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CALABI-YAU HYPERSURFACES IN THE DIRECT PRODUCT OF $\mathbb{P}^1$ AND INERTIA GROUPS

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ABSTRACT. We produce the family of Calabi-Yau hypersurfaces X_n of $(\mathbb{P}^1)^{n+1}$ in higher dimension whose inertia group contains non commutative free groups. This is completely different from Takahashi's result [4] for Calabi-Yau hypersurfaces M_n of $\mathbb{P}^{n+1}$.

1. Introduction

Throughout this paper, we work over $\mathbb{C}$. Given an algebraic variety X, it is natural to consider its birational automorphisms $\varphi : X \dashrightarrow X$. The set of these birational automorphisms forms a group $\mathrm{Bir}(X)$ with respect to the composition. Let V be an $(n+1)$-dimensional smooth projective rational manifold and $X \subset V$ a projective variety. The *decomposition group* of X is the group

$$\mathrm{Dec}(V, X) := \{f \in \mathrm{Bir}(V) \mid f(X) = X \text{ and } f|_X \in \mathrm{Bir}(X)\}.$$

The *inertia group* of X is the group

$$\mathrm{Ine}(V, X) := \{f \in \mathrm{Dec}(V, X) \mid f|_X = \mathrm{id}_X\}. \tag{1.1}$$

In this paper, we treat $\mathrm{Ine}(V, X)$ of some hypersurface $X \subset V$ originated in [2].

In Section 2, we mention the result (Theorem 2.1) of Takahashi [4] about the smooth Calabi-Yau hypersurfaces M_n of $\mathbb{P}^{n+1}$ of degree $n+2$. It turns out that the inertia group of M_n is trivial (Theorem 1.1). Theorem 1.1 is a direct consequence of Takahashi's result:

Theorem 1.1. *Suppose $n \geq 3$. Let $M_n = (n+2) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+2$. Then*

$$\mathrm{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}.$$

In Section 3, we consider Calabi-Yau hypersurfaces

$$X_n = (2, 2, \ldots, 2) \subset (\mathbb{P}^1)^{n+1}.$$

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Let

$$\mathrm{UC}(N) := \overbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}^{N} = \mathop{\Huge *}_{i=1}^{N} \langle t_i \rangle$$

be the *universal Coxeter group* of rank N where $\mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2. There is no non-trivial relation between its N natural generators t_i. Let

$$p_i : X_n \to (\mathbb{P}^1)^n \quad (i = 1, \ldots, n+1)$$

be the natural projections which are obtained by forgetting the i-th factor of $(\mathbb{P}^1)^{n+1}$. Then, the $n+1$ projections p_i are generically finite morphism of degree 2. Thus, for each index i, there is a birational transformation

$$\iota_i : X_n \dashrightarrow X_n$$

that permutes the two points of general fibers of p_i and this provides a group homomorphism

$$\Phi : \mathrm{UC}(n+1) \to \mathrm{Bir}(X_n).$$

From now, we set $P(n+1) := (\mathbb{P}^1)^{n+1}$. Cantat–Oguiso proved the following theorem in [1].

Theorem 1.2. ([1, Theorem 1.3 (2)]) *Let X_n be a generic hypersurface of multidegree $(2, 2, \ldots, 2)$ in $P(n+1)$ with $n \geq 3$. Then the morphism Φ that maps each generator t_j of $\mathrm{UC}(n+1)$ to the involution ι_j of X_n is an isomorphism from $\mathrm{UC}(n+1)$ to $\mathrm{Bir}(X_n)$.*

Here "generic" means X_n belongs to the complement of some countable union of proper closed subvarieties of the complete linear system $\big|(2, 2, \ldots, 2)\big|$.

Ludmil Katzarkov asked that how many do the lifts of ι_j exist? Our main result is following theorem, answering a question asked by Ludmil Katzarkov:

Theorem 1.3. *Let $X_n \subset P(n+1)$ be an irreducible hypersurface of multidegree $(2, 2, \ldots, 2)$ and $n \geq 3$. Then there are $n+1$ elements ρ_i $(1 \leq i \leq n+1)$ of $\mathrm{Ine}(P(n+1), X_n)$ such that*

$$\langle \rho_1, \rho_2, \ldots, \rho_{n+1} \rangle \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n+1} \subset \mathrm{Ine}(P(n+1), X_n).$$

In particular, $\mathrm{Ine}(P(n+1), X_n)$ is an infinite non-commutative group, i.e. the lifts of ι_j are infinitely.

Our proof of Theorem 1.3 is based on an explicit computation of elementary flavour.

It is interesting that the inertia groups of $X_n \subset P(n+1) = (\mathbb{P}^1)^{n+1}$ and $M_n \subset \mathbb{P}^{n+1}$ have completely different structures though both X_n and M_n are Calabi-Yau hypersurfaces in rational Fano manifolds.

2. Calabi-Yau hypersurfaces in $\mathbb{P}^{n+1}$

Our goal, in this section, is to prove Theorem 1.1 (i.e. Theorem 2.2). Before that, we introduce the result of Takahashi [4].

Theorem 2.1. ([4, Theorem 2.3]) *Let X be a Fano manifold (i.e. a manifold whose anti-canonical divisor $-K_X$ is ample,) with $\dim X \geq 3$ and $\dim_{\mathbb{Q}} \operatorname{Pic}(X) = 1$, $S \in |-K_X|$ a smooth hypersurface with $\operatorname{Pic}(X) \to \operatorname{Pic}(S)$ surjective. Let $\Phi : X \dashrightarrow X'$ be a birational map to a $\mathbb{Q}$-factorial terminal variety X' with $\dim_{\mathbb{Q}} \operatorname{Pic}(X') = 1$ which is not an isomorphism, and $S' = \Phi_* S$. Then $K_{X'} + S'$ is ample.*

After that, we consider n-dimensional *Calabi-Yau manifold* X in this paper. It is a projective manifold which is simply connected,

$$H^0(X, \Omega_X^i) = 0 \quad (0 < i < \dim X = n) \quad \text{and} \quad H^0(X, \Omega_X^n) = \mathbb{C}\omega_X,$$

where ω_X is a nowhere vanishing holomorphic n-form.

The following theorem is a consequence of Theorem 2.1, which is same as Theorem 1.1. This provides an example of the Calabi-Yau hypersurface M_n whose inertia group consists of only identity transformation.

Theorem 2.2. *Suppose $n \geq 3$. Let $M_n = (n+2) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+2$, which is a Calabi-Yau manifold of dimension n. Then*

(1) $\operatorname{Dec}(\mathbb{P}^{n+1}, M_n) = \{f \in \operatorname{PGL}(n+2, \mathbb{C}) = \operatorname{Aut}(\mathbb{P}^{n+1}) \mid f(M_n) = M_n\}$.
(2) $\operatorname{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}$.
(3) $\operatorname{Dec}(\mathbb{P}^{n+1}, M_n) \cong \operatorname{Bir}(M_n) = \operatorname{Aut}(M_n)$.

Proof. It is obvious that the set on the left side of (1) contains the set on the right of (1). We will show the converse. Assume that $f \in \operatorname{Dec}(\mathbb{P}^{n+1}, M_n)$. Then $f_*(M_n) = M_n$ and $K_{\mathbb{P}^{n+1}} + M_n = 0$. Thus by Theorem 2.1, $f \in \operatorname{Aut}(\mathbb{P}^{n+1}) = \operatorname{PGL}(n+2, \mathbb{C})$. This proves (1).

From here, we will show (2). For two points $x, y \in \mathbb{P}^{n+1}$, we denote the linear subspace on $\mathbb{P}^{n+1}$ of dimenion 1, which is defined by x and y by $C_{x,y}$. Then $C_{x,y} \cong \mathbb{P}^1$. Since the degree of M_n is $n+2$ and M_n is smooth, for a general point $x \in \mathbb{P}^{n+1}$, there is a point $y \in M_n$ such that $C_{x,y} \cap M_n$ is a set of $n+2$ points. Let $f \in \operatorname{Ine}(\mathbb{P}^{n+1}, M_n)$. Since $f \in \operatorname{PGL}(n+2, \mathbb{C})$ by (1), we have that $f(C_{x,y}) = C_{f(x),f(y)}$, i.e. $f(C_{x,y})$ is the linear subspace on $\mathbb{P}^{n+1}$ of dimenion 1, which is defined by $f(x)$ and $f(y)$. Since $f|_{M_n} = \mathrm{id}_{M_n}$, we get that $C_{x,y} \cap f(C_{x,y})$ contains at least $n+2$ points. Since $C_{x,y}$ and $f(C_{x,y})$ are linear subspaces on $\mathbb{P}^{n+1}$ of dimension 1, we obtain $C_{x,y} = f(C_{x,y})$. Thus f induces an automorphim $f|_{C_{x,y}}$ of $C_{x,y}$. If $f|_{C_{x,y}} \neq \mathrm{id}_{C_{x,y}}$, then the fixed points of $f|_{C_{x,y}}$ are at most 2 points since $C_{x,y} \cong \mathbb{P}^1$. Therefore, since $C_{x,y} \cap f(C_{x,y})$ contains at least $n+2$ points, and $n \geq 3$, we have $f|_{C_{x,y}} = \mathrm{id}_{C_{x,y}}$. Thus we obtain $f = \mathrm{id}_{\mathbb{P}^{n+1}}$, i.e. $\operatorname{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}$.

We will show (3). By Lefschetz hyperplane section theorem for $n \geq 3$, we have that $\pi_1(M_n) \simeq \pi_1(\mathbb{P}^{n+1}) = \{\mathrm{id}\}$, and $\mathrm{Pic}(M_n) = \mathbb{Z}h$ where h is the hyperplane class. By $\mathrm{Pic}(M_n) = \mathbb{Z}h$, there is no small projective contraction of M_n, in particular, M_n has no flop. Thus by Kawamata [3], we get $\mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$. By $\mathrm{Pic}(M_n) = \mathbb{Z}h$, for $g \in \mathrm{Aut}(M_n)$ we have $g = \tilde{g}|_{M_n}$ for some $\tilde{g} \in \mathrm{PGL}(n+2, \mathbb{C})$. Therefore, from (2) we get $\mathrm{Dec}(\mathbb{P}^{n+1}, M_n) \cong \mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$. □

3. Calabi-Yau hypersurfaces in $(\mathbb{P}^1)^{n+1}$

As in above section, the Calabi-Yau hypersurface M_n of $\mathbb{P}^{n+1}$ with $n \geq 3$ has only identical transformation as the element of its inertia group. However, there exist some Calabi-Yau hypersurfaces in the product of $\mathbb{P}^1$ which does not satisfy this property; as result (Theorem 3.2) shows.

To simplify, we denote

$$P(n+1) := (\mathbb{P}^1)^{n+1} = \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \cdots \times \mathbb{P}^1_{n+1},$$
$$P(n+1)_i := \mathbb{P}^1_1 \times \cdots \times \mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i+1} \times \cdots \times \mathbb{P}^1_{n+1} \simeq P(n),$$

and

$$p^i : P(n+1) \to \mathbb{P}^1_i \simeq \mathbb{P}^1,$$
$$p_i : P(n+1) \to P(n+1)_i$$

as the natural projections. Let H_i be the divisor class of $(p^i)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, then $P(n+1)$ is a Fano manifold of dimension $n+1$ and its anti canonical divisor has the form $-K_{P(n+1)} = \sum_{i=1}^{n+1} 2H_i$. Therefore, by the adjunction formula, the smooth hypersurface $X_n \subset P(n+1)$ has trivial canonical divisor if and only if it has multidegree $(2, 2, \ldots, 2)$. More strongly, for $n \geq 3$, $X_n = (2, 2, \ldots, 2)$ becomes a Calabi-Yau manifold of dimension n and, for $n = 2$, a $K3$ surface (i.e. 2-dimensional Calabi-Yau manifold).

From now, X_n is an irreducible hypersurface of $P(n+1)$ of multidegree $(2, 2, \ldots, 2)$ with $n \geq 3$. Let us write $P(n+1) = \mathbb{P}^1_i \times P(n+1)_i$. Let $[x_{i1} : x_{i2}]$ be the homogeneous coordinates of $\mathbb{P}^1_i$. Hereafter, we consider the affine locus and denote by $x_i = \dfrac{x_{i2}}{x_{i1}}$ the affine coordinates of $\mathbb{P}^1_i$ and by $\mathbf{z}_i$ that of $P(n+1)_i$. When we pay attention to x_i, X_n can be written by following equation

$$X_n = \{F_{i,0}(\mathbf{z}_i)x_i^2 + F_{i,1}(\mathbf{z}_i)x_i + F_{i,2}(\mathbf{z}_i) = 0\} \tag{3.1}$$

where each $F_{i,j}(\mathbf{z}_i)$ $(j = 0, 1, 2)$ is a quadratic polynomial of $\mathbf{z}_i$. Now, we consider the two involutions of $P(n+1)$:

$$\tau_i : (x_i, \mathbf{z}_i) \to \left(-x_i - \frac{F_{i,1}(\mathbf{z}_i)}{F_{i,0}(\mathbf{z}_i)}, \mathbf{z}_i\right), \tag{3.2}$$

$$\sigma_i : (x_i, \mathbf{z}_i) \to \left(\frac{F_{i,2}(\mathbf{z}_i)}{x_i \cdot F_{i,0}(\mathbf{z}_i)}, \mathbf{z}_i\right). \tag{3.3}$$

We get two birational automorphisms of X_n:

$$\rho_i = \sigma_i \circ \tau_i : (x_i, \mathbf{z}_i) \to \left(\frac{F_{i,2}(\mathbf{z}_i)}{-x_i \cdot F_{i,0}(\mathbf{z}_i) - F_{i,1}(\mathbf{z}_i)},\ \mathbf{z}_i\right),$$

$$\rho_i' = \tau_i \circ \sigma_i : (x_i, \mathbf{z}_i) \to \left(-\frac{x_i \cdot F_{i,1}(\mathbf{z}_i) + F_{i,2}(\mathbf{z}_i)}{x_i \cdot F_{i,0}(\mathbf{z}_i)},\ \mathbf{z}_i\right).$$

The involution $\tau_i|_{X_n} = \sigma_i|_{X_n}$ is ι_i which is mentioned in the introduction, and the birational automorphism ρ_i satisfies $\rho_i|_{X_n} = id_{X_n}$, i.e. $\rho_i \in \mathrm{Ine}(P(n+1), X_n)$.

Lemma 3.1. *Each ρ_i has infinite order.*

Proof. We consider a matrix

$$M := \begin{pmatrix} 0 & F_{i,2} \\ -F_{i,0} & -F_{i,1} \end{pmatrix} \in \mathbf{M}_2(\overline{\mathbb{C}(\mathbf{z}_i)}),$$

where $\overline{\mathbb{C}(\mathbf{z}_i)}$ is the algebraic closure of the field $\mathbb{C}(\mathbf{z}_i)$. If there is an integer $k \in \mathbb{Z}$ such that $\rho_i^k = \mathrm{id}_{\mathbb{P}^{n+1}}$, then $M^k = \alpha I$, where I is the identity matrix and $\alpha \in \mathbb{C}^\times$. Since their eigenvalues of M are

$$\frac{-F_{i,1} \pm \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}{2},$$

by $M^k = \alpha I$, we have

$$\left(\frac{-F_{i,1} \pm \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}{2}\right)^k = \alpha \in \mathbb{C}.$$

Since $\mathbb{C}$ is an algebraically closed field, we have

$$\beta := \frac{-F_{i,1} \pm \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}{2} \in \mathbb{C}.$$

Then we obtain that $F_{i,1}^2 - 4F_{i,0}F_{i,2} = 4\beta^2 + 4\beta F_{i,1} + F_{i,1}^2$. Since each $F_{i,j}(\mathbf{z}_i)$ $(j = 0, 1, 2)$ is a quadratic polynomial of $\mathbf{z}_i$, we get $F_{i,1} = 0$, and hence $F_{i,0}F_{i,2} = 0$. Since $X_n \subset P(n+1)$ is an irreducible hypersurface of multidegree $(2, 2, \ldots, 2)$, this is a contradiction. Thus the order of ρ_i is infinity. $\square$

Our main result is the following (which is same as Theorem 1.3):

Theorem 3.2. *Let $X_n \subset P(n+1)$ be an irreducible hypersurface of multidegree $(2,2,\ldots,2)$ and $n \geq 3$. Then $n+1$ elements $\rho_i \in \mathrm{Ine}(P(n+1), X_n)$ $(1 \leq i \leq n+1)$ satisfy*

$$\langle \rho_1, \rho_2, \ldots, \rho_{n+1} \rangle \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n+1} \subset \mathrm{Ine}(P(n+1), X_n).$$

In particular, $\mathrm{Ine}(P(n+1), X_n)$ *is an infinite non-commutative group.*

Proof. By Lemma 3.1, it is sufficient to show that there is no non-trivial relation between its $n+1$ elements ρ_i. We show by arguing by contradiction.

Suppose to the contrary that there is a non-trivial relation between $n+1$ elements ρ_i, that is,

$$\rho_{i_1}^{n_1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l} = \mathrm{id}_{P(n+1)} \tag{3.4}$$

where l is a positive integer, $n_k \in \mathbb{Z} \setminus \{0\}$ $(1 \leq k \leq l)$, and each ρ_{i_k} denotes one of the $n+1$ elements ρ_i $(1 \leq i \leq n+1)$ and satisfies $\rho_{i_k} \neq \rho_{i_{k+1}}$ $(0 \leq k \leq l-1)$. Put $N = |n_1| + \cdots + |n_l|$.

In the affine coordinates $(x_i, \mathbf{z}_i)$ where x_i is the affine coordinates of i-th factor $\mathbb{P}^1_i$, we can choose a point $(\alpha, \mathbf{z}_i)$, which is not included in $X_n \cup \overline{\mathrm{Ind}(\rho_{i_1}^{n_1-1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l}) \cup (\rho_{i_1}^{n_1-1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l})^{-1}(\mathrm{Ind}(\rho_{i_1}))}$.

We put $(\beta, \mathbf{w}_i)$ by $\rho_{i_1}^{n_1-1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l}(\alpha, \mathbf{z}_i)$. By a suitable projective linear coordinate change of $\mathbb{P}^1_i$, we can set $\alpha = 0$ and $\beta \neq \infty$. When we pay attention to the i-th element x_i of the new coordinates, we put same letters $F_{i,j}(\mathbf{z}_i)$ for the definitional equation of X_n, that is, X_n can be written by

$$X_n = \{F_{i,0}(\mathbf{z}_i)x_i^2 + F_{i,1}(\mathbf{z}_i)x_i + F_{i,2}(\mathbf{z}_i) = 0\}.$$

From the assumption, the following equality holds:

$$\rho_{i_1}(\beta, \mathbf{w}_i) = (0, \mathbf{z}_i).$$

Then, by the definition of ρ_i, it maps β to 0. That is, the equation $F_{i,2}(\mathbf{w}_i) = 0$ is satisfied. On the other hand, the intersection of X_n and the hyperplane $(x_i = 0)$ is written by

$$X_n \cap (x_i = 0) = \{F_{i,2}(\mathbf{z}_i) = 0\}.$$

This implies $(0, \mathbf{w}_i) = \rho_{i_1}(\beta, \mathbf{w}_i) = (0, \mathbf{z}_i)$ is a point on X_n, a contradiction to the fact that $(0, \mathbf{z}_i) \notin X_n$. Therefore, we can conclude that there does not exist such N. This completes the proof of Theorem 3.2. □

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