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SURJECTIVE ISOMETRIES ON A BANACH SPACE OF ANALYTIC FUNCTIONS ON THE OPEN UNIT DISC

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Abstract. Let $\mathcal{S}_A$ be the complex linear space of all analytic functions on the open unit disc $\mathbb{D}$, whose derivative can be extended to the closed unit disc $\bar{\mathbb{D}}$. We give the characterization of surjective, not necessarily linear, isometries on $\mathcal{S}_A$ with respect to the norm $\|f\|_\sigma = |f(0)| + \sup\{|f'(z)| : z \in \mathbb{D}\}$ for $f \in \mathcal{S}_A$.

1. Introduction and main result

A mapping $T\colon M \to N$ between two normed linear spaces $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ is an isometry if and only if it preserves the distance of two points in M, that is,

$$\|T(a) - T(b)\|_N = \|a - b\|_M \qquad (a, b \in M).$$

The Mazur-Ulam theorem [16] states that every surjective isometry T between two normed linear spaces is real linear provided $T(0) = 0$.

We mention the characterization of isometries on several normed linear spaces. Isometries were studied on various spaces by many researchers, as for example in [3, 12, 13, 20, 21]. In 1932, isometries are studied by Banach [1, Theorem 3 in Chapter XI] (see also [23, Theorem 83]). There have been numerous papers on isometries defined on Banach spaces of analytic functions; see [2, 4, 5, 8, 11, 14].

Among the basic problems in analytic function spaces, Novinger and Oberlin, in [19], characterized complex linear isometries on a normed space $\mathcal{S}^p$. The underlying space $\mathcal{S}^p$ is a normed space consisting of analytic functions f on the open unit disc $\mathbb{D}$ whose derivative f' belongs to the classical Hardy space $(H^p(\mathbb{D}), \|\cdot\|_p)$ for $1 \le p < \infty$. They introduced the norm $|f(0)| + \|f'\|_p$ on the normed space $\mathcal{S}^p$.

In this paper we study surjective isometries on the Banach space $\mathcal{S}_A$ of analytic functions f defined on $\mathbb{D}$ whose derivative can be extended to the closed unit disc $\bar{\mathbb{D}}$, and endowed with the norm $\|f\|_\sigma = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|$. We denote by $A(\bar{\mathbb{D}})$

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the disc algebra, that is, the algebra of all analytic functions on $\mathbb{D}$ which can be extended to continuous functions on $\bar{\mathbb{D}}$.

In Section 2, we start by defining an embedding of $\mathcal{S}_A$ into a subspace B consisting of complex valued continuous functions. Then using the Arens-Kelley theorem (see [10, Corollary 2.3.6 and Theorem 2.3.8]), we give a characterization of extreme points of the unit ball B_1^* of the dual space B^* of B. Then we construct some maps to describe extreme points of B_1^* in Section 3.

We used an idea by Ellis for the characterization of surjective real linear isometries on uniform algebras (see [9]). An adjoint operator of a surjective real linear isometry on the dual space B^* preserves extreme points. The action of such adjoint operator on the set of extreme points gives a representation for the isometries on B. We show in Section 4 that the isometries of $\mathcal{S}_A$ are integral operators of weighted differential operators. The main result of this paper is as follows.

Theorem 1. *If $T\colon \mathcal{S}_A \to \mathcal{S}_A$ is a surjective, not necessarily linear, isometry with respect to the norm $\|f\|_\sigma = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|$ for $f \in \mathcal{S}_A$, then there exist constants $c_0, c_1, \lambda \in \mathbb{T}$ and $a \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 f'(\rho(\zeta))\, d\zeta, \qquad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}), \qquad \textit{or}$$

$$T(f)(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\rho(\zeta))\, d\zeta, \qquad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}), \qquad \textit{or}$$

$$T(f)(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\rho(\bar{\zeta}))}\, d\zeta, \qquad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}), \qquad \textit{or}$$

$$T(f)(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} \overline{c_1 f'(\rho(\bar{\zeta}))}\, d\zeta \qquad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

where $\rho(z) = \lambda \dfrac{z-a}{\bar{a}z-1}$ for all $z \in \bar{\mathbb{D}}$.

Conversely, each of the above forms is a surjective isometry on $\mathcal{S}_A$ with the norm $\|\cdot\|_\sigma$, where $T(0)$ is an arbitrary element of $\mathcal{S}_A$.

2. Preliminaries and extreme points

Let $A(\bar{\mathbb{D}})$ be the Banach space of all analytic functions on the open unit disc $\mathbb{D}$ that can be continuously extended to the closed unit disk $\bar{\mathbb{D}}$ with the supremum norm on $\mathbb{D}$. For each $v \in A(\bar{\mathbb{D}})$, v' means the derivative of v on $\mathbb{D}$, that is,

$$v'(z) = \lim_{h \to 0} \frac{v(z+h) - v(z)}{h} \qquad (z \in \mathbb{D}).$$

We define $\mathcal{S}_A$ by the linear space of all analytic functions f on $\mathbb{D}$ whose derivative f' belongs to $A(\bar{\mathbb{D}})$. By [4, Theorem 3.11], we see that $\mathcal{S}_A \subset A(\bar{\mathbb{D}})$. By the definition of $\mathcal{S}_A$, f' is an analytic function on $\mathbb{D}$ which can be extended to a continuous function on $\bar{\mathbb{D}}$. Let $\widehat{v}$ be the unique continuous extension of $v \in A(\bar{\mathbb{D}})$ to $\bar{\mathbb{D}}$; in fact, such an extension is unique since $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$. We define the norm $\|f\|_\sigma$ of $f \in \mathcal{S}_A$ by

$$\|f\|_\sigma = |f(0)| + \|\widehat{f'}\|_\infty \qquad (f \in \mathcal{S}_A), \tag{2.1}$$

where $\|\widehat{f'}\|_\infty = \sup\{|\widehat{f'}(z)| : z \in \bar{\mathbb{D}}\} = \sup\{|f'(z)| : z \in \mathbb{D}\}$. It is routine to check that $(\mathcal{S}_A, \|\cdot\|_\sigma)$ is a complex Banach space. In the rest of this paper, $\mathbb{T}$ denotes the unit circle in the complex number field. We define

$$\widetilde{f}(z, w) = f(0) + \widehat{f'}(z)w \tag{2.2}$$

for $f \in \mathcal{S}_A$ and $(z, w) \in \mathbb{T}^2$. Then the function $\widetilde{f}$ is continuous on $\mathbb{T}^2$ with the product topology. Let $C(\mathbb{T}^2)$ be the Banach space of all continuous complex valued functions on $\mathbb{T}^2$ with respect to the supremum norm $\|\cdot\|_\infty$ on $\mathbb{T}^2$. We set

$$B = \{\widetilde{f} \in C(\mathbb{T}^2) : f \in \mathcal{S}_A\}.$$

Then B is a normed linear subspace of $C(\mathbb{T}^2)$. Let $\mathbf{1} \in \mathcal{S}_A$ be the constant function with $\mathbf{1}(z) = 1$ for $z \in \mathbb{D}$. By (2.2), we see that B has the constant function $\widetilde{\mathbf{1}}$. Notice that B separates points of $\mathbb{T}^2$ in the following sense: for each pair of distinct points $x_1, x_2 \in \mathbb{T}^2$ there exists $\widetilde{f} \in B$ such that $\widetilde{f}(x_1) \neq \widetilde{f}(x_2)$. In fact, let $x_j = (z_j, w_j) \in \mathbb{T}^2$ for $j = 1, 2$ with $x_1 \neq x_2$. Let id be the identity function in $\mathcal{S}_A$. If $w_1 \neq w_2$, then by (2.2), $\mathrm{id} \in \mathcal{S}_A$ satisfies $\widetilde{\mathrm{id}}(x_1) = w_1 \neq w_2 = \widetilde{\mathrm{id}}(x_2)$. If $w_1 = w_2$, then we have $z_1 \neq z_2$. Let $f \in \mathcal{S}_A$ be such that $f(z) = z^2$ for all $z \in \mathbb{D}$. Then $\widetilde{f}(x_1) = 2z_1w_1 \neq 2z_2w_2 = \widetilde{f}(x_2)$ by the assumption. Consequently, $\widetilde{f}(x_1) \neq \widetilde{f}(x_2)$ for some $\widetilde{f} \in B$ as is claimed.

We denote by B^* the complex dual space of $(B, \|\cdot\|_\infty)$. Let $\delta_x \colon B \to \mathbb{C}$ be the point evaluation defined by $\delta_x(\widetilde{f}) = \widetilde{f}(x)$ for $\widetilde{f} \in B$ and $x \in \mathbb{T}^2$. Now we characterize extreme points of the unit ball of the dual space of B.

Proposition 2.1. *The set of all extreme points* $\mathrm{ext}(B_1^*)$ *of the closed unit ball* B_1^* *of the dual space of* B *is* $\{\lambda\delta_x \in B_1^* : \lambda \in \mathbb{T},\, x \in \mathbb{T}^2\}$.

Proof. Let $\mathrm{Ch}(B)$ be the Choquet boundary for $B \subset C(\mathbb{T}^2)$, that is, the set of all $x \in \mathbb{T}^2$ such that δ_x is an extreme point of B_1^*. By the Arens-Kelly theorem (see [10, Corollary 2.3.6 and Theorem 2.3.8]), $\mathrm{ext}(B_1^*) = \{\lambda\delta_x \in B_1^* : \lambda \in \mathbb{T},\ x \in \mathrm{Ch}(B)\}$. We need to show that $\mathrm{Ch}(B) = \mathbb{T}^2$. To this end, we will prove that $\mathbb{T}^2 \subset \mathrm{Ch}(B)$. Let $x_0 = (z_0, w_0) \in \mathbb{T}^2$, and we set $f_0(z) = \overline{z_0 w_0}\, z^2 + \overline{w_0}\, z + 1$ for $z \in \mathbb{D}$. Then $f_0 \in \mathcal{S}_A$

with $\widetilde{f_0}(z,w) = 1 + 2\overline{z_0 w_0}\, zw + \overline{w_0}\, w$ for $(z,w) \in \mathbb{T}^2$. We thus obtain $|\widetilde{f_0}| \le 4$ on $\mathbb{T}^2$. By the equality condition for the triangle inequality, we see that $|\widetilde{f_0}(z,w)| = 4$ if and only if $(z,w) = (z_0, w_0)$. Since $\mathrm{Ch}(B)$ is a boundary for B, the function $\widetilde{f_0}$ attains its maximum modulus on $\mathrm{Ch}(B)$ (see [10, Theorem 2.3.8]). Hence $(z_0, w_0) \in \mathrm{Ch}(B)$, and therefore, $\mathbb{T}^2 \subset \mathrm{Ch}(B)$. Consequently $\mathrm{Ch}(B) = \mathbb{T}^2$ has been proven. $\square$

Let $T\colon (\mathcal{S}_A, \|\cdot\|_\sigma) \to (\mathcal{S}_A, \|\cdot\|_\sigma)$ be a surjective isometry. Define $T_0\colon (\mathcal{S}_A, \|\cdot\|_\sigma) \to (\mathcal{S}_A, \|\cdot\|_\sigma)$ by $T_0 = T - T(0)$. By the Mazur-Ulam theorem, T_0 is a surjective, *real linear* isometry from $(\mathcal{S}_A, \|\cdot\|_\sigma)$ onto itself.

The mapping $U\colon (\mathcal{S}_A, \|\cdot\|_\sigma) \to (B, \|\cdot\|_\infty)$ defined by $U(f) = \widetilde{f}$ for $f \in \mathcal{S}_A$ is a complex linear isometry. Here, $\widetilde{f}$ is defined as in (2.2). In particular, $\widetilde{if} = i\widetilde{f}$ for $f \in \mathcal{S}_A$. We define a mapping $S\colon (B, \|\cdot\|_\infty) \to (B, \|\cdot\|_\infty)$ by $S = UT_0U^{-1}$. Since U is a surjective complex linear isometry from $(\mathcal{S}_A, \|\cdot\|_\sigma)$ onto $(B, \|\cdot\|_\infty)$, it is a bijection, and thus S is a well-defined, surjective real linear isometry on $(B, \|\cdot\|_\infty)$. The equality $SU = UT_0$ is rewritten as follows.

$$S(\widetilde{f}) = \widetilde{T_0(f)} \qquad (f \in \mathcal{S}_A). \tag{2.3}$$

$$\begin{tikzcd}
\mathcal{S}_A \arrow[r, "T_0"] \arrow[d, "U"'] & \mathcal{S}_A \arrow[d, "U"] \\
B \arrow[r, "S"'] & B
\end{tikzcd}$$

Let B^* be the complex dual space of B with the operator norm. We define a mapping $S_*\colon B^* \to B^*$ by

$$S_*(\eta)(\widetilde{f}) = \mathrm{Re}\,\eta(S(\widetilde{f})) - i\,\mathrm{Re}\,\eta(S(i\widetilde{f})) \tag{2.4}$$

for $\eta \in B^*$ and $\widetilde{f} \in B$, where $\mathrm{Re}\, z$ denotes the real part of a complex number z. Here we notice that the mapping S_* was used for the characterization of real linear isometries on uniform algebras by Ellis in [9]. Such techniques are introduced in [22, Proposition 5.17]. The mapping S_* is a surjective real linear isometry with respect to the operator norm on B^* (cf. [17, Proposition 1]). We observe that S_* preserves extreme points of B_1^*.

3. Construction of mappings

In the remainder of this paper, we assume that $S\colon B \to B$ is a surjective real linear isometry defined by (2.3), and $S_*\colon B^* \to B^*$ is a surjective real linear isometry given as in (2.4).

Proposition 3.1. *The set of all extreme points* $\mathrm{ext}(B_1^*)$ *of* B_1^* *with the relative weak*-topology is homeomorphic to* $\mathbb{T}^3$ *with the product topology.*

Proof. We define $V\colon \mathbb{T}\times\mathbb{T}^2 \to \mathrm{ext}(B_1^*)$ by

$$V(\lambda,x)=\lambda\delta_x \qquad ((\lambda,x)\in\mathbb{T}\times\mathbb{T}^2). \tag{3.1}$$

We see that V is a well-defined surjective map by Proposition 2.1. We show that V is a homeomorphism.

If $V(\lambda,x)=V(\mu,y)$, then $\lambda\delta_x=\mu\delta_y$ by the definition of V. By evaluating this equality at $\widetilde{\mathbf{1}}\in B$, we see $\lambda=\lambda\delta_x(\widetilde{\mathbf{1}})=\mu\delta_y(\widetilde{\mathbf{1}})=\mu$, and hence $\lambda=\mu$. As $\lambda\in\mathbb{T}$, we obtain $\delta_x=\delta_y$. Since B separates points of $\mathbb{T}^2$, we have $x=y$ and thus $(\lambda,x)=(\mu,y)$. Consequently, V is injective.

Let $\{(\lambda_n,x_n)\}_n$ be a sequence in $\mathbb{T}\times\mathbb{T}^2$ converging to $(\lambda_0,x_0)\in\mathbb{T}\times\mathbb{T}^2$. For each $\widetilde{f}\in B$, $\widetilde{f}$ is continuous on $\mathbb{T}^2$, and then

$$V(\lambda_n,x_n)(\widetilde{f})=\lambda_n\widetilde{f}(x_n)\to\lambda_0\widetilde{f}(x_0)=V(\lambda_0,x_0)(\widetilde{f})$$

as $n\to\infty$. Therefore $\{V(\lambda_n,x_n)\}_n$ converges to $V(\lambda_0,x_0)$ with respect to the relative weak*-topology on $\mathrm{ext}(B_1^*)$. Hence V is continuous.

The weak*-topology of B^* is a Hausdorff topology, and thus $\mathrm{ext}(B_1^*)$ is a Hausdorff space with the relative weak*-topology. By the compactness of $\mathbb{T}\times\mathbb{T}^2$, we see that V is a homeomorphism. Consequently, $\mathrm{ext}(B_1^*)$ is homeomorphic to $\mathbb{T}^3$, as is claimed. $\square$

Definition 1. Let V be the map defined as in (3.1), and let p_j be the projection from $\mathbb{T}\times\mathbb{T}^2$ onto the j-th coordinate of $\mathbb{T}\times\mathbb{T}^2$ for $j=1,2$. We define maps $\alpha\colon\mathbb{T}\times\mathbb{T}^2\to\mathbb{T}$ and $\Phi\colon\mathbb{T}\times\mathbb{T}^2\to\mathbb{T}^2$ by $\alpha=p_1\circ V^{-1}\circ S_*\circ V$ and $\Phi=p_2\circ V^{-1}\circ S_*\circ V$.

$$\begin{tikzcd}
\mathbb{T}\times\mathbb{T}^2 \arrow[r] \arrow[d, "V"'] & \mathbb{T}\times\mathbb{T}^2 \arrow[d, "V"] \\
\mathrm{ext}(B_1^*) \arrow[r, "S_*"'] & \mathrm{ext}(B_1^*)
\end{tikzcd}$$

Recall that V is a homeomorphism and S_* is a surjective real linear isometry, and thus $S_*(\mathrm{ext}(B_1^*))=\mathrm{ext}(B_1^*)$. Hence α and Φ are both well-defined, surjective continuous functions.

By the definitions of α and Φ, $(V^{-1}\circ S_*\circ V)(\lambda,x)=(\alpha(\lambda,x),\Phi(\lambda,x))$ for $(\lambda,x)\in\mathbb{T}\times\mathbb{T}^2$. Hence $(S_*\circ V)(\lambda,x)=V(\alpha(\lambda,x),\Phi(\lambda,x))$, which shows

$$S_*(\lambda\delta_x)=\alpha(\lambda,x)\delta_{\Phi(\lambda,x)} \tag{3.2}$$

for every $\lambda\in\mathbb{T}$ and $x\in\mathbb{T}^2$. For the sake of simplicity of notation, we denote $\alpha(\lambda,x)$ by $\alpha_\lambda(x)$ for $\lambda\in\mathbb{T}$ and $x\in\mathbb{T}^2$.

Lemma 3.2. *For each $x\in\mathbb{T}^2$, $\alpha_i(x)=i\alpha_1(x)$ or $\alpha_i(x)=-i\alpha_1(x)$.*

Proof. Let $x \in \mathbb{T}^2$ and $\lambda_0 = (1+i)/\sqrt{2} \in \mathbb{T}$. By the definitions of α and Φ, $S_*(\lambda_0 \delta_x) = \alpha_{\lambda_0}(x)\delta_{\Phi(\lambda_0,x)}$. Since S_* is real linear,

$$\begin{aligned}\sqrt{2}\,\alpha_{\lambda_0}(x)\delta_{\Phi(\lambda_0,x)} &= S_*((1+i)\delta_x) = S_*(\delta_x) + S_*(i\delta_x)\\ &= \alpha_1(x)\delta_{\Phi(1,x)} + \alpha_i(x)\delta_{\Phi(i,x)},\end{aligned}$$

and hence $\sqrt{2}\,\alpha_{\lambda_0}(x)\delta_{\Phi(\lambda_0,x)} = \alpha_1(x)\delta_{\Phi(1,x)} + \alpha_i(x)\delta_{\Phi(i,x)}$. By the evaluation of the last equality at $\widetilde{\mathbf{1}} \in B$, $\sqrt{2}\,\alpha_{\lambda_0}(x) = \alpha_1(x) + \alpha_i(x)$. Since $\alpha_\lambda(x) \in \mathbb{T}$ for $\lambda \in \mathbb{T}$, we have $\sqrt{2} = |\alpha_1(x) + \alpha_i(x)| = |1 + \alpha_i(x)\overline{\alpha_1(x)}|$, and thus $\alpha_i(x)\overline{\alpha_1(x)}$ is i or $-i$. Consequently $\alpha_i(x) = i\alpha_1(x)$ or $\alpha_i(x) = -i\alpha_1(x)$ as is claimed. □

Lemma 3.3. *There exists $\varepsilon_0 \in \{\pm 1\}$ such that $S_*(i\delta_x) = i\varepsilon_0\alpha_1(x)\delta_{\Phi(i,x)}$ for every $x \in \mathbb{T}^2$.*

Proof. We need to prove that $\alpha_i(x) = i\alpha_1(x)$ for all $x \in \mathbb{T}^2$, or $\alpha_i(x) = -i\alpha_1(x)$ for all $x \in \mathbb{T}^2$. Define two subsets E_+ and E_- of $\mathbb{T}^2$ by

$$E_+ = \{x \in \mathbb{T}^2 : \alpha_i(x) = i\alpha_1(x)\} \quad \text{and} \quad E_- = \{x \in \mathbb{T}^2 : \alpha_i(x) = -i\alpha_1(x)\}.$$

According to Lemma 3.2, $\mathbb{T}^2 = E_+ \cup E_-$. As $|\alpha_1(x)| = 1$ for $x \in \mathbb{T}^2$, $E_+ \cap E_- = \emptyset$. As noticed in Definition 1, the function α is continuous on $\mathbb{T}^3$. Hence $\alpha_1 = \alpha(1,\cdot)$ and $\alpha_i = \alpha(i,\cdot)$ are continuous on $\mathbb{T}^2$, and thus E_+ and E_- are closed subsets of $\mathbb{T}^2$. Since $\mathbb{T}^2$ is connected, $\mathbb{T}^2 = E_+$ or $\mathbb{T}^2 = E_-$. In other words, $\alpha_i(x) = i\alpha_1(x)$ for every $x \in \mathbb{T}^2$, or $\alpha_i(x) = -i\alpha_1(x)$ for every $x \in \mathbb{T}^2$ as is claimed. □

Lemma 3.4. *For each $\lambda = a + ib \in \mathbb{T}$, $a, b \in \mathbb{R}$, and $x \in \mathbb{T}^2$,*

$$\lambda^{\varepsilon_0}\widetilde{f}(\Phi(\lambda,x)) = a\widetilde{f}(\Phi(1,x)) + ib\varepsilon_0\widetilde{f}(\Phi(i,x)) \tag{3.3}$$

for all $\widetilde{f} \in B$.

Proof. Let $\lambda = a + ib \in \mathbb{T}$ and $x \in \mathbb{T}^2$. Recall that $S_*(\delta_x) = \alpha_1(x)\delta_{\Phi(1,x)}$, and $S_*(i\delta_x) = i\varepsilon_0\alpha_1(x)\delta_{\Phi(i,x)}$ for some $\varepsilon_0 \in \{\pm 1\}$ by Lemma 3.3. Since S_* is real linear,

$$\begin{aligned}\alpha_\lambda(x)\delta_{\Phi(\lambda,x)} &= S_*(\lambda\delta_x) = aS_*(\delta_x) + bS_*(i\delta_x)\\ &= a\alpha_1(x)\delta_{\Phi(1,x)} + ib\varepsilon_0\alpha_1(x)\delta_{\Phi(i,x)},\end{aligned}$$

and therefore,

$$\alpha_\lambda(x)\delta_{\Phi(\lambda,x)} = \alpha_1(x)(a\delta_{\Phi(1,x)} + ib\varepsilon_0\delta_{\Phi(i,x)}). \tag{3.4}$$

Evaluating the above equality at $\widetilde{\mathbf{1}} \in B$, $\alpha_\lambda(x) = (a + ib\varepsilon_0)\alpha_1(x)$. As $\lambda = a + ib \in \mathbb{T}$ and $\varepsilon_0 = 1$ or -1, we can write $a + ib\varepsilon_0 = (a+ib)^{\varepsilon_0} = \lambda^{\varepsilon_0}$, and hence $\alpha_\lambda(x) = \lambda^{\varepsilon_0}\alpha_1(x)$. Note that $\alpha_1(x) \in \mathbb{T}$, and we thus obtain, by (3.4), $\lambda^{\varepsilon_0}\widetilde{f}(\Phi(\lambda,x)) = a\widetilde{f}(\Phi(1,x)) + ib\varepsilon_0\widetilde{f}(\Phi(i,x))$ for all $\widetilde{f} \in B$. □

Definition 2. We define $\phi, \psi\colon \mathbb{T}^2 \to \mathbb{T}$ by $\phi = \pi_1 \circ \Phi$ and $\psi = \pi_2 \circ \Phi$, where $\pi_j\colon \mathbb{T}^2 \to \mathbb{T}$ is the projection to the j-th coordinate of $\mathbb{T}^2$ for $j = 1, 2$. Then $\Phi(\lambda, x) = (\phi(\lambda, x), \psi(\lambda, x))$ for every $\lambda \in \mathbb{T}$ and $x \in \mathbb{T}^2$. For each $\lambda \in \mathbb{T}$, we also denote $\phi_\lambda(x) = \phi(\lambda, x)$ and $\psi_\lambda(x) = \psi(\lambda, x)$ for all $x \in \mathbb{T}^2$. Since Φ is surjective and continuous, we see that both ϕ and ψ are surjective and continuous functions.

Lemma 3.5. *For each $\lambda \in \mathbb{T}$ and $x \in \mathbb{T}^2$, $\phi_\lambda(x) = \phi_1(x)$.*

Proof. Let $x \in \mathbb{T}^2$. First, we show that $\phi_\lambda(x) \in \{\phi_1(x), \phi_i(x)\}$ for all $\lambda \in \mathbb{T} \setminus \{1, i\}$. Suppose, on the contrary, that $\phi_\lambda(x) \notin \{\phi_1(x), \phi_i(x)\}$ for some $\lambda \in \mathbb{T} \setminus \{1, i\}$. Then there exists a polynomial $f \in \mathcal{S}_A$ such that

$$f(0) = 0 \quad \text{and} \quad \widehat{f'}(\phi_\lambda(x)) = 1, \quad \widehat{f'}(\phi_1(x)) = 0 = \widehat{f'}(\phi_i(x));$$

for example, let $z_\mu = \phi_\mu(x)$ for each $\mu \in \mathbb{T}$ and $k = (z_\lambda - z_1)(z_\lambda - z_i)$. Then $k \neq 0$ by our hypothesis. If we define $g(z) = k^{-1}(z - z_1)(z - z_i)$, then $g(z_\lambda) = 1$ and $g(z_1) = 0 = g(z_i)$. Choose a polynomial f so that $f' = g$ and $f(0) = 0$, and then $f \in \mathcal{S}_A$ is a desired function. By Definition 2 with (2.2),

$$\widetilde{f}(\Phi(\mu, x)) = f(0) + \widehat{f'}(\phi_\mu(x))\psi_\mu(x)$$

for $\mu \in \mathbb{T}$. Thus $\widetilde{f}(\Phi(\lambda, x)) = \psi_\lambda(x)$ and $\widetilde{f}(\Phi(1, x)) = 0 = \widetilde{f}(\Phi(i, x))$, which implies $\lambda^{\varepsilon_0}\psi_\lambda(x) = 0$ by (3.3). This leads to a contradiction since $\lambda, \psi_\lambda(x) \in \mathbb{T}$. Consequently, $\phi_\lambda(x) \in \{\phi_1(x), \phi_i(x)\}$ for all $\lambda \in \mathbb{T} \setminus \{1, i\}$, as is claimed. By the liberty of the choice of $x \in \mathbb{T}^2$, we have proven $\phi_\lambda(x) \in \{\phi_1(x), \phi_i(x)\}$ for all $\lambda \in \mathbb{T} \setminus \{1, i\}$ and $x \in \mathbb{T}^2$.

We next prove that $\phi_1(x) = \phi_i(x)$ for all $x \in \mathbb{T}^2$. Let $\lambda \in \mathbb{T} \setminus \{1, i\}$. The mapping $\phi_\lambda\colon \mathbb{T}^2 \to \mathbb{T}$ is continuous as remarked in Definition 2, and thus $\phi_\lambda(\mathbb{T}^2)$ is a connected subset of $\mathbb{T}$. Since $\phi_\lambda(x) \in \{\phi_1(x), \phi_i(x)\}$ for all $x \in \mathbb{T}^2$, we have $\phi_1(x) = \phi_i(x)$ for all $x \in \mathbb{T}^2$, as is claimed. Consequently, we obtain $\phi_\lambda(x) = \phi_1(x)$ for all $\lambda \in \mathbb{T}$ and $x \in \mathbb{T}^2$. □

Lemma 3.6. *Let ψ_1 and ψ_i be functions from Definition 2. There exists $\varepsilon_1 \in \{\pm 1\}$ such that $\psi_i(x) = \varepsilon_1\psi_1(x)$ for all $x \in \mathbb{T}^2$.*

Proof. Let $x \in \mathbb{T}^2$ and $\lambda_0 = (1 + i)/\sqrt{2} \in \mathbb{T}$. According to (3.3)

$$\sqrt{2}\,\lambda_0^{\varepsilon_0}\widetilde{f}(\Phi(\lambda_0, x)) = \widetilde{f}(\Phi(1, x)) + i\varepsilon_0\widetilde{f}(\Phi(i, x)) \tag{3.5}$$

for all $f \in \mathcal{S}_A$. By Lemma 3.5, $\Phi(\lambda, x) = (\phi_1(x), \psi_\lambda(x))$ for every $\lambda \in \mathbb{T}$. Therefore, equality (2.2) becomes

$$\widetilde{f}(\Phi(\lambda, x)) = f(0) + \widehat{f'}(\phi_1(x))\psi_\lambda(x) \tag{3.6}$$

for all $f \in \mathcal{S}_A$ and $\lambda \in \mathbb{T}$. Substitute $f = \mathrm{id} \in \mathcal{S}_A$ into (3.6) to get $\widetilde{\mathrm{id}}(\Phi(\lambda, x)) = \psi_\lambda(x)$ for all $\lambda \in \mathbb{T}$. For $f = \mathrm{id}$, the equality (3.5) reduces to

$$\sqrt{2}\,\lambda_0^{\varepsilon_0} \psi_{\lambda_0}(x) = \psi_1(x) + i\varepsilon_0 \psi_i(x).$$

As $\psi_\lambda(x) \in \mathbb{T}$ for $\lambda \in \mathbb{T}$, $\sqrt{2} = |\psi_1(x) + i\varepsilon_0\psi_i(x)| = |1 + i\varepsilon_0\psi_i(x)\overline{\psi_1(x)}|$. Then we have that $i\varepsilon_0\psi_i(x)\overline{\psi_1(x)}$ is i or $-i$. Thus, for each $x \in \mathbb{T}^2$, $\psi_i(x) = \varepsilon_0\psi_1(x)$ or $\psi_i(x) = -\varepsilon_0\psi_1(x)$. As we remarked in Definition 2, ψ_1 and ψ_i are continuous on the connected set $\mathbb{T}^2$. Hence $\psi_i(x) = \varepsilon_0\psi_1(x)$ for all $x \in \mathbb{T}^2$, or $\psi_i(x) = -\varepsilon_0\psi_1(x)$ for all $x \in \mathbb{T}^2$. □

In the rest of this paper, we denote $a + ib\varepsilon$ by $[a + ib]^\varepsilon$ for $a, b \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$. Thus, for each $\lambda \in \mathbb{C}$, $[\lambda]^\varepsilon = \lambda$ if $\varepsilon = 1$ and $[\lambda]^\varepsilon = \overline{\lambda}$ if $\varepsilon = -1$. Therefore, $[\lambda\mu]^\varepsilon = [\lambda]^\varepsilon[\mu]^\varepsilon$ for all $\lambda, \mu \in \mathbb{C}$. If, in addition, $\lambda \in \mathbb{T}$, then $[\lambda]^\varepsilon = \lambda^\varepsilon$.

Lemma 3.7. *For each $f \in \mathcal{S}_A$ and $x \in \mathbb{T}^2$,*

$$S(\widetilde{f})(x) = [\alpha_1(x)f(0)]^{\varepsilon_0} + [\alpha_1(x)\widehat{f'}(\phi_1(x))\psi_1(x)]^{\varepsilon_0\varepsilon_1}. \tag{3.7}$$

Proof. Let $f \in \mathcal{S}_A$ and $x \in \mathbb{T}^2$. On one hand, by the definition (2.4) of S_*, $\operatorname{Re} S_*(\eta)(\widetilde{f}) = \operatorname{Re} \eta(S(\widetilde{f}))$ for every $\eta \in B^*$. Taking $\eta = \delta_x$ and $\eta = i\delta_x$ into the last equality, we have

$$\operatorname{Re} S_*(\delta_x)(\widetilde{f}) = \operatorname{Re} S(\widetilde{f})(x) \quad \text{and} \quad \operatorname{Re} S_*(i\delta_x)(\widetilde{f}) = -\operatorname{Im} S(\widetilde{f})(x),$$

respectively, and therefore,

$$S(\widetilde{f})(x) = \operatorname{Re} S_*(\delta_x)(\widetilde{f}) - i \operatorname{Re} S_*(i\delta_x)(\widetilde{f}). \tag{3.8}$$

On the other hand, $S_*(\delta_x) = \alpha_1(x)\delta_{\Phi(1,x)}$ and $S_*(i\delta_x) = i\varepsilon_0\alpha_1(x)\delta_{\Phi(i,x)}$ by (3.2) and Lemma 3.3. Substitute these two equalities into (3.8) to obtain

$$S(\widetilde{f})(x) = \operatorname{Re}[\alpha_1(x)\widetilde{f}(\Phi(1,x))] + i \operatorname{Im}[\varepsilon_0\alpha_1(x)\widetilde{f}(\Phi(i,x))].$$

Lemmas 3.5 and 3.6 imply that $\Phi(1,x) = (\phi_1(x), \psi_1(x))$ and $\Phi(i,x) = (\phi_1(x), \varepsilon_1\psi_1(x))$. It follows from (2.2) that

$$\begin{aligned} S(\widetilde{f})(x) &= \operatorname{Re}[\alpha_1(x)\widetilde{f}(\phi_1(x), \psi_1(x))] + i\operatorname{Im}[\varepsilon_0\alpha_1(x)\widetilde{f}(\phi_1(x), \varepsilon_1\psi_1(x))] \\ &= \operatorname{Re}[\alpha_1(x)\{f(0) + \widehat{f'}(\phi_1(x))\psi_1(x)\}] \\ &\qquad + i\varepsilon_0\operatorname{Im}[\alpha_1(x)\{f(0) + \widehat{f'}(\phi_1(x))\varepsilon_1\psi_1(x)\}] \\ &= [\alpha_1(x)f(0)]^{\varepsilon_0} + [\alpha_1(x)\widehat{f'}(\phi_1(x))\psi_1(x)]^{\varepsilon_0\varepsilon_1}. \end{aligned}$$

Hence (3.7) holds for all $f \in \mathcal{S}_A$ and $x \in \mathbb{T}^2$. □

4. Characterization of the surjective isometries on $\mathcal{S}_A$

Lemma 4.1. *For each $z, w \in \mathbb{T}$, $\phi_1(z,w) = \phi_1(z,1)$.*

Proof. To show that $\phi_1(z,w) = \phi_1(z,1)$ for all $z, w \in \mathbb{T}$, suppose not, and then there exist $z_0, w_0 \in \mathbb{T}$ such that $\phi_1(z_0,w_0) \neq \phi_1(z_0,1)$. We set $w_1 = 1$ and $x_j = (z_0, w_j)$ for $j = 0, 1$, and then $\phi_1(x_0) \neq \phi_1(x_1)$. Since the function $\phi_1(z_0,\cdot)\colon \mathbb{T} \to \mathbb{T}$, which maps $w \in \mathbb{T}$ to $\phi_1(z_0, w)$, is continuous, the image $\phi_1(z_0, \mathbb{T})$ is a connected subset of $\mathbb{T}$. Thus, $\phi_1(z_0,\mathbb{T}) \setminus \{\phi_1(x_0), \phi_1(x_1)\}$ is a non-empty set. Then there exists $w_2 \in \mathbb{T}$ such that $\phi_1(z_0, w_2) \notin \{\phi_1(x_0), \phi_1(x_1)\}$. We see that w_0, w_1 and w_2 are mutually distinct. Set $x_2 = (z_0, w_2)$, and then $\phi_1(x_0), \phi_1(x_1)$ and $\phi_1(x_2)$ are mutually distinct. Then we can choose $f_0 \in \mathcal{S}_A$ such that

$$f_0(0) = 0 \quad \text{and} \quad \widehat{f_0'}(\phi_1(x_0)) = 1,\ \widehat{f_0'}(\phi_1(x_1)) = 0 = \widehat{f_0'}(\phi_1(x_2)).$$

Recall $S(\widetilde{f}) = \widetilde{T_0(f)}$ by (2.3), and then equality (3.7) implies

$$T_0(f_0)(0) + \widehat{T_0(f_0)'}(z_0)w_j = [\alpha_1(x_j)f_0(0)]^{\varepsilon_0} + [\alpha_1(x_j)\widehat{f_0'}(\phi_1(x_j))\psi_1(x_j)]^{\varepsilon_0\varepsilon_1}$$

for $j = 0, 1, 2$. By the choice of f_0, we get

$$\begin{aligned} T_0(f_0)(0) + \widehat{T_0(f_0)'}(z_0)w_0 &= [\alpha_1(x_0)\psi_1(x_0)]^{\varepsilon_0\varepsilon_1}, \\ T_0(f_0)(0) + \widehat{T_0(f_0)'}(z_0)w_1 &= 0 = T_0(f_0)(0) + \widehat{T_0(f_0)'}(z_0)w_2. \end{aligned}$$

Since $w_1 \neq w_2$, we deduce $\widehat{T_0(f_0)'}(z_0) = 0$, and thus $T_0(f_0)(0) = 0$. It follows that $[\alpha_1(x_0)\psi_1(x_0)]^{\varepsilon_0\varepsilon_1} = 0$, which contradicts $\alpha_1(x_0), \psi_1(x_0) \in \mathbb{T}$. We thus conclude that $\phi_1(z,w) = \phi_1(z,1)$ for all $z, w \in \mathbb{T}$. $\square$

Lemma 4.2. *There exists a surjective continuous function $\varphi\colon \mathbb{T} \to \mathbb{T}$ such that*

$$T_0(f)(0) + \widehat{T_0(f)'}(z)w = [\alpha_1(x)f(0)]^{\varepsilon_0} + [\alpha_1(x)\widehat{f'}(\varphi(z))\psi_1(x)]^{\varepsilon_0\varepsilon_1} \tag{4.1}$$

for all $f \in \mathcal{S}_A$ and $x = (z,w) \in \mathbb{T}^2$.

Proof. We define the mapping $\varphi\colon \mathbb{T} \to \mathbb{T}$ by

$$\varphi(z) = \phi_1(z,1) \qquad (\forall z \in \mathbb{T}).$$

Since ϕ is continuous, φ is continuous on $\mathbb{T}$. Equality (3.7) yields (4.1) for all $f \in \mathcal{S}_A$ and $x = (z,w) \in \mathbb{T}^2$. We prove that $\varphi\colon \mathbb{T} \to \mathbb{T}$ is surjective. Recall, by Definition 2, that ϕ is surjective. Thus, for each $\zeta \in \mathbb{T}$ there exist $\lambda_1 \in \mathbb{T}$ and $x_1 = (z_1, w_1) \in \mathbb{T}^2$ such that $\zeta = \phi(\lambda_1, x_1) = \phi_{\lambda_1}(z_1, w_1)$. Note that $\phi_{\lambda_1}(z_1,w_1) = \phi_1(z_1,w_1)$ by Lemma 3.5. In addition, Lemma 4.1 shows that $\phi_1(z_1,w_1) = \phi_1(z_1,1) = \varphi(z_1)$, and thus $\zeta = \varphi(z_1)$. This yields that φ is surjective as is claimed. $\square$

Proposition 4.3. *Let $p, q \in \mathbb{C}$. If $|p + \lambda q| = 1$ for all $\lambda \in \mathbb{T}$, then $pq = 0$ and $|p| + |q| = 1$.*

Proof. We show $pq = 0$. Suppose, on the contrary, $p \neq 0$ and $q \neq 0$. Choose $\lambda_1 \in \mathbb{T}$ so that $\lambda_1 q = p|q||p|^{-1}$, and set $\lambda_2 = -\lambda_1$. By hypothesis, $|p + \lambda_1 q| = 1 = |p + \lambda_2 q|$, that is,

$$\left| p + \frac{p|q|}{|p|} \right| = 1 = \left| p - \frac{p|q|}{|p|} \right|.$$

These equalities yield $|p| + |q| = 1 = \big| |p| - |q| \big|$. We may assume $|p| > |q|$, and then we have $|q| = 0$, a contradiction. This implies $pq = 0$, as is claimed. Then $|p| + |q| = 1$ by the initial assumption. $\square$

Lemma 4.4. *There exists $c_0 \in \mathbb{T}$ such that $[\widehat{T_0(\mathbf{1})}(z)]^{\varepsilon_0} = \alpha_1(x) = c_0$ for all $x = (z, w) \in \mathbb{T}^2$.*

Proof. Apply $f = \mathbf{1}, \mathrm{id}$ to (4.1) to get

$$T_0(\mathbf{1})(0) + \widehat{T_0(\mathbf{1})'}(z)w = [\alpha_1(x)]^{\varepsilon_0}, \tag{4.2}$$

$$T_0(\mathrm{id})(0) + \widehat{T_0(\mathrm{id})'}(z)w = [\alpha_1(x)\psi_1(x)]^{\varepsilon_0\varepsilon_1} \tag{4.3}$$

for every $x = (z, w) \in \mathbb{T}^2$. We show that $T_0(\mathbf{1})(0) \neq 0$. Assume that $T_0(\mathbf{1})(0) = 0$, and then $\widehat{T_0(\mathbf{1})'}(z)w = [\alpha_1(x)]^{\varepsilon_0}$. Substitute this equality and (4.3) into (4.1) to have

$$\begin{aligned} &T_0(f)(0) + \widehat{T_0(f)'}(z)w \\ &\qquad = \widehat{T_0(\mathbf{1})'}(z)w[f(0)]^{\varepsilon_0} + \{T_0(\mathrm{id})(0) + \widehat{T_0(\mathrm{id})'}(z)w\}[\widehat{f'}(\varphi(z))]^{\varepsilon_0\varepsilon_1}, \end{aligned}$$

where we have used $[\lambda\mu]^\varepsilon = [\lambda]^\varepsilon[\mu]^\varepsilon$ for $\lambda, \mu \in \mathbb{C}$ and $\varepsilon = 1$ or -1. Since the above equality holds for all $w \in \mathbb{T}$, we obtain

$$T_0(f)(0) = T_0(\mathrm{id})(0)[\widehat{f'}(\varphi(z))]^{\varepsilon_0\varepsilon_1} \tag{4.4}$$

for all $f \in \mathcal{S}_A$ and $z \in \mathbb{T}$. Taking $f = \mathrm{id}^2 \in \mathcal{S}_A$ in (4.4), we get $T_0(\mathrm{id}^2)(0) = 2T_0(\mathrm{id})(0)[\varphi(z)]^{\varepsilon_0\varepsilon_1}$ for all $z \in \mathbb{T}$. By Lemma 4.2, $\varphi\colon \mathbb{T} \to \mathbb{T}$ is surjective, and then we deduce $T_0(\mathrm{id})(0) = 0$. Equality (4.4) implies $T_0(f)(0) = 0$ for all $f \in \mathcal{S}_A$. This is impossible since T_0 is surjective, which shows $T_0(\mathbf{1})(0) \neq 0$, as is claimed.

By equality (4.2) with Proposition 4.3, we see that $\widehat{T_0(\mathbf{1})'}(z) = 0$ for all $z \in \mathbb{T}$. Since $\mathbb{T}$ is a boundary for $A(\bar{\mathbb{D}})$, we have $\widehat{T_0(\mathbf{1})'} = 0$ on $\bar{\mathbb{D}}$. Then there exists a constant $c \in \mathbb{C}$ such that $\widehat{T_0(\mathbf{1})} = c$ on $\bar{\mathbb{D}}$. Substitute $\widehat{T_0(\mathbf{1})'}(z) = 0$ into (4.2) to obtain $c = [\alpha_1(x)]^{\varepsilon_0}$ for all $x \in \mathbb{T}^2$. Thus $c \in \mathbb{T}$, and $\alpha_1(x) = [c]^{\varepsilon_0} = [\widehat{T_0(\mathbf{1})}(z)]^{\varepsilon_0}$ for all $x = (z, w) \in \mathbb{T}^2$. $\square$

By Lemma 4.4, equality (4.1) reduces to

$$T_0(f)(0) + \widehat{T_0(f)'}(z)w = [c_0 f(0)]^{\varepsilon_0} + [c_0\widehat{f'}(\varphi(z))\psi_1(z, w)]^{\varepsilon_0\varepsilon_1} \tag{4.5}$$

for every $f \in \mathcal{S}_A$ and $(z, w) \in \mathbb{T}^2$.

Lemma 4.5. *Let $c_0 \in \mathbb{T}$ be the constant from Lemma 4.4. Then $\widehat{T_0(\mathrm{id})'}(z) = [c_0\psi_1(z,1)]^{\varepsilon_0\varepsilon_1}$ and $\psi_1(z,w) = \psi_1(z,1)w^{\varepsilon_0\varepsilon_1}$ for all $z, w \in \mathbb{T}$.*

Proof. Let $z_0 \in \mathbb{T}$. It follows from (4.5) that

$$T_0(\mathrm{id})(0) + \widehat{T_0(\mathrm{id})'}(z_0)w = [c_0\psi_1(z_0,w)]^{\varepsilon_0\varepsilon_1} \tag{4.6}$$

for every $w \in \mathbb{T}$. Taking the modulus in (4.6), we have $|T_0(\mathrm{id})(0) + \widehat{T_0(\mathrm{id})'}(z_0)w| = 1$ for all $w \in \mathbb{T}$. Proposition 4.3 asserts that $T_0(\mathrm{id})(0) = 0$ or $\widehat{T_0(\mathrm{id})'}(z_0) = 0$. Suppose, on the contrary, that $\widehat{T_0(\mathrm{id})'}(z_0) = 0$. Equality (4.6) shows $T_0(\mathrm{id})(0) = [c_0\psi_1(z_0,w)]^{\varepsilon_0\varepsilon_1}$ for all $w \in \mathbb{T}$. Since T_0 is surjective, there exists $g \in \mathcal{S}_A$ such that $T_0(g)(0) = 0$ and $\widehat{T_0(g)'}(z_0) = 1$. Substitute these two equalities and $T_0(\mathrm{id})(0) = [c_0\psi_1(z_0,w)]^{\varepsilon_0\varepsilon_1}$ into (4.5) to obtain

$$w = T_0(g)(0) + \widehat{T_0(g)'}(z_0)w = [c_0 g(0)]^{\varepsilon_0} + T_0(\mathrm{id})(0)\,[\widehat{g'}(\varphi(z_0))]^{\varepsilon_0\varepsilon_1}$$

for every $w \in \mathbb{T}$. This is impossible since the rightmost hand side of the above equalities is independent of $w \in \mathbb{T}$. Consequently, we have $\widehat{T_0(\mathrm{id})'}(z_0) \neq 0$, and hence $T_0(\mathrm{id})(0) = 0$. By equality (4.6), $\widehat{T_0(\mathrm{id})'}(z_0)w = [c_0\psi_1(z_0,w)]^{\varepsilon_0\varepsilon_1}$ for all $w \in \mathbb{T}$. By the liberty of the choice of $z_0 \in \mathbb{T}$, we get $\widehat{T_0(\mathrm{id})'}(z)w = [c_0\psi_1(z,w)]^{\varepsilon_0\varepsilon_1}$ for all $z, w \in \mathbb{T}$. Taking $w = 1$ in this equality, we obtain $\widehat{T_0(\mathrm{id})'}(z) = [c_0\psi_1(z,1)]^{\varepsilon_0\varepsilon_1}$ for $z \in \mathbb{T}$. It follows that

$$w = \frac{\widehat{T_0(\mathrm{id})'}(z)w}{\widehat{T_0(\mathrm{id})'}(z)} = \frac{[c_0\psi_1(z,w)]^{\varepsilon_0\varepsilon_1}}{[c_0\psi_1(z,1)]^{\varepsilon_0\varepsilon_1}} = \frac{[\psi_1(z,w)]^{\varepsilon_0\varepsilon_1}}{[\psi_1(z,1)]^{\varepsilon_0\varepsilon_1}},$$

and consequently, $\psi_1(z,w) = \psi_1(z,1)w^{\varepsilon_0\varepsilon_1}$ for all $z, w \in \mathbb{T}$. ☐

Proof of Theorem 1. Let $f \in \mathcal{S}_A$ and $z_0 \in \mathbb{T}$. By Lemma 4.5, $\psi_1(z_0,w) = \psi_1(z_0,1)w^{\varepsilon_0\varepsilon_1}$ for all $w \in \mathbb{T}$. Substitute this equality into (4.5) to have

$$T_0(f)(0) + \widehat{T_0(f)'}(z_0)w = [c_0 f(0)]^{\varepsilon_0} + [c_0\widehat{f'}(\varphi(z_0))\psi_1(z_0,1)]^{\varepsilon_0\varepsilon_1}w$$

for all $w \in \mathbb{T}$. The above equality holds for every $w \in \mathbb{T}$, and then

$$T_0(f)(0) = [c_0 f(0)]^{\varepsilon_0} \tag{4.7}$$

and $\widehat{T_0(f)'}(z_0) = [c_0\widehat{f'}(\varphi(z_0))\psi_1(z_0,1)]^{\varepsilon_0\varepsilon_1}$. By the liberty of the choice of $f \in \mathcal{S}_A$ and $z_0 \in \mathbb{T}$, we deduce

$$\widehat{T_0(f)'}(z) = [c_0\widehat{f'}(\varphi(z))\psi_1(z,1)]^{\varepsilon_0\varepsilon_1} \tag{4.8}$$

for all $f \in \mathcal{S}_A$ and $z \in \mathbb{T}$.

For each $v \in A(\bar{\mathbb{D}})$, we define $I(v)$ by

$$I(v)(z) = \int_{[0,z]} v(\zeta)\,d\zeta \qquad (z \in \mathbb{D}),$$

where $[0, z]$ denotes the straight line interval from 0 to z in $\mathbb{D}$. Then $I(v) \in A(\bar{\mathbb{D}})$ satisfying

$$I(v)' = v \qquad \text{on} \quad \mathbb{D}, \tag{4.9}$$

and hence $I(v) \in \mathcal{S}_A$. We set $\widehat{A}(\bar{\mathbb{D}}) = \{\widehat{v} : v \in A(\bar{\mathbb{D}})\}$ and define $W\colon \widehat{A}(\bar{\mathbb{D}}) \to \widehat{A}(\bar{\mathbb{D}})$ by

$$W(\widehat{v})(z) = [\widehat{T_0(I(v))'}([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1} \qquad (v \in A(\bar{\mathbb{D}}),\ z \in \bar{\mathbb{D}}); \tag{4.10}$$

More precisely

$$W(\widehat{v})(z) = \begin{cases} \widehat{T_0(I(v))'}(z) & \text{if } \varepsilon_0\varepsilon_1 = 1, \\ \overline{\widehat{T_0(I(v))'}(\bar{z})} & \text{if } \varepsilon_0\varepsilon_1 = -1 \end{cases}$$

for $\widehat{v} \in \widehat{A}(\bar{\mathbb{D}})$ and $z \in \bar{\mathbb{D}}$. We see that the mapping W is well-defined. Equality (4.8) with $I(v)' = v$ shows that

$$W(\widehat{v})(z) = c_0\widehat{I(v)'}(\varphi([z]^{\varepsilon_0\varepsilon_1}))\psi_1([z]^{\varepsilon_0\varepsilon_1}, 1) = c_0\widehat{v}(\varphi([z]^{\varepsilon_0\varepsilon_1}))\psi_1([z]^{\varepsilon_0\varepsilon_1}, 1)$$

for $\widehat{v} \in \widehat{A}(\bar{\mathbb{D}})$ and $z \in \mathbb{T}$. Since $\mathbb{T}$ is a boundary for $\widehat{A}(\bar{\mathbb{D}})$ and $\varphi(\mathbb{T}) = \mathbb{T}$, we have $\|W(\widehat{v})\|_\infty = \|\widehat{v}\|_\infty$, where $\|\cdot\|_\infty$ denotes the supremum norm on $\bar{\mathbb{D}}$. Thus W is a complex linear isometry on $(\widehat{A}(\bar{\mathbb{D}}), \|\cdot\|_\infty)$.

We show that W is surjective. By the surjectivity of $T_0\colon \mathcal{S}_A \to \mathcal{S}_A$, for each $v_0 \in A(\bar{\mathbb{D}})$ there exists $g \in \mathcal{S}_A$ such that $T_0(g)(z) = [I(v_0)([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1}$ for all $z \in \mathbb{D}$, and hence

$$T_0(g)'(z) = [I(v_0)'([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1} = [v_0([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1} \tag{4.11}$$

for every $z \in \mathbb{D}$. Since $I(g')' = g'$ on $\mathbb{D}$ by (4.9), we see that $I(g') - g$ is constant on $\mathbb{D}$, say $d \in \mathbb{C}$. Equality (4.8) shows $\widehat{T_0(d)'} = 0$ on $\mathbb{T}$. Since $\mathbb{T}$ is a boundary for $\widehat{A}(\bar{\mathbb{D}})$, we see that $\widehat{T_0(d)'} = 0$ on $\bar{\mathbb{D}}$, and hence $T_0(d)' = 0$ on $\mathbb{D}$. Therefore, $T_0(I(g') - g)' = T_0(d)' = 0$ on $\mathbb{D}$. By the real linearity of T_0, $T_0(I(g'))' = T_0(g)'$ on $\mathbb{D}$. Substitute this equality into (4.11) to get $T_0(I(g)')'(z) = [v_0([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1}$ for all $z \in \mathbb{D}$. Thus $\widehat{T_0(I(g'))'}(z) = [\widehat{v_0}([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1}$ for all $z \in \bar{\mathbb{D}}$. Therefore, (4.10) shows that $W(\widehat{g'})(z) = [\widehat{T_0(I(g'))'}([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1} = \widehat{v_0}(z)$ for all $z \in \bar{\mathbb{D}}$, which yields the surjectivity of $W\colon \widehat{A}(\bar{\mathbb{D}}) \to \widehat{A}(\bar{\mathbb{D}})$. Hence W is a surjective complex linear isometry on the uniform algebra $(\widehat{A}(\bar{\mathbb{D}}), \|\cdot\|_\infty)$. By a theorem of deLeeuw, Rudin and Wermer [5, Theorem 3] (see also Nagasawa [18]), there exist an invertible element u of $\widehat{A}(\bar{\mathbb{D}})$ and an algebra automorphism $W_1\colon \widehat{A}(\bar{\mathbb{D}}) \to \widehat{A}(\bar{\mathbb{D}})$ such that $|u| = 1$ on the maximal ideal space $\bar{\mathbb{D}}$ of $\widehat{A}(\bar{\mathbb{D}})$ and that $W(\widehat{v}) = u \cdot W_1(\widehat{v})$ for all $v \in A(\bar{\mathbb{D}})$. The maximum modulus principle asserts that u is a constant function c_1 of modulus 1. It is well-known that every automorphism on $\widehat{A}(\bar{\mathbb{D}})$ is represented by a composition operator; more explicitly, there exists a homeomorphism $\rho\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that $W_1(\widehat{v}) = \widehat{v} \circ \rho$ for all $v \in A(\bar{\mathbb{D}})$. Letting $v = \mathrm{id}$ in the last equality, we have $\rho = W_1(\widehat{\mathrm{id}})$, and hence

ρ is analytic on $\mathbb{D}$. Since ρ is a homeomorphism on $\bar{\mathbb{D}}$, which is also analytic on $\mathbb{D}$, there exist $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$ such that

$$\rho(z) = \lambda \frac{z-a}{\bar{a}z-1} \qquad (z \in \bar{\mathbb{D}})$$

(see. [22, Theorem 12.6]). We obtain

$$W(\widehat{v})(z) = c_1 \widehat{v}(\rho(z)) \qquad (v \in A(\bar{\mathbb{D}}),\ z \in \bar{\mathbb{D}}). \tag{4.12}$$

For each $f \in \mathcal{S}_A$, $I(f')' = f'$ on $\mathbb{D}$ by (4.9), and thus $I(f') - f$ is constant on $\mathbb{D}$. By the definition of I, $I(f')(0) = 0$, and we obtain

$$I(f') = f - f(0) \qquad \text{on} \quad \mathbb{D} \tag{4.13}$$

for all $f \in \mathcal{S}_A$. Applying T_0 to (4.13), we have $T_0(I(f')) = T_0(f) - T_0(f(0))$ on $\mathbb{D}$, where we have used the real linearity of T_0. Therefore, $\widehat{T_0(I(f'))'} = \widehat{T_0(f)'} - \widehat{T_0(f(0))'}$ on $\bar{\mathbb{D}}$. Equality (4.8) shows $\widehat{T_0(f(0))'} = 0$ on $\mathbb{T}$, and thus $\widehat{T_0(f(0))'} = 0$ on $\bar{\mathbb{D}}$ since $\mathbb{T}$ is a boundary for $A(\bar{\mathbb{D}})$. We deduce $\widehat{T_0(I(f'))'} = \widehat{T_0(f)'}$ on $\bar{\mathbb{D}}$. By using (4.10) and (4.12), we have

$$\begin{aligned} \widehat{T_0(f)'}(z) &= \widehat{T_0(I(f'))'}(z) = [W(\widehat{f'})([z]^{\varepsilon_0\varepsilon_1})]^{\varepsilon_0\varepsilon_1} \\ &= [c_1 \widehat{f'}(\rho([z]^{\varepsilon_0\varepsilon_1}))]^{\varepsilon_0\varepsilon_1} \end{aligned}$$

for every $z \in \bar{\mathbb{D}}$. In particular,

$$T_0(f)'(z) = [c_1 f'(\rho([z]^{\varepsilon_0\varepsilon_1}))]^{\varepsilon_0\varepsilon_1} \qquad (z \in \mathbb{D}).$$

Equality (4.13), applied to $T_0(f)$ instead of f, shows that $I(T_0(f)') = T_0(f) - T_0(f)(0)$ on $\mathbb{D}$. Recall that $T_0(f)(0) = [c_0 f(0)]^{\varepsilon_0}$ by (4.7), and consequently

$$\begin{aligned} T_0(f)(z) &= T_0(f)(0) + I(T_0(f)')(z) \\ &= [c_0 f(0)]^{\varepsilon_0} + \int_{[0,z]} [c_1 f'(\rho([\zeta]^{\varepsilon_0\varepsilon_1}))]^{\varepsilon_0\varepsilon_1}\, d\zeta \end{aligned}$$

for all $f \in \mathcal{S}_A$ and $z \in \mathbb{D}$.

Conversely, let $T(0) \in \mathcal{S}_A$, and suppose that

$$T(f)(z) - T(0)(z) = [c_0 f(0)]^{\varepsilon_0} + \int_{[0,z]} [c_1 f'(\rho([\zeta]^{\varepsilon_0\varepsilon_1}))]^{\varepsilon_0\varepsilon_1}\, d\zeta$$

for all $f \in \mathcal{S}_A$ and $z \in \mathbb{D}$, where $c_0, c_1 \in \mathbb{T}$, $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$ and $\rho \in \widehat{A}(\bar{\mathbb{D}})$ is a homeomorphism with the above properties. Then we observe that the map $T - T(0)$ is a surjective real linear isometry on $(\mathcal{S}_A, \|\cdot\|_\sigma)$. This completes the proof. $\square$

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