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NOTE ON THE PROOF OF THE EXISTENCE OF $D(p)$ OF A THEOREM OF JAROSZ

OSAMU HATORI AND KYOHEI TANABE

Abstract. Tanabe [5] gives a geometrical proof of the existence of $\lim \frac{p(1,t)-1}{t}$ for a natural norm p on $\mathbb{R}^2$. Following his idea, a shorter proof is given.

1. Introduction

Jarosz and Pathak exhibited in [3, Example 8] that for every compact metric space K, a surjective complex-linear isometry on $\mathrm{Lip}(K)$ with the sum norm is of the form of a weighted composition operator. After the publication of [3] some authors expressed their suspicion about the argument there and the validity of the statement there had not been confirmed until quite recently. Applying a theorem of Jarosz [2, Theorem], Hatori and Oi [1] gave a complete proof of Example 8 in [3] which established an affirmative answer to the problem of Rao and Roy [4, p.188]. We point out that the theorem of Jarosz plays a substantial role in due course of the proof in [1]. Recall that the theorem of Jarosz is as follows.

Theorem of Jarosz (Jarosz [2]). *Let X and Y be compact Hausdorff spaces, let A and B be complex linear subspaces of $C(X)$ and $C(Y)$, respectively, and let p and q be natural norms. Assume A and B contain constant functions, and let $\|\cdot\|_A$, $\|\cdot\|_B$ be a p-norm and q-norm on A and B, respectively. Assume next that there is a linear isometry T from $(A, \|\cdot\|_A)$ onto $(B, \|\cdot\|_B)$ with $T1 = 1$. Then if $D(p) = D(q) = 0$, or if A and B are regular subspaces of $C(X)$ and $C(Y)$, respectively, then T is an isometry from $(A, \|\cdot\|_\infty)$ onto $(B, \|\cdot\|_\infty)$.*

We say that a norm p on $\mathbb{R}^2$ is a natural norm if $p(1, 0) = 1$. Jarosz [2] defines a quantity $D(p)$ by

$$D(p) = \lim_{t\to+0} \frac{p(1, t) - 1}{t}.$$

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In the proof of Theorem in [2] he made a use of $D(p)$ without any proof or comments that it does exists. In this short note we prove that $D(p)$ always exists and it is finite.

2. Proof of the existence of $D(p)$

Let $p(x, y)$ denote a norm on $\mathbb{R}^2$ such that $p(1, 0) = 1$. We prove that

$$\exists \lim_{t \to +0} \frac{p(1, t) - 1}{t} \in \mathbb{R}.$$

Put $f(t) = p(1, t)$. Then f is continuous and we have that

$$\begin{aligned} f\left(\frac{\alpha+\beta}{2}\right) &= p\left(1, \frac{\alpha+\beta}{2}\right) = p\left([(1, \alpha) + (1, \beta)]/2\right) \\ &\le p\left(\frac{(1, \alpha)}{2}\right) + p\left(\frac{(1, \beta)}{2}\right) = \frac{1}{2}p(1, \alpha) + \frac{1}{2}p(1, \beta) \\ &= \frac{1}{2}(f(\alpha) + f(\beta)) \quad (1) \end{aligned}$$

for every pair $\alpha, \beta \in \mathbb{R}$; f is convex. Put $F(t) = \frac{p(1,t)-1}{t} = \frac{f(t)-1}{t}$ for $t > 0$. As f is convex we have

$$f(rt) \le rf(t) + (1 - r)f(0), \quad 0 < r < 1$$

for every $t > 0$. As $f(0) = 1$ we have

$$F(rt) = \frac{f(rt) - f(0)}{rt} \le \frac{rf(t) - rf(0)}{rt} = \frac{f(t) - f(0)}{t} = F(t)$$

for every $t > 0$ and $0 < r < 1$. Thus $F(t)$ is increasing for $t > 0$. Hence we have

$$\exists \lim_{t \to +0} F(t) \ge -\infty.$$

We prove that $\lim_{t \to +0} F(t) > -\infty$. Let $x = ay + 1$ be the line through two points $(1, 0)$ and $(m, n) = \left(\frac{1}{p(1,-1)}, \frac{-1}{p(1,-1)}\right)$. Let (x, y) be a point which satisfies that $y \ge 0$ and $x > ay + 1$. We prove that $p(x, y) > 1$. Suppose not; $p(x, y) \le 1$. Let ℓ be a segment between (x, y) and (m, n). Then $p(u, v) \le 1$ for every point $(u, v) \in \ell$ since $p(m, n) = 1$ and $p(x, y) \le 1$. As $n < 0$ and $y \ge 0$, there is an intersection of the line ℓ and the x-axisis. Put the intersection $(k, 0)$. Thus $k = k(p(1, 0)) = p(k, 0) \le 1$. On the other hand, since $x > ay + 1$, we infer that $k > 1$, which is a contradiction. We have proved that

$$p(x, y) > 1 \text{ if } y \ge 0 \text{ and } x > ay + 1. \quad (2)$$

Let ℓ' be the line through $(0, 0)$ and $(1, t)$. The intersection of ℓ' and $x = ay + 1$ is $\left(\frac{1}{1-ta}, \frac{t}{1-ta}\right)$. For every $t > 0$ with $1 - ta > 0$, we have $p\left(\frac{1}{1-ta}, \frac{t}{1-ta}\right) \ge 1$ by (2). By a simple calculation we have

$$F(t) = \frac{p(1,t) - 1}{t} \geq -a$$

for every $t > 0$ with $1 - ta > 0$. Since F is increasing for $t > 0$, we conclude that

$$\lim_{t \to +0} F(t) \geq -a.$$

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(Osamu Hatori) Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan

Email address: `hatori@math.sc.niigata-u.ac.jp`

(Kyohei Tanabe) Graduate School of Science and Technology, Niigata University, Niigata 950–2181, Japan

Email address: `f17a041e@alumni.niigata-u.ac.jp`

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