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NOTES ON KERNELS OF RATIONAL HIGHER DERIVATIONS IN INTEGRALLY CLOSED DOMAINS

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Abstract. Let k be a field of characteristic $p \geq 0$ and $A = k[x_0, x_1, x_2, \ldots]$ the polynomial ring in countably many variables over k. We construct a rational higher k-derivation on A whose kernel is not the kernel of any higher k-derivation on A. This example extends [5, Example 4].

1. Introduction

For an integral domain R with unit, we denote by $R^{[n]}$ the polynomial ring in n variables over R and by $Q(R)$ the field of fractions of R.

Derivations on integral domains and their kernels have been studied by many mathematicians. In particular, we have many significant results on the kernels of derivations in polynomial rings of characteristic zero. See, e.g., [11], [1], [2] for excellent accounts. Recently, higher derivations on polynomial rings and their kernels have been studied by several mathematicians. See, e.g., [9], [6], [7] and their references. Some results on the kernels of derivations in polynomial rings of characteristic zero have been generalized to those on the kernels of higher derivations in polynomial rings of any characteristic.

Let R be an integral domain with unit and let A be an R-domain. Assume that $(R \setminus \{0\})^{-1}A$ is integrally closed. In this note, we study the kernels of some rational higher R-derivations in A. In Section 2, we recall some basic notions on higher derivations and some elementary results on the kernels of higher derivations. Then we generalize some results of [8, §2]. In Section 3, we construct a rational higher k-derivation $\overline{D}$ on the polynomial ring A in countably many variables over a field k such that its kernel $A^{\overline{D}}$ is not the kernel of any higher k-derivation on A. This example extends [5, Example 4].

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2. Kernels of some rational higher derivations

First of all, we recall some notions on higher derivations. For more details, we refer to [6] and [9].

Let R be an integral domain with unit and A an R-domain. Let F be an R-domain that contains A as an R-subalgebra. A set $D = \{D_\ell\}_{\ell \geq 0}$ of R-homomorphisms from A to F is called a *higher R-derivation on A into F* if the following conditions are satisfied:

(1) D_0 is the identity map of A.

(2) For any $a, b \in A$ and for any integer $\ell \geq 0$,

$$D_\ell(ab) = \sum_{i+j=\ell} D_i(a)D_j(b).$$

A higher R-derivation on A into F is called a *higher R-derivation on A* (resp. a *rational higher R-derivation on A*) if $F = A$ (resp. $F = Q(A)$).

For a higher R-derivation $D = \{D_\ell\}_{\ell \geq 0}$ on A into F, we define the kernel A^D of D by $\{a \in A \mid D_\ell(a) = 0 \text{ for any } \ell \geq 1\} = \cap_{\ell \geq 1} \operatorname{Ker} D_\ell$. D is said to be *trivial* if $A^D = A$.

Let $D = \{D_\ell\}_{\ell \geq 0}$ be a rational higher R-derivation on A and let $\varphi_D : A \to Q(A)[[t]]$, where $Q(A)[[t]]$ is the formal power series ring in one variable over $Q(A)$, be the mapping defined by $\varphi_D(a) = \sum_{\ell \geq 0} D_\ell(a)t^\ell$ for $a \in A$. Since D is a rational higher R-derivation on A, φ_D is an R-algebra homomorphism. We call the mapping φ_D the *homomorphism associated to D*.

For a rational higher R-derivation $D = \{D_\ell\}_{\ell \geq 0}$ on A, we have a unique higher $Q(R)$-derivation $\overline{D} = \{\overline{D}_\ell\}_{\ell \geq 0}$ on $Q(A)$ such that $\overline{D}_\ell|_A = D_\ell$ for any $\ell \geq 0$. We call $\overline{D}$ the *extension of D to $Q(A)$*. For more details on the construction of $\overline{D}$, we refer to [6, Section 1]. It is clear that $Q(A^D) \subset Q(A)^{\overline{D}}$.

We note that all the results of [9] remain true if we assume that D (with the notation of [9]) is a rational higher R-derivation.

Lemma 2.1. *Let R be an integral domain, A an R-domain and D a rational higher R-derivation on A. Then the following assertions hold true*:

(1) *A^D is integrally closed in A.*

(2) *$Q(A)/Q(A)^{\overline{D}}$ is a regular filed extension.*

(3) *$Q(A)^{\overline{D}} \cap A = A^D$.*

Proof. (1) See [9, Lemma 2.2].

(2) See [3, Theorem (2.3)].

(3) See [9, Lemma 2.3]. □

Lemma 2.2. *With the same notations and assumptions as in Lemma* 2.1*, assume further that* $(R \setminus \{0\})^{-1}A$ *is integrally closed. Then* $Q(A)^{\overline{D}} = Q(A^D)$ *if and only if* $Q(A)^{\overline{D}}/Q(A^D)$ *is an algebraic field extension.*

Proof. The following argument is the same as in [4]. The "only if" part is clear. We prove the "if" part. Let f be any polynomial in $Q(A)^{\overline{D}}$. Then f satisfies an equation

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n = 0$$

with $a_0, a_1, \ldots a_n \in A^D$ and $a_0 \neq 0$. Then $a_0 f$ is integral over A^D. Since $a_0 f \in Q(A)$ and $(R \setminus \{0\})^{-1}A$ is integrally closed, $a_0 f \in (R \setminus \{0\})^{-1}A$. Then $ba_0 f \in A$ for some $b \in R \setminus \{0\}$ and $ba_0 f$ is integral over A^D. Hence, $f = ba_0 f / ba_0 \in Q(A^D)$. □

The following lemma generalizes [6, Theorem 1.1].

Lemma 2.3. *With the same notations and assumptions as in Lemma* 2.1*, assume further that* $\mathrm{tr.deg}_R A := \mathrm{tr.deg}_{Q(R)} Q(A) < +\infty$ *and* $(R \setminus \{0\})^{-1}A$ *is integrally closed. Let* D *be a rational higher* R*-derivation on* A *and let* $\overline{D}$ *be the extension of* D *to* $Q(A)$*. If* $\mathrm{tr.deg}_R A^D \geq -1 + \mathrm{tr.deg}_R A$*, then* $Q(A^D) = Q(A)^{\overline{D}}$*.*

Proof. The following argument is almost the same as that in [4] and the proof of [6, Theorem 1.1]. Set $t = \mathrm{tr.deg}_R A$, which is a non-negative integer by the assumption on A, and $t' = \mathrm{tr.deg}_{Q(R)} Q(A)^{\overline{D}}$. Since $t - 1 \leq \mathrm{tr.deg}_R A^D \leq t' \leq t$, we have $t' = t - 1, t$. If $t' = t$, then we infer from Lemma 2.1 (2) that $\overline{D}$ is trivial since the field extension $Q(A)/Q(A)^{\overline{D}}$ is algebraic. So D is trivial and hence $Q(A^D) = Q(A) = Q(A)^{\overline{D}}$. If $t' = t - 1$, then the field extension $Q(A)^{\overline{D}}/Q(A^D)$ is algebraic. Hence the assertion follows from Lemma 2.2. □

The following result generalizes [8, Theorem 2.2].

Proposition 2.1. *Let* R *be an integral domain and let* A *be an* R*-domain. Assume that* $Q(A)$ *is a finitely generated field over* $Q(R)$ *and* $(R \setminus \{0\})^{-1}A$ *is integrally closed. Let* B *be an* R*-subalgebra of* A*. We consider the following three conditions.*

(1) *B is integrally closed in A, $Q(B) \cap A = B$ and $Q(A)/Q(B)$ is a separable field extension.*
(2) *There exists a rational higher R-derivation D on A such that $B = A^D$ and $Q(A)^{\overline{D}} = Q(B)$, where $\overline{D}$ is the extension of D to $Q(A)$.*
(3) *There exists a rational higher R-derivation D on A such that $B = A^D$.*

Then (1) $\Longleftrightarrow$ (2)*. Moreover, if* $\mathrm{tr.deg}_R B \geq -1 + \mathrm{tr.deg}_R A$*, then the above three conditions* (1) – (3) *are equivalent to each other.*

Proof. Set $K = Q(R)$. We prove (1) $\Longleftrightarrow$ (2), where the argument is almost the same as that in the proof of [8, Theorem 2.1].

(2) $\Longrightarrow$ (1) By Lemma 2.1 (1), B is integrally closed in A. By Lemma 2.1 (2) , $Q(A)/Q(B)$ is a regular field extension. In particular, the field extension $Q(A)/Q(B)$ is separable. We infer from Lemma 2.1 (3) that $Q(B) \cap A = Q(A)^{\overline{D}} \cap A = A^D = B$.

(1) $\Longrightarrow$ (2) Since $(R \setminus \{0\})^{-1}A$ is integrally closed, $R \subset B$ and B is integrally closed in A, we know that $Q(B)$ is algebraically closed in $Q(A)$. So $Q(A)/Q(B)$ is a regular field extension. Moreover, by the assumption on A, $Q(A)/Q(B)$ is a finitely generated field extension. It then follows from [12, Theorem 1] that there exists a higher K-derivation $\tilde{D} = \{\tilde{D}_\ell\}_{\ell \geq 0}$ on $Q(A)$ such that $Q(A)^{\tilde{D}} = Q(B)$. Set $D_\ell = \tilde{D}_\ell|_A$ for each non-negative integer ℓ and set $D = \{\tilde{D}_\ell\}_{\ell \geq 0}$. Then D is a rational higher R-derivation on A, $B \subset A^D$ and $\overline{D} = \tilde{D}$. Since $A^D = Q(A^D) \cap A \subset Q(A)^{\tilde{D}} \cap A = Q(B) \cap A = B$, we have $B = A^D$. Further, $Q(A^D) = Q(B) = Q(A)^{\tilde{D}} = Q(A)^{\overline{D}}$.

Finally, we prove (2) $\Longleftrightarrow$ (3) provided $\mathrm{tr.deg}_R B \geq -1 + \mathrm{tr.deg}_R A$. By the assumption on A, $\mathrm{tr.deg}_R A < +\infty$. The part "(2) $\Longrightarrow$ (3)" is clear. Let D be the one in (3). Since $\mathrm{tr.deg}_R A^D = \mathrm{tr.deg}_R B \geq -1 + \mathrm{tr.deg}_R A$, we infer from Lemma 2.3 that $Q(A^D) = Q(A)^{\overline{D}}$, where $\overline{D}$ is the extension of D to $Q(A)$. This proves the part "(3) $\Longrightarrow$ (2)". □

Remark 2.1. Let R and A be the same as in Proposition 2.1. The author does not know whether there exists a rational higher R-derivation D on A such that one of the following conditions is satisfied:

(1) The field extension $Q(A)/Q(A^D)$ is not separable.
(2) A^D is not the kernel of any *higher* R-derivation.

When the characteristic of R equals zero and A is a finitely generated R-domain, we have the following result.

Proposition 2.2. *Let R be an integral domain of characteristic zero and A a finitely generated R-domain. Let B be an R-subalgebra of A. We consider the following three conditions.*

(1) *B is integrally closed in A and $Q(B) \cap A = B$.*
(2) *There exists an R-derivation d on A such that $B = \mathrm{Ker}\, d$.*
(3) *There exists an R-derivation d on A such that $B = \mathrm{Ker}\, d$ and $\mathrm{Ker}\, \overline{d} = Q(B)$, where $\overline{d}$ is the extension of d to $Q(A)$.*

Then (1) $\Longleftrightarrow$ (2). *Moreover, if $(R \setminus \{0\})^{-1}A$ is normal, the above three conditions* (1)–(3) *are equivalent to each other.*

Proof. The part "(3) $\Longrightarrow$ (2)" is clear. The part "(1) $\Longleftrightarrow$ (2)" can be proved by using the same argument as in the proof of [8, Theorem 2.2]. (See also [10, Theorem 5.4].) However, for the reader's convenience, we reproduce the proof. We set $K = Q(R)$, $B_K = (R \setminus \{0\})^{-1}B$ and $A_K = (R \setminus \{0\})^{-1}A$.

(2) $\Longrightarrow$ (1) Suppose that $f \in A \setminus B$ is integral over $B = \operatorname{Ker} d$. Then f is integral over B_K and so there exist a positive integer m and $c_1, \ldots, c_m \in B_K$ such that

$$f^m + c_1 f^{m-1} + \cdots + c_{m-1} f + c_m = 0.$$

We may assume that m is minimal. Let $\overline{d}$ be the extension of d to $Q(A)$, which is a K-derivation on $Q(A)$. Since $\overline{d}|_A = d$, $\operatorname{Ker} \overline{d} \cap A = \operatorname{Ker} d = B$. We have

$$0 = \overline{d}(0) = (mf^{m-1} + (m-1)c_1 f^{m-2} + \cdots + c_{m-1})\overline{d}(f).$$

Since $mf^{m-1} + (m-1)c_1 f^{m-2} + \cdots + c_{m-1} \neq 0$ because m is minimal and $\operatorname{char} K = 0$, $\overline{d}(f) = 0$ and so $f \in \operatorname{Ker} \overline{d} \cap A = B$. This is a contradiction. Therefore, B is integrally closed in A. Since $B \subset \operatorname{Ker} \overline{d}$, $\operatorname{Ker} \overline{d}$ is a subfield of $Q(A)$ and $\operatorname{Ker} \overline{d} \cap A = B$, we have $Q(B) \cap A = B$.

(1) $\Longrightarrow$ (2) We may write as $A = R[a_1, \ldots, a_n]$. Then $A_K = K[a_1, \ldots, a_n]$ is a finitely generated K-domain. Since B is integrally closed in A, so is B_K in A_K. Since $Q(B_K) = Q(B)$ and $Q(B) \cap A = B$, we have $Q(B_K) \cap A_K = B_K$. We infer from [10, Theorem 5.4] that there exists a K-derivation δ on A_K such that $\operatorname{Ker} \delta = B_K$. Let b be a nonzero element of $R \setminus \{0\}$ such that $b\delta(a_1), \ldots, b\delta(a_n) \in A$ and set $d = b\delta|_A$. Then d is an R-derivation on A and $\operatorname{Ker} d = \operatorname{Ker} \delta \cap A = B_K \cap A = B$. This proves (2).

Finally, assuming that A_K is normal, we prove "(1) $\Longrightarrow$ (3)". Since A_K is normal, we know that $Q(B_K)$ is algebraically closed in $Q(A_K)$. In particular, $Q(A_K)/Q(B_K)$ is a regular field extension. By [12, Theorem 1] (or [10, Theorem 4.2]), there exists a K-derivation δ on $Q(A)$ such that $\operatorname{Ker} \delta = Q(B)$. Let b be a nonzero element of $A = R[a_1, \ldots, a_n]$ such that $b\delta(a_1), \ldots, b\delta(a_n) \in A$ and set $d = b\delta|_A$. Then d is an R-derivation on A and $\operatorname{Ker} d = \operatorname{Ker} \delta \cap A = Q(B) \cap A = B$. We easily see that $\overline{d} = b\delta$. So $\operatorname{Ker} \overline{d} = \operatorname{Ker} b\delta = \operatorname{Ker} \delta = Q(B)$. $\square$

3. Example of a rational higher derivation on $k[x_0, x_1, x_2, \ldots]$

Let k be a field of characteristic $p \geq 0$ and $A = k[x_0, x_1, x_2, \ldots]$ the polynomial ring in countably many variables over k. We construct a rational higher k-derivation Δ on A whose kernel A^Δ is not the kernel of any higher k-derivation on A.

We claim the following.

Claim 3.1. *There exists a higher k-derivation $\tilde{D} = \{\tilde{D}_\ell\}_{\ell \geq 0}$ on $Q(A)$ such that the following conditions are satisfied:*

(1) $\tilde{D}_1(x_0) = 1$, $\tilde{D}_\ell(x_0) = 0$ *for any* $\ell \geq 1$.
(2) *If* $p \neq 2$, *then* $x_n^2 - 2x_0 \in Q(A)^{\tilde{D}}$ *for any* $n \geq 1$.
(3) *If* $p = 2$, *then* $x_n^3 + x_0 \in Q(A)^{\tilde{D}}$ *for any* $n \geq 1$.

Proof. It suffices to construct a k-algebra homomorphism $\varphi : A \to Q(A)[[t]]$ such that the following conditions are satisfied.

1) $\varphi(f)|_{t=0} = f$ for any $f \in A$.
2) $\varphi(x_0) = x_0 + t$.
3) If $p \neq 2$, then $\varphi(x_n^2 - 2x_0) = x_n^2 - 2x_0$ for any $n \geq 1$.
4) If $p = 2$, then $\varphi(x_n^3 + x_0) = x_n^3 + x_0$ for any $n \geq 1$.

In fact, such a homomorphism φ gives a k-algebra homomorphism $Q(A) \to Q(A)[[t]]$ that defines a higher k-derivation $\tilde{D}$ on $Q(A)$ satisfying the conditions (1)–(3). Since φ is determined by the image of $\{x_0, x_1, x_2, \ldots\}$, it suffices to determine the image of $\{x_1, x_2, \ldots\}$ via φ. For any integer $n \geq 1$, set $\varphi(x_n) = x_n + \sum_{\ell=1}^{\infty} a_\ell t^\ell$, where the constant term of $\varphi(x_n)$ has to be x_n by the condition 1).

Assume that $p \neq 2$. Since φ is a ring homomorphism, $\varphi(x_n^2) = \varphi(x_n)^2$ and so the coefficient of t^ℓ in $\varphi(x_n^2)$ is as follows:

$$\begin{cases} 2(a_\ell x_n + a_1 a_{\ell-1} + \cdots + a_m a_{m+1}) & \text{if } \ell = 2m+1 \quad (m \geq 0), \\ 2(a_\ell x_n + a_1 a_{\ell-1} + \cdots + a_{m-1} a_{m+1}) + a_m^2 & \text{if } \ell = 2m \quad (m \geq 1), \end{cases}$$

where we set $a_0 = 0$. Then we can determine $a_\ell \in k(x_n)$ ($\ell = 1, 2, \ldots$) inductively such that $\varphi(x_n^2 - 2x_0) = x_n^2 - 2x_0$. For example, $a_1 = \frac{1}{x_n}$, $a_2 = -\frac{1}{2x_n^3}$, $a_3 = \frac{1}{2x_n^5}$, etc.

Assume that $p = 2$. By using similar argument as in the previous paragraph, for any integer $n \geq 1$, we obtain $a_\ell \in k(x_n)$ ($\ell = 1, 2, \ldots$) such that $\varphi(x_n) = x_n + \sum_{\ell=1}^{\infty} a_\ell t^\ell$ and $\varphi(x_n^3 + x_0) = x_n^3 + x_0$. For example, $a_1 = \frac{1}{x_n^2}$, $a_2 = \frac{1}{x_n^5}$, $a_3 = \frac{1}{x_n^8}$, $a_4 = a_5 = a_6 = a_7 = 0$, $a_8 = \frac{1}{x_n^{23}}$, etc. $\square$

Claim 3.2. *Let $\tilde{D}$ be a higher k-derivation on $Q(A)$ satisfying the conditions* (1)–(3) *of Claim* 3.1 *and set $B = Q(A)^{\tilde{D}} \cap A$. Then the ring B is not the kernel of any higher k-derivation on A.*

Proof. We consider the case $p \neq 2$. The case $p = 2$ can be treated similarly. Suppose to the contrary that there exists a higher k-derivation $D = \{D_\ell\}_{\ell \geq 0}$ on A such that $B = A^D$.

Since $x_i^2 - 2x_0 \in B$ for any $i \geq 1$, we see that $D_\ell(x_i^2 - 2x_0) = 0$ for any $i, \ell \geq 1$. For any $i \geq 1$, $D_1(x_i^2 - 2x_0) = D_1(x_i^2) - 2D_1(x_0) = 2x_i D_1(x_i) - 2D_1(x_0) = 0$ and so $x_i \mid D_1(x_0)$. Hence $D_1(x_i) = 0$ for any $i \geq 0$. For an integer $j \geq 2$, assume that $D_k(x_i) = 0$ for $k = 1, 2, \ldots, j-1$ and for any $i \geq 0$. Then

$$0 = D_j(x_i^2 - 2x_0) = D_j(x_i^2) - 2D_j(x_0) = \sum_{\ell+\ell'=j} D_\ell(x_i) D_{\ell'}(x_i) - 2D_j(x_0).$$

By the inductive assumption, $\sum_{\ell+\ell'=j} D_\ell(x_i) D_{\ell'}(x_i) = 2x_i D_j(x_i)$. Then $x_i \mid D_j(x_0)$ for any $i \geq 1$. Hence $D_j(x_i) = 0$ for any $i \geq 0$. Therefore, by the induction on j, we know that $x_0, x_1, x_2, \ldots \in A^D = B$. In particular, $B = A$. This is a contradiction because $x_0 \notin Q(A)^{\tilde{D}} \cap A = B$. $\square$

Therefore, $\Delta := \{\tilde{D}_\ell|_A\}_{\ell \geq 0}$ is a rational higher k-derivation on A satisfying the conditions stated in the first paragraph of this section.

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