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FIXED POINT THEOREMS FOR ASYMPTOTIC MAPPINGS OF A GENERALIZED CONTRACTIVE TYPE IN COMPLETE METRIC SPACES

MASASHI TOYODA AND TOSHIKAZU WATANABE

Abstract. We consider an asymptotic version of α-ψ contractive mappings. We show the existence and uniqueness of fixed points. Caccioppoli's fixed point theorem is deduced from main results in this paper. Moreover, we discuss an asymptotic version of mappings related with (c)-comaprison functions.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers.

Let (X, d) be a metric space. In [20], Samet, Vetro and Vetro introduced the notion of α-ψ contractive mappings: A mapping T of X into itself is α-ψ contractive if there exist a mapping α of $X \times X$ into $[0, \infty)$ and a nondecreasing mapping ψ of $[0, \infty)$ into itself such that the series $\sum_{n=1}^{\infty} \psi^n(t)$ converges for all $t > 0$ and for any $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty) \leq \psi\left(d(x, y)\right), \tag{1}$$

where we say that ψ is nondecreasing if for any $t, s \in [0, \infty)$, $t \leq s$ implies $\psi(t) \leq \psi(s)$. If $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = rt$ for all $t \geq 0$ and some $r \in [0, 1)$, then the mapping (1) is a Banach contraction, i.e.,

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. By the Banach contraction principle, a Banach contraction of a complete metric space into itself has a unique fixed point. Samet, Vetro and Vetro in [20] consider the existence and uniqueness of fixed points for α-ψ contractive mappings. Their results imply the Banach contraction principle. In fact, Theorems 2.1 and 2.3 in [20] imply the Banach contraction principle; see Theorem 3.1 in [20]. Theorems in [20] were extended by a number of authors; see [1, 6, 7, 8, 9, 10, 12, 13, 16] and the references therein.

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On the other hand, Caccioppoli proved the following fixed point theorem: Let (X, d) be a complete metric space and T a mapping of X into itself. If there exists a nonnegative sequence $\{r_n\}$ such that the series $\sum_{n=1}^{\infty} r_n$ converges and for any $x, y \in X$ and $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq r_n d(x, y), \tag{2}$$

then T has a unique fixed point; see [14] and the references therein. See [23] also. The mapping (2) is an asymptotic version of Banach contractions. In [15], Kirk considered an asymptotic version of Boyd-Wong contractions [4]; see [11] also. Moreover, in [21], Suzuki considered an asymptotic version of Meir-Keeler contractions [17].

In this paper, we consider an asymptotic version of α-ψ contractive mappings. We show the existence and uniqueness of fixed points. Caccioppoli's fixed point theorem is deduced from main results in this paper. Moreover, we discuss an asymptotic version of mappings related with (c)-comaprison functions [3].

2. Main Results

Let (X, d) be a metric space and T a mapping of X into itself. We say that T is α-ψ_n contractive if there exist a mapping α of $X \times X$ into $[0, \infty)$ and a sequence of nondecreasing mappings $\{\psi_n\}$ of $[0, \infty)$ into itself such that the series $\sum_{n=1}^{\infty} \psi_n(t)$ converges for all $t > 0$ and for any $x, y \in X$, $n \in \mathbb{N}$,

$$\alpha(x, y) d(T^n x, T^n y) \leq \psi_n \left(d(x, y) \right).$$

For α-ψ_n contractive mappings, we obtain the following fixed point theorems.

Theorem 1. *Let (X, d) be a complete metric space and T an α-ψ_n contractive mapping of X into itself. Assume that the following hold.*

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

(ii) *T is continuous.*

Then T has a fixed point.

Proof. By (i), there exists x_0 such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. Thus we can assume that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$.

It follows that for any $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x_0, T^n x_1) \\ &\leq \alpha(x_0, x_1) d(T^n x_0, T^n x_1) \\ &\leq \psi_n \left(d(x_0, x_1) \right). \end{aligned}$$

For $\varepsilon > 0$, there exists n_0 such that

$$\sum_{n=n_0}^{\infty} \psi_n \left(d(x_0, x_1)\right) < \varepsilon.$$

For n, m with $m > n \geq n_0$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi_k \left(d(x_0, x_1)\right) \\ &\leq \sum_{k=n_0}^{\infty} \psi_k \left(d(x_0, x_1)\right) < \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By (ii), we have $Tx_n \to Tu$ as $n \to \infty$. Since $Tx_n = x_{n+1} \to u$ as $n \to \infty$, we have $u = Tu$. □

The next theorem does not require the continuity of T. Let (X, d) be a metric space, α a mapping of $X \times X$ into $[0, \infty)$ and T a mapping of X into itself. We say that T is α-admissible [20] if for any $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$. There are examples for α-admissible mappings in [13, 20]. For example, let $X = (0, \infty)$ and T a mapping of X into itself defined by $Tx = \ln x$ for all $x \in X$. If α is a mapping of $X \times X$ into $[0, \infty)$ defiend by $\alpha(x, y) = 2$ $(x \geq y)$ and $\alpha(x, y) = 0$ $(x < y)$, then T is α-admissible; see Example 2.1 in [20].

Theorem 2. *Let (X, d) be a complete metric space and T an α-ψ_n contractive and α-admissible mapping of X into itself. Assume that the following hold.*

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*
(ii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.*
(iii) *For any $t > 0$, $\lim_{t\to 0} \psi_1(t) = 0$.*

Then T has a fixed point.

Proof. Following the proof of Theorem 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots,$ converges to some $u \in X$. By (i), we have $\alpha(x_0, x_1) \geq 1$. Since T is α-admissible, we have $\alpha(Tx_0, Tx_1) \geq 1$. Then $\alpha(x_1, x_2) \geq 1$. Moreover we have $\alpha(Tx_1, Tx_2) \geq 1$. Inductively, we have for any $n = 0, 1, 2, \ldots,$

$$\alpha(x_n, x_{n+1}) \geq 1.$$

Then by (ii), there exists a subsequence $\{x_{n_k}\}$ such that

$$\alpha(x_{n_k}, u) \geq 1$$

for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_{n_k+1}, Tu) &= d(Tx_{n_k}, Tu) \\ &\leq \alpha(x_{n_k}, u) d(Tx_{n_k}, Tu) \\ &\leq \psi_1 \left(d(x_{n_k}, u) \right) \end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ in the above inequality, we obtain

$$d(u, Tu) \leq 0$$

by (iii). Thus we have $d(u, Tu) = 0$, that is, $u = Tu$. □

For the uniqueness of fixed points of an α-ψ_n contractive mapping, we obtain the following.

Theorem 3. *To the hypotheses of Theorem 1 (or Theorem 2), we assume that for any $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Then the fixed point of T is unique.*

Proof. By Theorems 1 and 2, there exists a fixed point u of T. Suppose that v is another fixed point of T. Then there exists $z \in X$ such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. Define the sequence $\{z_n\}$ in X by $z_0 = z$ and $z_{n+1} = Tz_n$ for all $n = 0, 1, 2, \ldots$. For $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} d(u, z_{n+1}) &= d(T^n u, T^n z) \\ &\leq \alpha(u, z) d(T^n u, T^n z) \\ &\leq \psi_n \left(d(u, z) \right). \end{aligned}$$

Hence we have for any $n = 0, 1, 2, \ldots$,

$$d(u, z_{n+1}) \leq \psi_n \left(d(u, z) \right).$$

Similarly, we have for any $n = 0, 1, 2, \ldots$,

$$d(v, z_{n+1}) \leq \psi_n \left(d(v, z) \right).$$

Suppose that $d(u, z) > 0$ and $d(v, z) > 0$. Since $\sum_{n=1}^{\infty} \psi_n \left(d(u, z) \right) < \infty$ and $\sum_{n=1}^{\infty} \psi_n \left(d(v, z) \right) < \infty$, we have

$$\lim_{n \to \infty} \psi_n(d(u, z)) = 0$$

and

$$\lim_{n \to \infty} \psi_n(d(v, z)) = 0.$$

Thus we get

$$\lim_{n\to\infty} d(u, z_{n+1}) = 0$$

and

$$\lim_{n\to\infty} d(v, z_{n+1}) = 0.$$

Hence we have $u = v$.

Finally, we consider other cases. Suppose that $d(u, z) = 0$ and $d(v, z) = 0$. Then we have $u = z$ and $v = z$. Hence we have $u = v$. Suppose that $d(u, z) = 0$ and $d(v, z) > 0$. Then we have $u = z$ and $z_n \to v$. Since $z_n = u$ for all $n = 0, 1, 2, \ldots$, we have $u = v$. Suppose that $d(u, z) > 0$ and $d(v, z) = 0$. Then we have $v = z$ and $z_n \to u$. Since $z_n = v$ for all $n = 0, 1, 2, \ldots$, we have $u = v$.

Thus we obtain the uniqueness of fixed points of T. □

In Theorem 3, putting $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi_n(t) = r_n t$ for all $n \in \mathbb{N}$, $t > 0$, where $r_1, r_2, r_3, \ldots$ are some nonnegative real numbers, we obtain the following Caccioppoli's fixed point theorem.

Corollary 4. *Let (X, d) be a complete metric space and T a mapping of X into itself. Assume that there exists a nonnegative sequence $\{r_n\}$ such that the series $\sum_{n=1}^{\infty} r_n$ converges and for any $x, y \in X$ and $n \in \mathbb{N}$, (2) holds. Then T has a unique fixed point.*

3. Asymptotic version of mappings related with (c)-comaprison functions

In this section, we consider an asymptotic version of mappings of Theorem 2.8 in [3].

In Theorem 3, putting $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following.

Corollary 5. *Let (X, d) be a complete metric space and T a mapping of X into itself. Assume that there exists a sequence of nondecreasing mappings $\{\psi_n\}$ of $[0, \infty)$ into itself such that the series $\sum_{n=1}^{\infty} \psi_n(t)$ converges and $\lim_{t\to 0} \psi_1(t) = 0$ for all $t > 0$ and for any $x, y \in X$ and $n \in \mathbb{N}$,*

$$d(T^n x, T^n y) \leq \psi_n\left(d(x, y)\right).$$

Then T has a unique fixed point.

Proof. If $\alpha(x, y) = 1$ for all $x, y \in X$, then T is α-admissible and conditions (i) and (ii) in Theorem 2 hold. Moreover X satisfies the assumption of Theorem 3. Hence T has a unique fixed point. □

Before deducing Theorem 2.8 in [3], we show the relation between α-ψ contractive mappings and α-ψ_n contractive mappings.

Lemma 6. *Let (X, d) be a metric space and T an α-ψ contractive mapping of X into itself such that for any $x, y \in X$ and $n \in \mathbb{N}$, $1 \leq \alpha(x, y) \leq \alpha(T^n x, T^n y)$. If $\psi_n(t) = \psi^n(t)$ for all $t > 0$ and $n \in \mathbb{N}$ where ψ^n is the n-th iterate of ψ, then T is α-ψ_n contractive.*

Proof. For $x, y \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\alpha(x,y)d(T^n x, T^n y) &\leq \alpha(T^{n-1}x, T^{n-1}y)d(T^n x, T^n y) \\
&\leq \psi\left(d(T^{n-1}x, T^{n-1}y)\right) \\
&\leq \psi\left(\alpha(T^{n-2}x, T^{n-2}y)d(T^{n-1}x, T^{n-1}y)\right) \\
&\leq \psi^2\left(d(T^{n-2}x, T^{n-2}y)\right) \\
&\leq \psi^2\left(\alpha(T^{n-3}x, T^{n-3}y)d(T^{n-2}x, T^{n-2}y)\right) \\
&\leq \cdots \\
&\leq \psi^n\left(d(x,y)\right).
\end{aligned}$$

If we put $\psi_n(t) = \psi^n(t)$ for all $t > 0$ and $n \in \mathbb{N}$, then T is α-ψ_n contractive. □

Now, we deduce Theorem 2.8 in [3]. A function ψ of $[0, \infty)$ into itself is a (c)-comparison function [3] if ψ is nondecreasing and the series $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$. If ψ is (c)-comparison, then $\psi(t) < t$ for all $t > 0$; see Lemmas 2.1 and 2.2 in [3]. By Theorem 3, we obtain the following which is Theorem 2.8 in [3].

Corollary 7. *Let (X, d) be a complete metric space and ψ a (c)-comparison mapping of $[0, \infty)$ into itself. If T is a mapping of X into itself such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \psi\left(d(x,y)\right),$$

then T has a unique fixed point.

Proof. Define a mapping α of $X \times X$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then T is α-admissible and conditions (i) and (ii) in Theorem 2 hold. Moreover we define mappings $\{\psi_n\}$ by $\psi_n(t) = \psi^n(t)$ for all $t > 0$ and $n \in \mathbb{N}$. Since ψ is (c)-comaprison, condition (iii) of Theorem 2 holds. By Lemma 6, T is α-ψ_n contractive. Since all the hypotheses of Theorem 3 are satisfied, T has a unique fixed point. □

Remark 8. *Ran and Reurings* [19] *considered the Banach fixed point theorem in the setting of metric spaces with a partial order; see* [18] *also. For an asymptotic version of mappings of theorems in* [19, 18], *see* [22].

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(Masashi Toyoda) Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: `mss-toyoda@is.sci.toho-u.ac.jp`

(Toshikazu Watanabe) College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo, 101-8308, Japan
E-mail address: `twatana@edu.tuis.ac.jp`

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