v_0|^2}{4} t + \sum_{1<j\leqq l} \Big(\frac{|P_j v_0|^2}{4\lambda_j(1 - \lambda_j t)} - \frac{n_j}{2} \log(1 - \lambda_j t) \Big) + \delta_2$$

with some constants δ_1 and δ_2. Thus

$$f_0(t) = |P_1 a_0|^2 t + \sum_{1<j\leqq k} \frac{|P_j a_0|^2}{\lambda_j(1 - \lambda_j t)} + \delta_0$$

with some constant δ_0. Put

$$\alpha_j = \begin{cases} |P_j a_0|, & j = 1, \\ \dfrac{|P_j a_0|}{|\lambda_j|}, & j > 1, \end{cases} \qquad e_j = \begin{cases} \dfrac{P_j a_0}{|P_j a_0|}, & j = 1, \\ \dfrac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|}, & j > 1, \end{cases}$$

$$\beta_j = \frac{1}{\lambda_j}, \quad 1 < j \leqq k.$$

Note that β_j are mutually distinct. Adding $m-k$ eigenvectors of $U(0)$ to $\{e_1, \dots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis $\{e_1, \dots, e_m\}$ of $\mathbb{R}^m$. If $j > k$ and $U(0)e_j = \lambda_i e_j$ for some $\lambda_i \neq 0$, we put $\beta_j = 1/\lambda_i$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \dots, e_m\}$, we write $x = (x_1, \dots, x_m)$ again for every $x \in \mathbb{R}^m$.
Putting $\gamma_j = e_j \cdot (P_1 v_0 + \sum_{1<i\leqq l} P_i v_0/\lambda_i)$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \alpha_1(x_1 + \gamma_1 t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j\in J_0} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{j\in J_1} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)},$$

where $J_0 = \{j; U(0)e_j = 0\}$, $J_1 = \{j; U(0)e_j \neq 0\}$ and where C is a positive constant.
Put

$$h(t,x) = C \prod_{\substack{j\in J_0\\ k<j\leqq m}} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{\substack{j\in J_1\\ k<j\leqq m}} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

Then $h = h(t, x_{k+1}, \dots, x_m)$ is a positive caloric function and

$$\varphi(t,x) = h \exp\left[\frac{\gamma_1^2}{4}t + \frac{\gamma_1}{2}x_1\right] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

Finally, if $\lambda_{k+1} = 0$, then $b(t)$, $w(t)$ are given by

$$b(t) = \sum_{1\leqq j\leqq k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j(1-\lambda_j t)} + \delta_1,$$

$$w(t) = \frac{|P_{k+1}v_0|^2}{4}t + \sum_{j \neq k+1} \Big(\frac{|P_j v_0|^2}{4\lambda_j(1-\lambda_j t)} - \frac{n_j}{2}\log(1-\lambda_j t) \Big) + \delta_2$$

with some constants δ_1 and δ_2. Thus

$$f_0(t) = \sum_{1 \leqq j \leqq k} \frac{|P_j a_0|^2}{\lambda_j(1-\lambda_j t)} + \delta_0$$

with some constant δ_0. Put

$$\alpha_j = \frac{|P_j a_0|}{|\lambda_j|}, \quad e_j = \frac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|}, \quad \beta_j = \frac{1}{\lambda_j}, \quad 1 \leqq j \leqq k.$$

Note that β_j are mutually distinct. Adding $m-k$ eigenvectors of $U(0)$ to $\{e_1, \dots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis $\{e_1, \dots, e_m\}$ of $\mathbb{R}^m$. If $j > k$ and $U(0)e_j = \lambda_i e_j$ for some $\lambda_i \neq 0$, we put $\beta_j = 1/\lambda_i$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \dots, e_m\}$, we write $x = (x_1, \dots, x_m)$ again for every $x \in \mathbb{R}^m$.
Putting $\gamma_j = e_j \cdot (P_{k+1}v_0 + \sum_{1 \leqq i \leqq l,\, i \neq k+1} P_i v_0/\lambda_i)$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j \in J_0} \exp\Big[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\Big] \prod_{j \in J_1} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)},$$

where $J_0 = \{j; U(0)e_j = 0\}$, $J_1 = \{j; U(0)e_j \neq 0\}$ and where C is a positive constant.
Since $1, \dots, k \in J_1$,

$$h(t,x) = C \prod_{j \in J_0} \exp\Big[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\Big] \prod_{\substack{j \in J_1 \\ k < j \leqq m}} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

is a positive caloric function and

$$\varphi(t,x) = h \prod_{1 \leqq j \leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

□

For the proof of Theorem 7, we may assume that f is a caloric morphism of the form

$$f_i(t,x) = \sum_{j=1}^{m} a_{ij}(t)x_j + b_i(t), \quad 1 \leqq i \leqq n, \tag{11}$$

by virtue of Proposition 6. Denote by $a_i(t)$ the row-vector $(a_{i1}(t), \dots, a_{im}(t))$.

We introduce the functions $p_k(t)$, $q_k(t)$, $k \geqq 1$ which will be used in the proof of Theorem 7. We define $p_1(t)$ and $q_1(t)$ by

$$p_1(t) = \frac{f_0''(t)}{2f_0'(t)}, \quad q_1(t) = \frac{1}{\sqrt{3}}(p_1'(t) - p_1(t)^2)^{1/2}.$$

(Recall that $f_0'(t) > 0$ for all t by virtue of Corollary 4). For $k \geqq 2$, we define $p_k(t)$ and $q_k(t)$ inductively by

$$p_k(t) = \frac{q_{k-1}'(t)}{kq_{k-1}(t)} + \frac{k-2}{k}p_{k-1}(t), \tag{12}$$

$$q_k(t) = \frac{k}{\sqrt{2k+1}}\Big(p_k'(t) - p_k^2(t) + \frac{2k-3}{(k-1)^2}q_{k-1}^2(t)\Big)^{1/2}, \tag{13}$$

if $q_{k-1}(t) \neq 0$. We put $r_i(t) \in \mathbb{R}^m$, $1 \leqq i \leqq n$ by

$$r_i(t) = \frac{1}{|a_i(t)|}a_i(t),$$

(Note that $|a_i(t)| = \sqrt{f_0'(t)} > 0$ for all i and t because of (4)). And we put $r_{n+1}(t), \dots, r_{kn}(t)$ inductively by

$$r_{i+n}(t) = \begin{cases} \dfrac{1}{q_1(t)}r_i'(t), & 1 \leqq i \leqq n, \\ \dfrac{1}{q_j(t)}(r_i'(t) + q_{j-1}(t)r_{i-n}(t)), & \\ \qquad (j-1)n+1 \leqq i \leqq jn,\ 2 \leqq j \leqq k-1, & \end{cases} \tag{14}$$

if $q_j(t) \neq 0$, $1 \leqq j \leqq k-1$.

The following is the key lemma to prove Theorem 7.

Lemma 9. *Let l be a positive integer. Assume that $q_1, \dots, q_l$ are defined on an open interval $I \subset \mathbb{R}$. Then the following statements hold.*

(i) *If $q_l \neq 0$ on I, then $r_1(t), \dots, r_{(l+1)n}(t)$ defined in (14) are orthonormal C^∞-vectors of $\mathbb{R}^m$. Adding arbitrary C^∞-vectors $r_{(l+1)n+1}(t), \dots, r_m(t)$*

such that $\{r_1(t),\dots,r_m(t)\}$ *forms an orthonormal basis of* $\mathbb{R}^m$ *for each* $t\in I$*, in the case of* $m \geqq (l+1)n+1$*, we take the change of variables*

$$\begin{cases} \tau = t, \\ \xi_j = r_j(t)\cdot x, \quad 1\leqq j\leqq m, \end{cases}$$

on $D\cap(I\times\mathbb{R}^m)$*. Then there exists a* C^∞*-function* $\psi_{l+1}(\tau,\xi_{ln+1},\dots,\xi_m)$ *on* $D\cap(I\times\mathbb{R}^m)$ *such that*

$$\begin{aligned}\log\varphi(\tau,\xi) &= \sum_{k=1}^{l}\Big(\sum_{i=(k-1)n+1}^{kn}\frac{1}{4}p_k(\tau)\xi_i^2+\frac{1}{2k}q_k(\tau)\xi_i\xi_{i+n}+\beta_i(\tau)\xi_i+\rho_i(\tau)\Big)\\ &\qquad + \psi_{l+1}(\tau,\xi_{ln+1},\dots,\xi_m),\\ \frac{\partial\psi_{l+1}}{\partial\xi_i} &= \frac{1}{2}p_{l+1}(\tau)\xi_i+\frac{1}{2(l+1)}\sum_{j=ln+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_j+\beta_i(\tau),\\ &\qquad\qquad ln+1\leqq i\leqq (l+1)n,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\psi_{l+1}}{\partial\tau}-\Delta_\xi\psi_{l+1}-\sum_{k=ln+1}^{m}\frac{\partial\psi_{l+1}}{\partial\xi_k}\Big(\frac{\partial\psi_{l+1}}{\partial\xi_k}-\sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\Big)&\\ +\sum_{i=ln+1}^{(l+1)n}\Big(\frac{2l-1}{4l^2}q_l(\tau)^2\xi_i^2+\frac{l-1}{l}q_l(\tau)\beta_{i-n}(\tau)\xi_i\Big)&=0,\end{aligned}$$

where

$$\beta_i=\begin{cases}\dfrac{b_i'}{2\sqrt{f_0'}}, & 1\leqq i\leqq n,\\[2ex] \dfrac{1}{2q_1}(\beta_{i-n}'-p_1\beta_{i-n}), & n+1\leqq i\leqq 2n,\\[2ex] \dfrac{k}{(k+1)q_k}(\beta_{i-n}'-p_k\beta_{i-n}+\dfrac{k-2}{k-1}q_{k-1}\beta_{i-2n}), & \\ \qquad\qquad kn+1\leqq i\leqq (k+1)n, 2\leqq k\leqq l,\end{cases}$$

and

$$\rho_i(\tau)=\int\Big(\frac{1}{2}p_k(\tau)+\beta_i^2(\tau)\Big)d\tau, \qquad (k-1)n+1\leqq i\leqq kn, 1\leqq k\leqq l. \tag{15}$$

(ii) *If $q_l(t) = 0$ for all $t \in I$, then $r_1(t), \dots, r_{ln}(t)$ defined in* (14) *are orthonormal C^∞-vectors of $\mathbb{R}^m$ and satisfies the equations*

$$(16) \quad r'_{(l-1)n+i}(t) = \begin{cases} 0, & \text{if } l = 1, \\ -q_{l-1}(t) r_{(l-2)n+i}(t), & \text{if } l \geqq 2, \end{cases} \qquad 1 \leqq i \leqq n,$$

for all $t \in I$. Add arbitrary C^∞-vectors $r_{ln+1}(t), \dots, r_m(t)$ such that $\{r_1(t), \dots, r_m(t)\}$ forms an orthonormal basis of $\mathbb{R}^m$ for each $t \in I$, if necessary. We take the change of variables $(t, x) \mapsto (\tau, \xi)$ defined in (1). *Then there exists a C^∞-function $\psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m)$ on $D \cap (I \times \mathbb{R}^m)$ such that*

$$\begin{aligned}(17)\ \log \varphi(\tau, \xi) \\ &= \sum_{k=1}^{l-1} \Big(\sum_{i=(k-1)n+1}^{kn} \frac{1}{4} p_k(\tau) \xi_i^2 + \frac{1}{2k} q_k(\tau) \xi_i \xi_{i+n} + \beta_i(\tau) \xi_i + \rho_i(\tau) \Big) \\ &\quad + \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau) \xi_i^2 + \beta_i(\tau) \xi_i + \rho_i(\tau) \Big) \\ &\quad + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m),\end{aligned}$$

and

$$(18)\ \frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - |\nabla_\xi \psi_{l+1}|^2 + \sum_{j,k=ln+1}^{m} (r'_k(\tau) \cdot r_j(\tau)) \xi_j \frac{\partial \psi_{l+1}}{\partial \xi_k} = 0,$$

where β_i and ρ_i, $1 \leqq i \leqq ln$ are defined in (i).

Proof. We shall show the lemma by induction.

First we shall deal with the case of $l = 1$. By (4) and Corollary 4,

$$a_i(t) \cdot a_j(t) = \nabla f_i(t, x) \cdot \nabla f_j(t, x) = \delta_{ij} f'_0(t) > 0, \quad 1 \leqq i \leqq n,$$

which shows that $\{r_1(t), \dots, r_n(t)\}$ is an orthonormal system of $\mathbb{R}^m$ for each t. Let $r_{n+1}(t), \dots, r_m(t)$ be $m - n$ orthonormal C^∞-vectors such that $\{r_1(t), \dots, r_m(t)\}$ is an orthonormal basis of $\mathbb{R}^m$. By the chain rule,

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial t} \frac{\partial}{\partial \xi_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} r'_j(\tau) \cdot x \frac{\partial}{\partial \xi_j} \\ &= \frac{\partial}{\partial \tau} + \sum_{j,k=1}^{m} (r'_j(\tau) \cdot r_k(\tau)) \xi_k \frac{\partial}{\partial \xi_j}, \\ \frac{\partial}{\partial x_i} &= \frac{\partial \tau}{\partial x_i} \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{m} r_{ji}(\tau) \frac{\partial}{\partial \xi_j},\end{aligned}$$

where $r_i(\tau) = (r_{i1}(\tau), \ldots, r_{im}(\tau))$, $1 \leqq i \leqq m$. Since $r_1(\tau), \ldots, r_m(\tau)$ is orthonormal, we have

$$\Delta_x = \Delta_\xi,$$
$$\nabla_x u \cdot \nabla_x v = \nabla_\xi u \cdot \nabla_\xi v.$$

Since (f, φ) is a caloric morphism, Theorem 1 (2) and Proposition 6 imply

$$2\nabla \log \varphi \cdot \nabla f_i = \frac{\partial f_i}{\partial t}, \quad 1 \leqq i \leqq n. \tag{19}$$

By (11) we have

$$f_i(\tau, \xi) = \sqrt{f_0'(\tau)}\xi_i + b_i(\tau) \tag{20}$$

and hence

$$Hf_i = \frac{\partial f_i}{\partial t} = \frac{f_0''(\tau)}{2\sqrt{f_0'(\tau)}}\xi_i + \sqrt{f_0'(\tau)}\sum_{j=1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_j + b_i'(\tau).$$

Then (19) becomes

$$\frac{\partial \log \varphi}{\partial \xi_i} = \frac{1}{2}p_1(\tau)\xi_i + \frac{1}{2}\sum_{j=1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_j + \beta_i(\tau). \tag{21}$$

Hence we have

$$r_i'(\tau) \cdot r_j(\tau) = r_i(\tau) \cdot r_j'(\tau), \quad 1 \leqq i, j \leqq n, \tag{22}$$

because $(\partial/\partial\xi_j)(\partial \log \varphi/\partial \xi_i) = r_i'(\tau) \cdot r_j(\tau)$. On the other hand, $r_i(\tau) \cdot r_j(\tau) = \delta_{ij}$ implies

$$r_i'(\tau) \cdot r_j(\tau) = -r_i(\tau) \cdot r_j'(\tau), \quad 1 \leqq i, j \leqq m. \tag{23}$$

Therefore

$$r_i'(\tau) \cdot r_j(\tau) = 0, \quad 1 \leqq i, j \leqq n. \tag{24}$$

Then by (21) and (24),

$$\psi_2 = \log \varphi - \sum_{i=1}^{n} \Big(\frac{1}{4}p_1(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big)$$

is a C^∞-function of $\tau, \xi_{n+1}, \ldots, \xi_m$. Thus we have

$$\begin{aligned}(25)\quad &\log\varphi(\tau,\xi)\\ &= \sum_{i=1}^{n}\Big(\frac{1}{4}p_1(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big)\\ &\quad + \psi_2(\tau,\xi_{n+1},\ldots,\xi_m).\end{aligned}$$

On the other hand, $\psi_1 := \log\varphi$ satisfies

$$\frac{\partial\psi_1}{\partial t} - \Delta\psi_1 - |\nabla\psi_1|^2 = 0$$

because φ is a positive caloric function. In the coordinate $(\tau, \xi_1, \ldots, \xi_m)$, the above equation is

$$(26)\qquad \frac{\partial\psi_1}{\partial\tau} + \sum_{j,k=1}^{m}(r_j'(\tau)\cdot r_k(\tau))\xi_k\frac{\partial\psi_1}{\partial\xi_j} - \Delta_\xi\psi_1 - |\nabla_\xi\psi_1|^2 = 0.$$

Then from (25), we have

$$\frac{\partial\psi_1}{\partial\tau} = \sum_{i=1}^{n}\Big(\frac{1}{4}p_1'(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))'\xi_i\xi_j + \beta_i'(\tau)\xi_i + \rho_i'(\tau)\Big) + \frac{\partial\psi_2}{\partial\tau},$$

$$\frac{\partial\psi_1}{\partial\xi_k} = \begin{cases} \dfrac{1}{2}p_1(\tau)\xi_k + \dfrac{1}{2}\displaystyle\sum_{j=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), & 1 \leqq k \leqq n,\\ \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}(r_i'(\tau)\cdot r_k(\tau))\xi_i + \dfrac{\partial\psi_2}{\partial\xi_k}, & n+1 \leqq k \leqq m,\end{cases}$$

$$\Delta_\xi\psi_1 = \frac{n}{2}p_1(\tau) + \Delta_\xi\psi_2.$$

Substituting these into (26) and comparing the coefficients with respect to $\xi_1, \ldots, \xi_n$, we obtain the following:

$$(27)\quad \frac{1}{4}(p_1'(\tau) - p_1^2(\tau))\delta_{ij} - \frac{3}{4}\sum_{k=n+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_j'(\tau)\cdot r_k(\tau)) = 0, \qquad 1 \leqq i, j \leqq n,$$

$$(28)\quad \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))'\xi_j + (\beta_i'(\tau) - p_1(\tau)\beta_i(\tau))$$

$$- 2 \sum_{k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau)) \frac{\partial \psi_2}{\partial \xi_k}$$
$$+ \frac{1}{2} \sum_{j,k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau))(r_k'(\tau) \cdot r_j(\tau))\xi_j = 0, \quad 1 \leqq i \leqq n,$$

and

$$\frac{\partial \psi_2}{\partial \tau} - \Delta_\xi \psi_2 - \sum_{k=n+1}^{m} \frac{\partial \psi_2}{\partial \xi_k}\Big(\frac{\partial \psi_2}{\partial \xi_k} - \sum_{j=n+1}^{m} (r_k'(\tau) \cdot r_j(\tau))\xi_j\Big) + \frac{1}{4}\sum_{i=1}^{n} \sum_{j,k=n+1}^{m} (r_i'(\tau) \cdot r_j(\tau))(r_i'(\tau) \cdot r_k(\tau))\xi_j\xi_k = 0. \tag{29}$$

Since $r_i'(\tau) \cdot r_j(\tau) = 0$, $1 \leqq i, j \leqq n$, $r_i'(\tau) = \sum_{k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau)) r_k(\tau)$ for $1 \leqq i \leqq n$. Hence (27) gives

$$r_i'(\tau) \cdot r_j'(\tau) = q_1(\tau)^2 \delta_{ij}, \quad 1 \leqq i, j \leqq n. \tag{30}$$

(Note that $q_1(\tau)^2 = |r_i'(\tau)|^2 \geqq 0$.)

If $q_1 \neq 0$ on an open interval I, then (24) and (30) show that $r_1(\tau), \dots, r_n(\tau), r_1'(\tau), \dots, r_n'(\tau)$ are linearly independent for all $\tau \in I$. Therefore $m \geqq 2n$. Putting

$$r_{i+n}(\tau) = \frac{r_i'(\tau)}{q_1(\tau)}, \quad 1 \leqq i \leqq n,$$

we have an orthonormal system $\{r_1(\tau), \dots, r_{2n}(\tau)\}$ of $\mathbb{R}^m$. Adding $m - 2n$ C^∞-vectors $r_{2n+1}(\tau), \dots, r_m(\tau)$ if $m \geqq 2n+1$, we obtain an orthonormal basis $\{r_1(\tau), \dots, r_m(\tau)\}$ of $\mathbb{R}^m$. Then

$$r_i'(\tau) \cdot r_j(\tau) = q_1(\tau) r_{i+n}(\tau) \cdot r_j(\tau) = q_1(\tau)\delta_{i+n,\,j}, \quad 1 \leqq i \leqq n,\ n+1 \leqq j \leqq m.$$

By (25), (28) and (29)

$$\log \varphi(\tau, \xi) = \sum_{i=1}^{n} \Big(\frac{1}{4} p_1(\tau)\xi_i^2 + \frac{1}{2} q_1(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_2(\tau, \xi_{n+1}, \dots, \xi_m),$$

$$\frac{1}{2}q_1'(\tau)\xi_{i+n} + \beta_i'(\tau) - p_1(\tau)\beta_i(\tau) - 2q_1(\tau)\frac{\partial\psi_2}{\partial\xi_{i+n}} + \frac{1}{2}q_1(\tau)\sum_{j=n+1}^{m}(r_{i+n}'(\tau)\cdot r_j(\tau))\xi_j = 0, \qquad 1 \leqq i \leqq n,$$

and

$$\frac{\partial\psi_2}{\partial\tau} - \Delta_\xi\psi_2 - \sum_{k=n+1}^{m}\frac{\partial\psi_2}{\partial\xi_k}\Big(\frac{\partial\psi_2}{\partial\xi_k} - \sum_{j=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\Big) + \frac{1}{4}q_1(\tau)^2\sum_{i=1}^{n}\xi_{i+n}^2 = 0.$$

If $q_1(\tau) = 0$ for all $\tau \in I$, then by (30), $r_i' = 0$, $1 \leqq i \leqq n$ on I so that

$$\log\varphi(\tau,\xi) = \sum_{i=1}^{n}\Big(\frac{1}{4}p_1(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_2(\tau,\xi_{n+1},\dots,\xi_m),$$

and

$$\frac{\partial\psi_2}{\partial\tau} - \Delta_\xi\psi_2 - |\nabla_\xi\psi_2|^2 + \sum_{j,k=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\frac{\partial\psi_2}{\partial\xi_k} = 0.$$

Thus the assertion in the case of $l = 1$ is shown.

Assume $l \geqq 2$ and that the assertion for $1,\dots,l-1$ holds. Suppose that $q_1 \neq 0,\dots,q_{l-1} \neq 0$ on some open interval I. Then q_l is defined on I and $r_1(\tau),\dots,r_{ln}(\tau)$ defined in (14) are orthonormal C^∞-vectors on $\mathbb{R}^m$. By the assumption on $1,\dots,l-1$, there exists a C^∞-function $\psi_l(\tau,\xi_{(l-1)n+1},\dots,\xi_m)$ such that

$$\begin{aligned}(31)\quad &\log\varphi(\tau,\xi)\\ &= \sum_{k=1}^{l-1}\Big(\sum_{i=(k-1)n+1}^{kn}\frac{1}{4}p_k(\tau)\xi_i^2 + \frac{1}{2k}q_k(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big)\\ &\quad + \psi_l(\tau,\xi_{(l-1)n+1},\dots,\xi_m),\end{aligned}$$

$$(32)\qquad \frac{\partial\psi_l}{\partial\xi_i} = \frac{1}{2}p_l(\tau)\xi_i + \frac{1}{2l}\sum_{j=(l-1)n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_j + \beta_i(\tau), \qquad (l-1)n+1 \leqq i \leqq ln,$$

and

$$(33)\quad \frac{\partial\psi_l}{\partial\tau} - \Delta_\xi\psi_l - \sum_{k=(l-1)n+1}^{m} \frac{\partial\psi_l}{\partial\xi_k}\Big(\frac{\partial\psi_l}{\partial\xi_k} - \sum_{j=(l-1)n+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j\Big)$$
$$+ \sum_{i=(l-1)n+1}^{ln} \Big(\frac{2l-3}{4(l-1)^2} q_{l-1}(\tau)^2\xi_i^2 + \frac{l-2}{l-1} q_{l-1}(\tau)\beta_{i-n}(\tau)\xi_i\Big) = 0.$$

By (23) and (32)

$$(34)\qquad r_i'(\tau)\cdot r_j(\tau) = 0, \quad (l-1)n+1 \leqq i,\, j \leqq ln$$

for $\tau \in I$. Put

$$(35)\quad \psi_{l+1}$$
$$= \psi_l - \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4}p_l(\tau)\xi_i^2 - \frac{1}{2l}\sum_{j=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big).$$

Then ψ_{l+1} is a C^∞-function of $\tau, \xi_{ln+1}, \ldots, \xi_m$ (in the case of $m = ln$, we have $(1/2l)\sum_{j=ln+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_j = 0$ and ψ_{l+1} depends only on τ). From (35) follow

$$\frac{\partial\psi_l}{\partial\tau} = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4}p_l'(\tau)\xi_i^2 + \frac{1}{2l}\sum_{j=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))'\xi_i\xi_j + \beta_i'(\tau)\xi_i + \rho_i'(\tau)\Big)$$
$$+ \frac{\partial\psi_{l+1}}{\partial\tau},$$

$$\frac{\partial\psi_l}{\partial\xi_k} = \begin{cases} \dfrac{1}{2}p_l(\tau)\xi_k + \dfrac{1}{2l}\displaystyle\sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), \\ \qquad\qquad\qquad\qquad (l-1)n+1 \leqq k \leqq ln, \\ \dfrac{1}{2l}\displaystyle\sum_{i=(l-1)n+1}^{ln}(r_i'(\tau)\cdot r_k(\tau))\xi_i + \dfrac{\partial\psi_{l+1}}{\partial\xi_k}, \quad ln+1 \leqq k \leqq m, \end{cases}$$

$$\frac{\partial\psi_l}{\partial\xi_k} - \sum_{j=(l-1)n+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j$$
$$= \begin{cases} \dfrac{1}{2}p_l(\tau)\xi_k - \dfrac{2l-1}{2l}\displaystyle\sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), \\ \hfill (l-1)n+1 \leqq k \leqq ln, \\ \dfrac{2l+1}{2l}\displaystyle\sum_{i=(l-1)n+1}^{ln}(r_i'(\tau)\cdot r_k(\tau))\xi_i - \displaystyle\sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \dfrac{\partial\psi_{l+1}}{\partial\xi_k}, \\ \hfill ln+1 \leqq k \leqq m, \end{cases}$$

and

$$\Delta_\xi\psi_l = \frac{n}{2}p_l(\tau) + \Delta_\xi\psi_{l+1}.$$

Substituting these into (33) and comparing the coefficients with respect to $\xi_{(l-1)n+1}, \ldots, \xi_{ln}$, we obtain the following:

$$\begin{aligned}(36)\quad & \frac{1}{4}\Big(p_l'(\tau) - p_l(\tau)^2 + \frac{2l-3}{(l-1)^2}q_{l-1}(\tau)^2\Big)\delta_{ij} \\ & \quad - \frac{2l+1}{4l^2}\sum_{k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_j'(\tau)\cdot r_k(\tau)) = 0, \\ & \hfill (l-1)n+1 \leqq i,j \leqq ln,\end{aligned}$$

$$\begin{aligned}(37)\quad & \frac{l+1}{l}\sum_{k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))\frac{\partial\psi_{l+1}}{\partial\xi_k} \\ & \quad = \frac{1}{2l}\sum_{j=ln+1}^{m}\{(r_i'(\tau)\cdot r_j(\tau))' + (l-1)p_l(\tau)(r_i(\tau)'\cdot r_j(\tau))\}\xi_j \\ & \quad\quad + \frac{1}{2l}\sum_{j,k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_k'(\tau)\cdot r_j(\tau))\xi_j \\ & \quad\quad + \beta_i'(\tau) - p_l(\tau)\beta_i(\tau) + \frac{l-2}{l-1}q_{l-1}(\tau)\beta_{i-n}(\tau), \\ & \hfill (l-1)n+1 \leqq i \leqq ln,\end{aligned}$$

and

$$\frac{\partial\psi_{l+1}}{\partial\tau} - \Delta_\xi \psi_{l+1} - \sum_{k=ln+1}^{m} \frac{\partial\psi_{l+1}}{\partial\xi_k}\Big(\frac{\partial\psi_{l+1}}{\partial\xi_k} - \sum_{j=ln+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j\Big)$$
$$+ \frac{2l-1}{4l^2} \sum_{i=(l-1)n+1}^{ln} \sum_{j,k=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))(r_i'(\tau)\cdot r_k(\tau))\xi_j\xi_k$$
$$+ \frac{l-1}{l} \sum_{i=(l-1)n+1}^{ln} \sum_{j=ln+1}^{m} \beta_i(\tau)(r_i'(\tau)\cdot r_j(\tau))\xi_j = 0. \tag{38}$$

Let $P_l = P_l(\tau)$ be the orthogonal projection of $\mathbb{R}^m$ to the orthogonal complement of the subspace generated by $\{r_1(\tau), \dots, r_{ln}(\tau)\}$. By (36) and (13), we have

$$P_l r_i' \cdot P_l r_j' = q_l^2 \delta_{ij}, \quad (l-1)n+1 \leqq i, j \leqq ln. \tag{39}$$

We shall show that

$$P_l r_i' = r_i' + q_{l-1} r_{i-n}, \quad (l-1)n+1 \leqq i \leqq ln. \tag{40}$$

By recalling the definition of P_l, (34) implies

$$P_l r_i' = r_i' - \sum_{j=1}^{(l-1)n} (r_i' \cdot r_j) r_j.$$

If $1 \leqq j \leqq (l-1)n$, then by (14),

$$r_j' = \begin{cases} q_1 r_{j+n}, & 1 \leqq j \leqq n, \\ q_k r_{j+n} - q_{k-1} r_{j-n}, & (k-1)n+1 \leqq j \leqq kn,\ 2 \leqq k \leqq l-1, \end{cases}$$

and so

$$r_i' \cdot r_j = -r_i \cdot r_j' = -q_{l-1}\delta_{i,\,j+n}, \qquad (l-1)n+1 \leqq i \leqq ln,\ 1 \leqq j \leqq (l-1)n. \tag{41}$$

Thus (40) holds.

If $q_l(t) \neq 0$ for all $t \in I$, then (39) and (41) imply that $r_1(\tau), \dots, r_{(l+1)n}(\tau)$ defined in (14) are orthonormal C^∞-vectors of $\mathbb{R}^m$ on I where

$$r_{i+n}(\tau) = \frac{1}{q_l(\tau)}(r_i'(\tau) + q_{l-1}(\tau) r_{i-n}(\tau)), \quad (l-1)n+1 \leqq i \leqq ln.$$

In the case of $m > (l+1)n$, we choose arbitrary C^∞-vectors $r_{(l+1)n+1}(\tau), \dots,$ $r_m(\tau)$ such that $\{r_1(\tau), \dots, r_m(\tau)\}$ forms an orthonormal basis of $\mathbb{R}^m$ for each $t \in I$. Then we have

$$r_i'(\tau) \cdot r_j(\tau) = q_l(\tau)\delta_{i+n,\,j} \quad (l-1)n+1 \leqq i \leqq ln,\ ln+1 \leqq j \leqq m.$$

From (35) follows

$$\begin{aligned}
&\psi_l(\tau, \xi_{(l-1)n+1,\dots,}\xi_m) \\
&\qquad = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4}p_l(\tau)\xi_i^2 - \frac{1}{2l}q_l(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\
&\qquad\qquad + \psi_{l+l}(\tau, \xi_{ln+1}, \dots, \xi_m),
\end{aligned}$$

which implies

$$\begin{aligned}
&\log\varphi(\tau,\xi) \\
&\quad = \sum_{k=1}^{l} \sum_{i=(k-1)n+1}^{kn} \Big(\frac{1}{4}p_k(\tau)\xi_i^2 + \frac{1}{2k}q_k(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\
&\qquad\qquad + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m).
\end{aligned}$$

From (37) and (38) follow

$$\begin{aligned}
\frac{\partial\psi_{l+1}}{\partial\xi_i} &= \frac{1}{2(l+1)}\Big(\frac{q_l'(\tau)}{q_l(\tau)} - (l-1)p_l(\tau)\Big)\xi_i \\
&\quad + \frac{1}{2(l+1)} \sum_{j=(l+1)n+1}^{m} (r_i'(\tau)\cdot r_j(\tau))\xi_j + \beta_i(\tau), \\
&\qquad\qquad\qquad\qquad ln+1 \leqq i \leqq (l+1)n,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial\psi_{l+1}}{\partial\tau} - \Delta_\xi\psi_{l+1} - \sum_{k=ln+1}^{m} \frac{\partial\psi_{l+1}}{\partial\xi_k}\Big(\frac{\partial\psi_{l+1}}{\partial\xi_k} - \sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\Big) \\
&\qquad + \sum_{i=ln+1}^{(l+1)n} \Big(\frac{2l-1}{4l^2}q_l(\tau)^2\xi_i^2 + \frac{l-1}{l}q_l(\tau)\beta_{i-n}(\tau)\xi_i\Big) = 0.
\end{aligned}$$

Assume $q_l(t) = 0$ for all $t \in I$. Then (39) gives

$$P_l r_i' = 0, \quad (l-1)n+1 \leqq i \leqq ln.$$

This and (40) show

$$r_i'(\tau) = -q_{l-1}(\tau) r_{i-n}(\tau), \quad (l-1)n+1 \leqq i \leqq ln.$$

Substituting this into (35), we have

$$\psi_l(\tau, \xi_{(l-1)n+1}, \dots, \xi_m) = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\ + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m),$$

which implies

$$\begin{aligned} &\log \varphi(\tau, \xi) \\ &= \sum_{k=1}^{l-1} \sum_{i=(k-1)n+1}^{kn} \Big(\frac{1}{4} p_k(\tau)\xi_i^2 + \frac{1}{2k} q_k(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\ &\quad + \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m). \end{aligned}$$

From (38) follows

$$\frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - |\nabla_\xi \psi_{l+1}|^2 + \sum_{j,k=ln+1}^{m} (r_k'(\tau) \cdot r_j(\tau))\xi_j \frac{\partial \psi_{l-1}}{\partial \xi_k} = 0.$$

Thus the assertion for l is shown. $\square$

Proof of Theorem 7. For each $t \in D$, there exists a positive integer $l \leqq m/n$ such that $q_l(t) = 0$. In fact, if $q_1(t) \neq 0, \dots, q_k(t) \neq 0$, then by Lemma 9, $(k+1)n \leqq m$.

Assume that $q_1 \neq 0, \dots, q_{l-1} \neq 0$ and $q_l = 0$ on an open interval I. Then by (14) and (16), we obtain n systems of linear differential equations:

$$\text{(42)} \quad \frac{d}{dt} \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix} = \begin{pmatrix} 0 & q_1 & & \mathbf{0} \\ -q_1 & 0 & \ddots & \\ & \ddots & \ddots & q_{l-1} \\ \mathbf{0} & & -q_{l-1} & 0 \end{pmatrix} \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix} \\ =: Q \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix},$$

for $1 \leqq i \leqq n$. Fix arbitrary $t_0 \in I$ and let $S(t) = (s_{jk}(t))_{j,k=1}^{l}$ be the solution of the initial value problem

$$\tag{43} \begin{cases} \dfrac{d}{dt}S(t) = Q(t)S(t), \\ S(t_0) = I_l, \end{cases}$$

where I_l is the (l, l) unit matrix. Then $S(t)$ is an orthogonal matrix for every $t \in I$, because $Q(t)$ is skew symmetric. Then by (42), we have

$$\begin{pmatrix} r_i(t) \\ r_{n+i}(t) \\ \vdots \\ r_{(l-1)n+i}(t) \end{pmatrix} = S(t) \begin{pmatrix} r_i(t_0) \\ r_{n+i}(t_0) \\ \vdots \\ r_{(l-1)n+i}(t_0) \end{pmatrix}, \quad 1 \leqq i \leqq n.$$

This means that $r_1(t), r_2(t), \dots, r_{ln}(t)$ are contained in the ln-dimensional space V spanned by the constant vectors $r_1(t_0), r_2(t_0), \dots, r_{ln}(t_0)$ for every t. Therefore we can choose constant vectors $r_{ln+1}, \dots, r_m$ which are the orthonormal basis of the orthogonal complement of V. Put $x_j = r_j(t_0) \cdot x$, $1 \leqq j \leqq m$ for $x \in \mathbb{R}^m$. Then

$$\tag{44} \xi_{(j-1)n+i} = \sum_{k=1}^{l} s_{jk}(t) x_{(k-1)n+i}, \quad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq l,$$

and if $m \geqq ln + 1$,

$$\xi_j = x_j, \quad ln + 1 \leqq j \leqq m.$$

Then ψ_{l+1} is a C^∞-function of $t, x_{ln+1}, \dots, x_m$ and so the equation (18) reduces to

$$\frac{\partial \psi_{l+1}}{\partial t} - \Delta \psi_{l+1} - |\nabla \psi_{l+1}|^2 = 0.$$

Therefore $\varphi_{l+1}(t, x_{ln+1}, \dots, x_m) = \exp \psi_{l+1}$ is a positive caloric function (in the case of $m = ln$, ψ_{l+1} is equal to a constant). From (20) follows

$$f_i = \sum_{k=1}^{l} \lambda(t) s_{1k}(t) x_{(k-1)n+i} + b_i(t),$$

where $\lambda(t) = \sqrt{f_0'(t)}$. On the other hand, by (17) and (44) we have

$$
\begin{aligned}
\log\varphi
&= \sum_{i=1}^{n}\Big[\sum_{j,k=1}^{l}\frac{1}{4}u_{jk}(t)x_{(j-1)n+i}x_{(k-1)n+i} + \sum_{j=1}^{l}\frac{1}{2}v_{ij}(t)x_{(j-1)n+i} + w_i(t)\Big] \\
&\quad + \psi_{l+1},
\end{aligned}
$$

where

$$u_{ij} = \sum_{k=1}^{l} p_k s_{ki}s_{kj} + \sum_{k=1}^{l-1}\frac{q_k}{k}(s_{ki}s_{k+1,j} + s_{k+1,i}s_{kj}), \quad 1 \leqq i,j \leqq l,$$

and

$$v_{ij} = \sum_{k=1}^{l} 2\beta_{(k-1)n+i}s_{kj}, \quad w_i = \sum_{k=1}^{l}\rho_{(k-1)n+i}, \quad 1 \leqq i \leqq n, 1 \leqq j \leqq l.$$

Put

$$
\begin{aligned}
&(45) \quad g_{i1}(t,x_1,\dots,x_l) = \sum_{j=1}^{l}\lambda(t)s_{1j}(t)x_j + b_i(t), \quad 1 \leqq i \leqq n, \\
&\qquad g_i(t,x_1,\dots,x_l) = (f_0(t), g_{i1}(t,x_1,\dots,x_l)), \quad 1 \leqq i \leqq n, \\
&(46) \quad \varphi_i(t,x_1,\dots,x_l) = \exp\Big[\sum_{j,k=1}^{l}\frac{1}{4}u_{jk}(t)x_jx_k + \sum_{j=1}^{l}\frac{1}{2}v_{ij}(t)x_j + w_i(t)\Big], \\
&\qquad\qquad 1 \leqq i \leqq n.
\end{aligned}
$$

Then

$$
\begin{aligned}
f_i(t,x) &= g_{i1}(t, x_i, x_{n+i}, \dots, x_{(l-1)n+i}), \\
\varphi(t,x) &= \varphi_{l+1}\prod_{i=1}^{n}\varphi_i(t, x_i, x_{n+i}, \dots, x_{(l-1)n+i}).
\end{aligned}
$$

We shall prove that each pair (g_i, φ_i), $1 \leqq i \leqq n$ is a caloric morphism from $I \times \mathbb{R}^l$ to $\mathbb{R}^{1+1}$. By $Hg_{i1} = \partial g_{i1}/\partial t$ and (43), we have

$$
\begin{aligned}
Hg_{i1} &= \sum_{j=1}^{n}(\lambda'(t)s_{1j}(t)x_j + \lambda(t)s'_{1j}(t)x_j) + b'_i(t) \\
&= \sum_{j=1}^{n}(\lambda'(t)s_{1j}(t)x_j + \lambda(t)q_1(t)s_{2j}(t)x_j) + b'_i(t).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
2\nabla \log \varphi_i \cdot \nabla g_{i1} &= \sum_{j,k=1}^{l} \frac{1}{2}\lambda(u_{jk}s_{1k} + u_{kj}s_{1k})x_j + \sum_{j=1}^{l} \lambda v_{ij}s_{1j} \\
&= \sum_{j=1}^{l} \lambda(p_1 s_{1j}x_j + q_1 s_{2j}x_j + 2\beta_i),
\end{aligned}
$$

because $u_{ij} = u_{ji}$ and S is orthogonal. Hence

$$
Hg_{i1} = 2\nabla \log \varphi_i \cdot \nabla g_{i1}, \quad 1 \leqq i \leqq n.
$$

Since $f_0' = \lambda^2$,

$$
\frac{df_0}{dt} = |\nabla g_{i1}|^2.
$$

By the assumption, $\varphi(t,x)$ and φ_{l+1} are caloric functions, φ_{l+1} is independent of $x_1, \dots, x_{ln}$ and

$$
\prod_{i=1}^{n} \varphi_i(t, x_i, x_{n+1}, \dots, x_{(l-1)n+i})
$$

is a caloric function. Hence we have

$$
\sum_{i=1}^{n} (K\varphi_i)(t, x_i, x_{n+i}, \dots, x_{(l-1)n+i}) = 0,
$$

where $K\varphi_i = (1/\varphi_i)H\varphi_i$. We have also $K\varphi_i = (\partial \log \varphi_i/\partial t) - \Delta \log \varphi_i - |\nabla \log \varphi_i|^2$. Comparing the coefficients with respect to x_j, we see that $K\varphi_i$ depends only on t. Therefore

$$
\frac{\partial \log \varphi_i}{\partial t} - \Delta \log \varphi_i - |\nabla \log \varphi_i|^2 = \sum_{j=1}^{l} \left(\rho'_{(j-1)n+i} - \frac{1}{2}u_{jj} - \frac{1}{4}v_{ij}^2\right).
$$

Since

$$
\begin{pmatrix} u_{11} & \dots & u_{1l} \\ \vdots & \ddots & \vdots \\ u_{l1} & \dots & u_{ll} \end{pmatrix} = {}^tS \begin{pmatrix} p_1 & q_1 & & \text{\Large 0} \\ q_1 & p_2 & \ddots & \\ & \ddots & \ddots & \frac{q_{l-1}}{l-1} \\ \text{\Large 0} & & \frac{q_{l-1}}{l-1} & p_l \end{pmatrix} S,
$$

and

$$(v_{i1}, \dots, v_{il}) = 2(\beta_i, \beta_{n+i}, \dots, \beta_{(l-1)n+i})S,$$

we have

$$\sum_{j=1}^{l} \Big(\rho'_{(j-1)n+i} - \frac{1}{2}u_{jj} - \frac{1}{4}v_{ij}^2\Big) = \sum_{j=1}^{l} \Big(\rho'_{(j-1)n+i} - \frac{p_j}{2} - \beta^2_{(j-1)n+i}\Big) = 0$$

by the definition of ρ_j in (15). Therefore each φ_i is a positive caloric function. Thus (g_i, φ_i) is a caloric morphism. By (45) and (46), each (g_i, φ_i) satisfies the assumption of Lemma 8. Therefore there exist a positive integer $k \leqq l$, an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \dots, x_m)$ again and positive caloric functions $h_i = h_i(t, x_{kn+i}, \dots, x_{(l-1)n+i})$, $1 \leqq i \leqq n$ (in the case of $k = l$, $h_1, \dots, h_n$ are positive constants) such that f and φ are of form (1) or (2) with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $0 \leqq i \leqq n$ and γ_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$:

(1)

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_i(t,x) = g_{i1}(t, x_i, \dots, x_{(l-1)n+i}) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i,$$

$$\begin{aligned}\varphi(t,x) &= \varphi_{l+1} \prod_{i=1}^{n} \varphi_i(t, x_i, \dots, x_{(l-1)n+i}) \\ &= \varphi_{l+1} \prod_{i=1}^{n} h_i \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)},\end{aligned}$$

(2)

$$f_0(t) = \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$\begin{aligned}f_i(t,x) &= g_{i1}(t, x_i, \dots, x_{(l-1)n+i}) \\ &= \alpha_1(x_i + \gamma_{i1}t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i,\end{aligned}$$

$$
\begin{aligned}
\varphi(t,x) &= \varphi_{l+1} \prod_{i=1}^{n} \varphi_i(t, x_i, \dots, x_{(l-1)n+i}) \\
&= \varphi_{l+1} \prod_{i=1}^{n} h_i \exp\left[\frac{\gamma_{i1}^2}{4} t + \frac{\gamma_{i1}}{2} x_i\right] \\
&\qquad \times \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)}.
\end{aligned}
$$

Put $h = \varphi_{l+1} h_1 \cdots h_n$. Then $h = h(t, x_{kn+1}, \dots, x_m)$ is a positive caloric function. We obtain the required form of (f, φ) on $D \cap (I \times \mathbb{R}^m)$. Since f_0 is of C^∞, the form of (f, φ) holds on the closure $\bar{I}$ of I, if $\bar{I}$ is contained in the interval where f_0 is defined. Thus (f, φ) has the required form on each open interval where $q_1 > 0, \dots, q_{l-1} > 0$. Fix an open interval I such that $q_1 > 0, \dots, q_{l-2} > 0$. The analyticity of f_0 and (13) implies that q_{l-1} is an analytic function on I. Therefore, the zero-points of q_{l-1} is discrete, which is denoted by $\{\sigma_\nu\}_{\nu=M}^{N}$ (M, N may be $-\infty, \infty$, respectively). For each ν, f_0 is of form

$$
f_0(t) = \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, & t \in (\sigma_{\nu-1}, \sigma_\nu], \\ \displaystyle\sum_{j=1}^{\tilde{k}} \frac{\tilde{\alpha}_j^2}{\tilde{\beta}_j - t} + \tilde{\delta}_0, & t \in [\sigma_\nu, \sigma_{\nu+1}), \end{cases}
$$

in the case of (1). Then $\tilde{k} = k$, $\tilde{\alpha}_j = \alpha_j$, $\tilde{\beta}_j = \beta_j$ and $\tilde{\delta}_0 = \delta_0$, because f_0 is of C^∞. Therefore (f, φ) has the required form on each interval where $q_1 > 0, \dots, q_{l-2} > 0$. In the case of (2), the same argument holds. Consequently, (f, φ) is of a required form on D. This completes the proof of Theorem 7. □

Corollary 10. *Let (f, φ) be the same as in Theorem 7. Then (f, φ) is equal to the composition of a the direct sum of k caloric morphisms of $\mathbb{R}^{n+1}$ and a projection $\mathbb{R}^{m+1} \to \mathbb{R}^{kn+1}$.*

Proof. In the case of (I), we put

$$
g_{j0}(t) = \begin{cases} \dfrac{\alpha_1^2}{\beta_1 - t} + \delta_0, & j = 1, \\ \dfrac{\alpha_j^2}{\beta_j - t}, & j > 1, \end{cases}
$$

$$g_{ji}(t,x_1,\dots,x_n)=\begin{cases}\dfrac{\alpha_1}{\beta_1-t}(x_i+\gamma_{ij})+\delta_i, & j=1,\\ \dfrac{\alpha_j}{\beta_j-t}(x_i+\gamma_{ij}), & j>1,\end{cases}$$

$$\varphi_j(t,x_1,\dots,x_n)=\frac{1}{|\beta_j-t|^{n/2}}\exp\sum_{i=1}^{n}\frac{(x_i+\gamma_{ij})^2}{4(\beta_j-t)},$$

for $1\leqq i\leqq n$ and $1\leqq j\leqq k$. In the case of (II), we put

$$g_{j0}(t)=\begin{cases}\alpha_1^2t+\delta_0, & j=1,\\ \dfrac{\alpha_j^2}{\beta_j-t}, & j>1,\end{cases}$$

$$g_{ji}(t,x_1,\dots,x_n)=\begin{cases}\alpha_1(x_i+\gamma_{i1}t)+\delta_1, & j=1,\\ \dfrac{\alpha_j}{\beta_j-t}(x_i+\gamma_{ij}), & j>1,\end{cases}$$

$$\varphi_j(t,x_1,\dots,x_n)=\begin{cases}\exp\sum_{i=1}^{n}\left[\dfrac{\gamma_{i1}^2}{4}t+\dfrac{\gamma_{i1}}{2}x_i\right], & j=1,\\ \dfrac{1}{|\beta_j-t|^{n/2}}\exp\sum_{i=1}^{n}\dfrac{(x_i+\gamma_{ij})^2}{4(\beta_j-t)}, & j>1,\end{cases}$$

for $1\leqq i\leqq n$ and $1\leqq j\leqq k$. Then each pair $(g_j,\varphi_j)=((g_{j0},\dots,g_{jn}),\varphi_j)$, $1\leqq j\leqq k$ is a caloric morphism. (g_1,φ_1) is defined on $\mathbb{R}^n\setminus\{t\neq\beta_1\}$ in the case of (I) and on $\mathbb{R}^n$ in the case of (I). For $j>1$, (g_j,φ_j) is defined on $\mathbb{R}^n\setminus\{t\neq\beta_j\}$. Let (p,ψ) be the projection $\mathbb{R}^{m+1}\to\mathbb{R}^{kn+1}$ such that $p_0(t)=t$, $p_i(t,x_1,\dots,x_m)=x_i$, $1\leqq i\leqq kn$ and $\psi(t,x_1,\dots,x_m)=h(t,x_{kn+1},\dots,x_m)$. Then (f,φ) is equal to the composition of the direct sum of (g_1,φ_1),...,(g_k,φ_k) and (p,ψ). $\square$

References

[1] P. Appell, *Sur l'équation $\partial^2z/\partial x^2-\partial z/\partial y$ et la théorie de la chaleur*, J. Math. Pures Appl., **8** (1892), 186–216.

[2] B. Fuglede, *Harmonic Morphisms Between Riemannian Manifolds*, Ann. Inst. Fourier(Grenoble), **28** (1978), no. 2, 107–144.

[3] B. Fuglede, *Harmonic Morphisms*, Complex Analysis, Joensuu 1978, Lecture Notes in Math., **747** (1979), Springer-Verlag, Berlin (1979), 123–131.

[4] B. Fuglede, *Harmonic morphisms between semi-riemannian manifolds*, Ann. Acad. Sci. Fenn. Math., **21** (1996), 31–50.

[5] F. W. Gehring and H. Haahti, *The transformations which preserve the harmonic functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **293** (1960), 1–12.

[6] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., **19** (1979), no. 2, 215–229.

[7] H. Leutwiler, *On the Appell Transformation*, Potential Theory (1988), Plenum, New York, 215–222.

[8] K. Shimomura, *Caloric morphisms and a generalization of the Appell transformation*, Proceedings of the Seventh International Colloquium on Differential Equations (1997), VSP Press, Netherlands, 389–394.

[9] F. Trèves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.

Department of Mathematical Sciences
Ibaraki University
Mito, Ibaraki, 310
Japan
shimomur@mito.ipc.ibaraki.ac.jp