Nihonkai Math. J.
Vol.29(2018), 1–19

THE QUADRATIC QUANTUM f-DIVERGENCE OF CONVEX FUNCTIONS AND MATRICES

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ABSTRACT. In this paper we introduce the concept of *quadratic quantum f-divergence measure* for a continuos function f defined on the positive semi-axis of real numbers, the invertible matrix T and matrix V by

$$\mathcal{S}_f(V,T) := \operatorname{tr}\left[|T^*|^2 f\left(\left|VT^{-1}\right|^2\right)\right].$$

Some fundamental inequalities for this quantum f-divergence in the case of convex functions are established. Applications for particular quantum divergence measures of interest are also provided.

1. Introduction

Let $\mathcal{M}$ denote the algebra of all $n \times n$ matrices with complex entries and $\mathcal{M}^+$ the subclass of all positive matrices.

Consider the complex Hilbert space $(\mathcal{M}, \langle \cdot, \cdot \rangle_2)$, where the *Hilbert-Schmidt inner product* is defined by

$$\langle U, V \rangle_2 := \operatorname{tr}(V^*U), \ U, \ V \in \mathcal{M}.$$

We denote by $\mathcal{S}_2(\mathcal{M})$ the set of all matrices $A \in \mathcal{M}$ with $\|A\|_2 = 1$. In terms of trace, this is equivalent to $\operatorname{tr}\left(|A|^2\right) = \operatorname{tr}\left(|A^*|^2\right) = 1$.

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function on $[0, \infty)$. By utilising the *continuous functional calculus for selfadjoint operators* in Hilbert spaces, we can define the following *quadratic quantum f-divergence* for matrices T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible, by

$$\begin{aligned}\mathcal{S}_f(V,T) &:= \operatorname{tr}\left[T^* f\left((T^*)^{-1} V^*VT^{-1}\right) T\right] \\ &= \operatorname{tr}\left[T^* f\left(\left|VT^{-1}\right|^2\right) T\right] = \operatorname{tr}\left[|T^*|^2 f\left(\left|VT^{-1}\right|^2\right)\right].\end{aligned} \tag{S}$$

2010 *Mathematics Subject Classification.* Primary 47A63, 47A30; Secondary 15A60, 26D15, 26D10.

Key words and phrases. Operator perspective, convex functions, operator inequalities, arithmetic mean-geometric mean operator inequality, relative operator entropy.

If we take $V = Q^{1/2}$, $T = P^{1/2}$ with $\operatorname{tr}(P) = \operatorname{tr}(Q) = 1$, P invertible, then we have

$$\mathcal{S}_f(V,T) := \operatorname{tr}\left[P^{1/2} f\left(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}\right) P^{1/2}\right] = \operatorname{tr}\left[P f\left(\left|Q^{1/2} P^{-1/2}\right|^2\right)\right] =: \mathcal{D}_f(Q,P)$$

that shows that the quadratic quantum divergence $\mathcal{S}_f$ is an extension of the quantum divergence $\mathcal{D}_f$ defined above.

If we take the convex function $f(t) = t^2 - 1$, $t \geq 0$, then we get

$$\begin{aligned}\mathcal{S}_f(V,T) &= \operatorname{tr}\left[T^*\left((T^*)^{-1} V^* V T^{-1}\right)^2 T - |T^*|^2\right] = \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^4\right) - 1 \\ &= \operatorname{tr}\left(|V|^4 |T|^{-2}\right) - 1 =: \chi_2^2(V,T),\end{aligned}$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible, which, we call, the *quadratic* χ^2*-divergence* for matrices (V,T).

More general, if we take the convex function $f(t) = t^n - 1$, $t \geq 0$ and n a natural number with $n \geq 2$, then we get

$$\mathcal{S}_f(V,T) = \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^{2n}\right) - 1 =: D_{\tilde{\chi}_2^n}(V,T)$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

If we take the convex function $f(t) = t \ln t$ for $t > 0$ and $f(0) := 0$, then we get

$$\mathcal{S}_f(V,T) = \operatorname{tr}\left[|T^*|^2 \left|VT^{-1}\right|^2 \ln\left(\left|VT^{-1}\right|^2\right)\right] =: D_{KL}(V,T)$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

If we take the convex function $f(t) = -\ln t$ for $t > 0$, then we get

$$\begin{aligned}\mathcal{S}_f(V,T) &= -\operatorname{tr}\left[|T^*|^2 \ln\left(\left|VT^{-1}\right|^2\right)\right] = \operatorname{tr}\left[|T^*|^2 \ln\left(\left|(V^*)^{-1} T^*\right|^2\right)\right] \\ &=: \tilde{D}_{KL}(V,T)\end{aligned}$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T, V invertible.

If we take the convex function $f(t) = |t-1|$, $t \geq 0$, then we get

$$\begin{aligned}\mathcal{S}_f(V,T) &= \operatorname{tr}\left(|T^*|^2 \left|\left|VT^{-1}\right|^2 - 1_H\right|\right) \\ &= \operatorname{tr}\left[|T^*|^2 \left|(T^*)^{-1}\left(|V|^2 - |T|^2\right) T^{-1}\right|\right] =: D_V(V,T)\end{aligned}$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

If we consider the convex function $f(t) = \frac{1}{t} - 1$, $t > 0$, then

$$\begin{aligned}\mathcal{S}_f(V,T) &= \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^{-2}\right) - 1 = \operatorname{tr}\left(|T|^2 |V|^{-2} |T|^2\right) - 1 \\ &= \operatorname{tr}\left(|T|^4 |V|^{-2}\right) - 1 = \chi_2^2(T,V)\end{aligned}$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T, V invertible.

If we take the convex function $f(t) = f_q(t) = \frac{1-t^q}{1-q}$, $q \in (0,1)$, then we get

$$\mathcal{S}_{f_q}(V,T) = \frac{1}{1-q}\left[1 - \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^{2q}\right)\right]$$
$$= \frac{1}{1-q}\left[1 - \operatorname{tr}\left(T^* \left|VT^{-1}\right|^{2q} T\right)\right] = \frac{1}{1-q}\left[1 - \operatorname{tr}\left(T ⓢ_q V\right)\right],$$

with

$$T ⓢ_q V := T^* \left|VT^{-1}\right|^{2q} T = \left|\left|VT^{-1}\right|^q T\right|^2$$

is the *quadratic weighted operator geometric mean* of (T,V) introduced in [25], where several properties were established.

For the classical concept of quantum f-divergence and its properties, see the recent papers [24], [27], [28], [36], [37] and the references therein.

For inequalities for classical f-divergence measures, see [5], [12]–[22].

For some classical trace inequalities see [7], [9], [34] and [45], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [7], [26], [30], [32], [33], [39] and [42].

In this paper we introduce the concept of *quadratic quantum f-divergence measure* for a continuos function f defined on the positive semi-axis of real numbers, the invertible matrix T and matrix V on a Hilbert space. Some fundamental inequalities for this quantum f-divergence in the case of convex functions are established. Applications for particular quadratic quantum divergence measures of interest are also provided.

2. Inequalities for quadratic f-divergence measure

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,\ y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I. \tag{1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following fundamental result holds:

Theorem 2.1. *Let f be a continuous convex function on $[0,\infty)$ with $f(1) = 0$. Then we have*

$$0 \leq \mathcal{S}_f(V,T) \tag{2}$$

for any T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

If, in addition, f is continuously differentiable on $(0,\infty)$, then we also have

$$(0 \leq)\, \mathcal{S}_f(V,T) \leq \mathcal{S}_{\ell f'}(V,T) - \mathcal{S}_{f'}(V,T), \tag{3}$$

where ℓ is the identity function.

Proof. For any $x \geq 0$ we have from the gradient inequality (1) that

$$f(x) \geq f(1) + (x-1) f'_+(1)$$

and since f is normalized, then

$$f(x) \geq (x-1) f'_+(1). \tag{4}$$

Utilising the continuous functional calculus for the positive matrix X we have by (4) that

$$f(X) \geq f'_+(1)(X - 1_H) \tag{5}$$

in the operator order of $\mathcal{M}$.

Let T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible, then by taking $X = |VT^{-1}|^2 \geq 0$ in (5) we have

$$f\left(\left|VT^{-1}\right|^2\right) \geq f'_+(1)\left(\left|VT^{-1}\right|^2 - 1_H\right). \tag{6}$$

So, if we multiply (6) at left with T^* and at right with T, then we get

$$\begin{aligned} T^* f\left(\left|VT^{-1}\right|^2\right) T &\geq f'_+(1)\, T^*\left(\left|VT^{-1}\right|^2 - 1_H\right) T \\ &= f'_+(1)\left(|V|^2 - |T|^2\right) \end{aligned}$$

and by taking the trace in this inequality, we get

$$\begin{aligned} \operatorname{tr}\left(T^* f\left(\left|VT^{-1}\right|^2\right) T\right) &\geq f'_+(1)\operatorname{tr}\left(|V|^2 - |T|^2\right) \\ &= f'_+(1)\left[\operatorname{tr}\left(|V|^2\right) - \operatorname{tr}\left(|T|^2\right)\right] = 0, \end{aligned}$$

since T, $V \in \mathcal{S}_2(\mathcal{M})$, namely $\operatorname{tr}\left(|V|^2\right) = \operatorname{tr}\left(|T|^2\right) = 1$. This proves (2).

From the gradient inequality we also have for any $x \geq 0$ that

$$(x-1) f'(x) + f(1) \geq f(x)$$

and since f is normalized, then

$$(x-1) f'(x) \geq f(x)$$

which, as above, implies that

$$\left|VT^{-1}\right|^2 f'\left(\left|VT^{-1}\right|^2\right) - f'\left(\left|VT^{-1}\right|^2\right) \geq f\left(\left|VT^{-1}\right|^2\right) \tag{7}$$

for T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

If we multiply (7) at left with T^* and at right with T, then we get the desired result (3). □

Remark 2.1. If we take $f(t) = -\ln t$, $t > 0$ in Theorem 2.1 then we get

$$0 \leq \tilde{D}_{KL}(V,T) \leq \chi_2^2(V,T) \tag{8}$$

for any T, $V \in \mathcal{S}_2(\mathcal{M})$ with T invertible.

The following lemma is of interest in itself since it provides a reverse of Schwarz inequality for trace:

Lemma 2.1. *Let S be a selfadjoint operator such that $\gamma 1_H \leq S \leq \Gamma 1_H$ for some real constants $\Gamma \geq \gamma$. Then for any $P > 0$ and $\operatorname{tr}(P) < \infty$ we have*

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \\ &\leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right) \\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)^2. \end{aligned} \tag{9}$$

Proof. For the sake of completeness, we give here a simple proof.

Observe that

$$\begin{aligned} &\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right) \\ &= \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(PS\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right) \\ &- \frac{\Gamma+\gamma}{2}\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right) \\ &= \frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \end{aligned} \tag{10}$$

since, obviously

$$\operatorname{tr}\left(P\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right) = 0.$$

Now, since $\gamma 1_H \leq S \leq \Gamma 1_H$ then

$$\left|S - \frac{\Gamma+\gamma}{2}1_H\right| \leq \frac{1}{2}(\Gamma-\gamma)1_H.$$

Taking the modulus in (10) and using the properties of trace, we have

$$\begin{aligned}
&\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \\
&= \frac{1}{\operatorname{tr}(P)} \left|\operatorname{tr}\left(P\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right)\right| \\
&\le \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left|\left(S - \frac{\Gamma+\gamma}{2}1_H\right)\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)\right|\right) \\
&\le \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right),
\end{aligned} \tag{11}$$

which proves the first part of (9).

By Schwarz inequality for trace we also have

$$\begin{aligned}
&\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right) \\
&\le \left[\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)^2\right)\right]^{1/2} \\
&= \left[\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}.
\end{aligned} \tag{12}$$

From (11) and (12) we get

$$\begin{aligned}
&\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2 \\
&\le \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2},
\end{aligned}$$

which implies that

$$\left[\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \le \frac{1}{2}(\Gamma-\gamma).$$

By (12) we then obtain

$$\begin{aligned}
&\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(P\left|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right) \\
&\le \left[\frac{\operatorname{tr}(PS^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\le \frac{1}{2}(\Gamma-\gamma)
\end{aligned}$$

that proves the last part of (9). □

We denote by $\mathcal{M}^{-1}$ the class of all invertible matrices $n \times n$ with complex entries. The following simple fact also holds, see [25]:

Lemma 2.2. *Let $T, V \in \mathcal{M}^{-1}$ and $0 < m < M < \infty$. Then the following statements are equivalent:*
(i) The inequality

$$m \|Tx\| \leq \|Vx\| \leq M \|Tx\| \tag{13}$$

holds for any $x \in \mathbb{C}^n$;
(ii) We have the operator inequality

$$m 1_H \leq \left|VT^{-1}\right| \leq M 1_H. \tag{14}$$

Corollary 2.2. Let $T, V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \leq 1 \leq M < \infty$ such that either (13), or, equivalently (14) is valid. Then

$$\begin{aligned} 0 &\leq \chi_2^2(V,T) \\ &\leq \frac{1}{2}\left(M^2 - m^2\right) D_V(V,T) \\ &\leq \frac{1}{2}\left(M^2 - m^2\right) \chi_2(V,T) \\ &\leq \frac{1}{4}\left(M^2 - m^2\right)^2. \end{aligned} \tag{15}$$

Proof. We write the inequality (9) for $P = |T^*|^2$, $S = \left|VT^{-1}\right|^2$, $\gamma = m^2$ and $\Gamma = M^2$ to get

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^4\right) - \left(\operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^2\right)\right)^2 \\ &\leq \frac{1}{2}\left(M^2 - m^2\right) \operatorname{tr}\left(|T^*|^2 \left|\left|VT^{-1}\right|^2 - \operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^2\right) 1_H\right|\right) \\ &\leq \frac{1}{2}\left(M^2 - m^2\right) \left[\operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^4\right) - \left(\operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^2\right)\right)^2\right]^{1/2} \\ &\leq \frac{1}{4}\left(M^2 - m^2\right)^2. \end{aligned} \tag{16}$$

Since

$$\operatorname{tr}\left(|T^*|^2 \left|VT^{-1}\right|^2\right) = \operatorname{tr}\left(TV^*VT^{-1}\right) = \operatorname{tr}\left(V^*V\right) = 1,$$

hence (16) can be written as

$$
\begin{aligned}
0 &\le \operatorname{tr}\left(|T^*|^2\left|VT^{-1}\right|^4\right)-1\\
&\le \frac{1}{2}\left(M^2-m^2\right)\operatorname{tr}\left(|T^*|^2\left|\left|VT^{-1}\right|^2-1_H\right|\right)\\
&\le \frac{1}{2}\left(M^2-m^2\right)\left[\operatorname{tr}\left(|T^*|^2\left|VT^{-1}\right|^4\right)-1\right]^{1/2}\\
&\le \frac{1}{4}\left(M^2-m^2\right)^2,
\end{aligned}
$$

which is equivalent to the desired result (15). □

The following result provides a simple upper bound for the quantum f-divergence $\mathcal{S}_f(V,T)$.

Theorem 2.3. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$. If T, $V\in\mathcal{M}^{-1}\cap\mathcal{S}_2(\mathcal{M})$ and $0<m\le 1\le M<\infty$ such that either (13), or, equivalently (14) is valid, then we have*

$$
\begin{aligned}
0 &\le \mathcal{S}_f(V,T)\\
&\le \frac{1}{2}\left[f'_-\left(M^2\right)-f'_+\left(m^2\right)\right]D_V(V,T) \qquad (17)\\
&\le \frac{1}{2}\left[f'_-\left(M^2\right)-f'_+\left(m^2\right)\right]\chi_2(V,T)\\
&\le \frac{1}{4}\left(M^2-m^2\right)\left[f'_-\left(M^2\right)-f'_+\left(m^2\right)\right].
\end{aligned}
$$

Proof. Without loosing the generality, we prove the inequality in the case when f is continuously differentiable on $(0,\infty)$.

We have

$$
\begin{aligned}
&\operatorname{tr}\left[|T^*|^2\left(\left|VT^{-1}\right|^2-1_H\right)\left[f'\left(\left|VT^{-1}\right|^2\right)-\lambda 1_H\right]\right] \qquad (18)\\
&=\operatorname{tr}\left[|T^*|^2\left(\left|VT^{-1}\right|^2-1_H\right)f'\left(\left|VT^{-1}\right|^2\right)\right]
\end{aligned}
$$

for any $\lambda\in\mathbb{R}$ and for any T, $V\in\mathcal{M}^{-1}\cap\mathcal{S}_2(\mathcal{M})$.

Since f' is monotonic nondecreasing on $[m^2,M^2]$, then

$$
f'_+\left(m^2\right)\le f'(x)\le f'_-\left(M^2\right) \text{ for any } x\in\left[m^2,M^2\right].
$$

This implies that

$$
\left|f'(x)-\frac{f'_-(M^2)+f'_+(m^2)}{2}\right|\le\frac{1}{2}\left[f'_-\left(M^2\right)-f'_+\left(m^2\right)\right]
$$

for any $x \in [m^2, M^2]$, therefore by using the continuous functional calculus for the selfadjoint matrix $|VT^{-1}|^2$ with $m^2 1_H \le |VT^{-1}|^2 \le M^2 1_H$, we have

$$\left| f'\left(|VT^{-1}|^2\right) - \frac{f'_-(M^2) + f'_+(m^2)}{2} 1_H \right| \le \frac{1}{2}\left[f'_-(M^2) - f'_+(m^2)\right] 1_H. \tag{19}$$

From (3), (18), (19) and properties of trace, we have

$$\begin{aligned}
0 &\le \operatorname{tr}\left[|T^*|^2 f\left(|VT^{-1}|^2\right)\right] \le \operatorname{tr}\left[|T^*|^2\left(|VT^{-1}|^2 - 1_H\right) f'\left(|VT^{-1}|^2\right)\right] \\
&= \operatorname{tr}\left[|T^*|^2\left(|VT^{-1}|^2 - 1_H\right)\left[f'\left(|VT^{-1}|^2\right) - \frac{f'_-(M^2) + f'_+(m^2)}{2} 1_H\right]\right] \\
&= \left|\operatorname{tr}\left[|T^*|^2\left(|VT^{-1}|^2 - 1_H\right)\left[f'\left(|VT^{-1}|^2\right) - \frac{f'_-(M^2) + f'_+(m^2)}{2} 1_H\right]\right]\right| \\
&\le \operatorname{tr}\left[|T^*|^2\left|\left(|VT^{-1}|^2 - 1_H\right)\left[f'\left(|VT^{-1}|^2\right) - \frac{f'_-(M^2) + f'_+(m^2)}{2} 1_H\right]\right|\right] \\
&\le \frac{1}{2}\left[f'_-(M^2) - f'_+(m^2)\right] \operatorname{tr}\left[|T^*|^2\left||VT^{-1}|^2 - 1_H\right|\right] \\
&= \frac{1}{2}\left[f'_-(M^2) - f'_+(m^2)\right] D_V(V,T),
\end{aligned}$$

which proves the first inequality in (17).

The rest follows by (15). □

Example 1. Let T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid.

1) If we take $f(t) = -\ln t$, $t > 0$ in Theorem 2.3, then we get

$$\begin{aligned}
0 &\le \tilde{D}_{KL}(V,T) \le \frac{M^2 - m^2}{2m^2 M^2} D_V(V,T) \\
&\le \frac{M^2 - m^2}{2m^2 M^2} \chi_2(V,T) \le \frac{(M^2 - m^2)^2}{4m^2 M^2}.
\end{aligned} \tag{20}$$

2) If we take $f(t) = t \ln t$, $t > 0$ in Theorem 2.3, then we get

$$\begin{aligned}
0 &\le D_{KL}(V,T) \le \ln\left(\frac{M}{m}\right) D_V(V,T) \\
&\le \ln\left(\frac{M}{m}\right) \chi_2(V,T) \le \frac{1}{2}\left(M^2 - m^2\right) \ln\left(\frac{M}{m}\right).
\end{aligned} \tag{21}$$

3) If we take in (17) $f(t) = f_q(t) = \frac{1-t^q}{1-q}$, then we get

$$
\begin{aligned}
0 &\le D_{f_q}(V,T) \le \frac{q}{2(1-q)} \left(\frac{M^{2(1-q)} - m^{2(1-q)}}{M^{2(1-q)} m^{2(1-q)}} \right) D_V(V,T) \\
&\le \frac{q}{2(1-q)} \left(\frac{M^{2(1-q)} - m^{2(1-q)}}{M^{2(1-q)} m^{2(1-q)}} \right) \chi_2(V,T) \\
&\le \frac{q}{4(1-q)} \left(\frac{M^{2(1-q)} - m^{2(1-q)}}{M^{2(1-q)} m^{2(1-q)}} \right) \left(M^2 - m^2\right).
\end{aligned} \tag{22}
$$

3. Some related inequalities

We have the following upper bound as well:

Theorem 3.1. *Let f be a continuous convex function on $[0,\infty)$ with $f(1) = 0$. If T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid, then we have*

$$
0 \le \mathcal{S}_f(V,T) \le \frac{(M^2-1) f(m^2) + (1-m^2) f(M^2)}{M^2 - m^2}. \tag{1}
$$

Proof. By the convexity of f we have

$$
\begin{aligned}
f(t) &= f\left(\frac{(M^2-t) m^2 + (t-m^2) M^2}{M^2-m^2} \right) \\
&\le \frac{(M^2-t) f(m^2) + (t-m^2) f(M^2)}{M^2-m^2}
\end{aligned}
$$

for any $t \in [m^2, M^2]$.

This inequality implies the following inequality in the operator order of $\mathcal{B}(H)$

$$
f\left(\left|VT^{-1}\right|^2\right) \le \frac{\left(M^2 - \left|VT^{-1}\right|^2\right) f(m^2) + \left(\left|VT^{-1}\right|^2 - m^2\right) f(M^2)}{M^2-m^2}, \tag{2}
$$

for any T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that the condition (13) is satisfied.

Utilising the property of trace we get from (2) that

$$\begin{aligned} \operatorname{tr}\left[\left|T^*\right|^2 f\left(\left|VT^{-1}\right|^2\right)\right] &\le \frac{f\left(m^2\right)}{M^2-m^2} \operatorname{tr}\left[\left|T^*\right|^2\left(M^2 1_H-\left|VT^{-1}\right|^2\right)\right] \\ &+ \frac{f\left(M^2\right)}{M^2-m^2} \operatorname{tr}\left[\left|T^*\right|^2\left(\left|VT^{-1}\right|^2-m^2 1_H\right)\right] \\ &= \frac{f\left(m^2\right)}{M^2-m^2}\left(M^2 \operatorname{tr}\left(\left|T^*\right|^2\right)-\operatorname{tr}\left(\left|T^*\right|^2\left|VT^{-1}\right|^2\right)\right) \\ &+ \frac{f\left(M^2\right)}{M^2-m^2}\left(\operatorname{tr}\left(\left|T^*\right|^2\left|VT^{-1}\right|^2\right)-m^2 \operatorname{tr}\left(\left|T^*\right|^2\right)\right) \\ &= \frac{\left(M^2-1\right) f\left(m^2\right)+\left(1-m^2\right) f\left(M^2\right)}{M^2-m^2}, \end{aligned} \tag{3}$$

and the inequality (1) is thus proved. □

Example 2. Let $T, V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid.

1) If we take in (1) $f(t) = t^2 - 1$, then we get

$$0 \le \chi_2^2(V,T) \le \left(M^2-1\right)\left(1-m^2\right) \frac{M^2+m^2+2}{M^2-m^2}. \tag{4}$$

2) If we take in (1) $f(t) = t \ln t$, then we get the inequality

$$0 \le D_{KL}(V,T) \le 2 \ln\left[m^{\frac{\left(M^2-1\right)m^2}{M^2-m^2}} M^{\frac{M^2\left(1-m^2\right)}{M^2-m^2}}\right]. \tag{5}$$

3) If we take in (1) $f(t) = -\ln t$, then we get the inequality

$$0 \le \tilde{D}_{KL}(V,T) \le 2 \ln\left[m^{\frac{1-M^2}{M^2-m^2}} M^{\frac{m^2-1}{M^2-m^2}}\right]. \tag{6}$$

We have the following upper bounds as well:

Theorem 3.2. *With the assumptions of Theorem 3.1, the following inequalities hold:*

$$\begin{aligned} (0 \le)\, \mathcal{S}_f(V,T) &\le \frac{\left(M^2-1\right)\left(1-m^2\right)}{M^2-m^2} \Psi_f\left(1; m^2, M^2\right) \\ &\le \frac{\left(M^2-1\right)\left(1-m^2\right)}{M^2-m^2} \sup_{t \in (m^2, M^2)} \Psi_f\left(t; m^2, M^2\right) \\ &\le \left(M^2-1\right)\left(1-m^2\right) \frac{f'_-\left(M^2\right)-f'_+\left(m^2\right)}{M^2-m^2} \\ &\le \frac{1}{4}\left(M^2-m^2\right)\left[f'_-\left(M^2\right)-f'_+\left(m^2\right)\right], \end{aligned} \tag{7}$$

where $\Psi_f(\cdot; m^2, M^2) : (m^2, M^2) \to \mathbb{R}$ *is defined by*

$$\Psi_f\left(t; m^2, M^2\right) = \frac{f(M^2) - f(t)}{M^2 - t} - \frac{f(t) - f(m^2)}{t - m^2}. \tag{8}$$

We also have

$$\begin{aligned}(0 \leq) \mathcal{S}_f(V, T) &\leq \frac{(M^2-1)(1-m^2)}{M^2-m^2} \Psi_f\left(1; m^2, M^2\right) \\ &\leq \frac{1}{4}\left(M^2 - m^2\right) \Psi_f\left(1; m^2, M^2\right) \\ &\leq \frac{1}{4}\left(M^2 - m^2\right) \sup_{t \in (m^2, M^2)} \Psi_f\left(t; m^2, M^2\right) \\ &\leq \frac{1}{4}\left(M^2 - m^2\right)\left[f'_-\left(M^2\right) - f'_+\left(m^2\right)\right].\end{aligned} \tag{9}$$

Proof. By denoting

$$\Delta_f\left(t; m^2, M^2\right) := \frac{(t - m^2) f(M^2) + (M^2 - t) f(m^2)}{M^2 - m^2} - f(t), \quad t \in \left[m^2, M^2\right]$$

we have

$$\begin{aligned}&\Delta_f\left(t; m^2, M^2\right) \\ &= \frac{(t-m^2) f(M^2) + (M^2 - t) f(m^2) - (M^2 - m^2) f(t)}{M^2 - m^2} \\ &= \frac{(t - m^2)\left[f(M^2) - f(t)\right] - (M^2 - t)\left[f(t) - f(m^2)\right]}{M^2 - m^2} \\ &= \frac{(M^2 - t)(t - m^2)}{M^2 - m^2} \Psi_f\left(t; m^2, M^2\right)\end{aligned} \tag{10}$$

for any $t \in (m^2, M^2)$.

From the proof of Theorem 3.1 and since $f(1) = 0$, we have

$$\begin{aligned}\operatorname{tr}\left[|T^*|^2 f\left(\left|VT^{-1}\right|^2\right)\right] &\leq \frac{(M^2-1) f(m^2) + (1-m^2) f(M^2)}{M^2 - m^2} - f(1) \\ &= \frac{(M^2-1)(1-m^2)}{M^2-m^2} \Psi_f\left(1; m^2, M^2\right)\end{aligned}$$

for any T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$, such that (13) is valid.

Since

$$
\begin{aligned}
&\Psi_f\left(1; m^2, M^2\right) \qquad (11)\\
&\le \sup_{t\in(m^2,M^2)} \Psi_f\left(t; m^2, M^2\right)\\
&= \sup_{t\in(m^2,M^2)} \left[\frac{f\left(M^2\right)-f\left(t\right)}{M^2-t} - \frac{f\left(t\right)-f\left(m^2\right)}{t-m^2}\right]\\
&\le \sup_{t\in(m^2,M^2)} \left[\frac{f\left(M^2\right)-f\left(t\right)}{M^2-t}\right] + \sup_{t\in(m^2,M^2)} \left[-\frac{f\left(t\right)-f\left(m^2\right)}{t-m^2}\right]\\
&= \sup_{t\in(m^2,M^2)} \left[\frac{f\left(M^2\right)-f\left(t\right)}{M^2-t}\right] - \inf_{t\in(m^2,M^2)} \left[\frac{f\left(t\right)-f\left(m^2\right)}{t-m^2}\right]\\
&= f'_-\left(M^2\right) - f'_+\left(m^2\right),
\end{aligned}
$$

and, obviously

$$
\frac{1}{M^2-m^2}\left(M^2-1\right)\left(1-m^2\right) \le \frac{1}{4}\left(M^2-m^2\right), \tag{12}
$$

then by (10)–(12) we have the desired result (7).

The rest is obvious. □

Example 3. Let $T,\ V \in \mathcal{M}^{-1} \cap \mathcal{S}_2\left(\mathcal{M}\right)$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid.

1) If we consider the convex normalized function $f\left(t\right) = t^2 - 1$, then

$$
\Psi_f\left(t; m^2, M^2\right) = \frac{M^4-t^2}{M^2-t} - \frac{t^2-m^4}{t-m^2} = M^2 - m^2,\ t \in \left(m^2, M^2\right)
$$

and we get from (7) the simple inequality

$$
0 \le \chi_2^2\left(V, T\right) \le \left(M^2-1\right)\left(1-m^2\right). \tag{13}
$$

This inequality is better than (4).

2) If we take the convex normalized function $f\left(t\right) = t^{-1} - 1$, then we have

$$
\Psi_f\left(t; m^2, M^2\right) = \frac{M^{-2}-t^{-1}}{M^2-t} - \frac{t^{-1}-m^{-2}}{t-m^2} = \frac{M^2-m^2}{m^2M^2t},\ t \in \left[m^2, M^2\right].
$$

Also

$$
S_f\left(V, T\right) = \chi_2^2\left(T, V\right).
$$

Using (7) we get

$$
\left(0 \le\right) \chi_2^2\left(T, V\right) \le \frac{\left(M^2-1\right)\left(1-m^2\right)}{M^2m^2}. \tag{14}
$$

3) If we consider the convex function $f(t) = -\ln t$ defined on $[m^2, M^2] \subset (0,\infty)$, then

$$\Psi_f\left(t; m^2, M^2\right) = \frac{-\ln M^2 + \ln t}{M^2 - t} - \frac{-\ln t + \ln m^2}{t - m^2}$$
$$= \ln\left(\frac{t^{M^2-m^2}}{m^{2(M^2-t)} M^{2(t-m^2)}}\right)^{\frac{1}{(M^2-t)(t-m^2)}}, \ t \in \left(m^2, M^2\right).$$

Then by (7) we have

$$(0 \le)\, \tilde{D}_{KL}(V,T) \le 2\ln\left[m^{\frac{1-M^2}{M^2-m^2}} M^{\frac{m^2-1}{M^2-m^2}}\right] \le \frac{\left(M^2-1\right)\left(1-m^2\right)}{m^2 M^2}. \tag{15}$$

4) If we consider the convex function $f(t) = t\ln t$ defined on $[m^2, M^2] \subset (0,\infty)$, then

$$\Psi_f\left(t; m^2, M^2\right) = \frac{M^2 \ln M^2 - t\ln t}{M^2 - t} - \frac{t\ln t - m^2 \ln m^2}{t - m^2}, \ t \in \left(m^2, M^2\right),$$

which gives that

$$\Psi_f\left(1; m^2, M^2\right) = \frac{\ln\left[\left(M^2\right)^{M^2\left(1-m^2\right)}\left(m^2\right)^{m^2\left(M^2-1\right)}\right]}{\left(M^2-1\right)\left(1-m^2\right)}.$$

Using (7) we get

$$(0 \le)\, D_{KL}(V,T) \le \frac{\ln\left[\left(M^2\right)^{M^2\left(1-m^2\right)}\left(m^2\right)^{m^2\left(M^2-1\right)}\right]}{M^2 - m^2} \tag{16}$$
$$\le 2\left(M^2-1\right)\left(1-m^2\right)\ln\left[\left(\frac{M}{m}\right)^{\frac{1}{M^2-m^2}}\right].$$

Finally, we have:

Theorem 3.3. *Let f be a continuous convex function on $[0,\infty)$ with $f(1) = 0$. If T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid, then we have*

$$0 \le \mathcal{S}_f(V,T) \tag{17}$$
$$\le 2\max\left\{\frac{M^2-1}{M^2-m^2}, \frac{1-m^2}{M^2-m^2}\right\}\left[\frac{f\left(m^2\right)+f\left(M^2\right)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right]$$
$$\le 2\left[\frac{f\left(m^2\right)+f\left(M^2\right)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right].$$

Proof. We recall the following result (see for instance [11]) that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$n \min_{i\in\{1,\dots,n\}} \{p_i\} \left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \tag{18}$$
$$\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$
$$\leq n \max_{i\in\{1,\dots,n\}} \{p_i\} \left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right],$$

where $f : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (18) that

$$2\min\{s, 1-s\} \left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \tag{19}$$
$$\leq s f(x) + (1-s) f(y) - f(sx + (1-s) y)$$
$$\leq 2\max\{s, 1-s\} \left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]$$

for any $x, y \in C$ and $s \in [0,1]$.

Now, if we use the second inequality in (19) for $x = m^2$, $y = M^2$, $s = \frac{M^2-t}{M^2-m^2}$ with $t \in [m^2, M^2]$, then we have

$$\frac{(M^2-t) f(m^2) + (t-m^2) f(M^2)}{M^2-m^2} - f(t) \tag{20}$$
$$\leq 2\max\left\{\frac{M^2-t}{M^2-m^2}, \frac{t-m^2}{M^2-m^2}\right\}$$
$$\times \left[\frac{f(m^2)+f(M^2)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right]$$
$$\leq 2\left[\frac{f(m^2)+f(M^2)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right]$$

for any $t \in [m^2, M^2]$.

This implies that

$$\begin{aligned}
&\operatorname{tr}\left[|T^*|^2 f\left(\left|VT^{-1}\right|^2\right)\right] \\
&\le \frac{(M^2-1)\, f(m^2) + (1-m^2)\, f(M^2)}{M^2-m^2} \\
&\le 2\max\left\{\frac{M^2-1}{M^2-m^2}, \frac{1-m^2}{M^2-m^2}\right\}\left[\frac{f(m^2)+f(M^2)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right] \\
&\le 2\left[\frac{f(m^2)+f(M^2)}{2} - f\left(\frac{m^2+M^2}{2}\right)\right]
\end{aligned}$$

and the proof is completed. □

Example 4. Let T, $V \in \mathcal{M}^{-1} \cap \mathcal{S}_2(\mathcal{M})$ and $0 < m \le 1 \le M < \infty$ such that either (13), or, equivalently (14) is valid.

1) If we take in (17) $f(t) = t^{-1} - 1$, then we have

$$0 \le \chi_2^2(T,V) \le \max\left\{M^2-1, 1-m^2\right\} \frac{M^2-m^2}{m^2M^2(m^2+M^2)}. \tag{21}$$

2) If we take in (17) $f(t) = -\ln t$, then we have

$$\begin{aligned}
0 \le \tilde{D}_{KL}(V,T) &\le \max\left\{\frac{M^2-1}{M^2-m^2}, \frac{1-m^2}{M^2-m^2}\right\} \ln\left(\frac{(M^2+m^2)^2}{4m^2M^2}\right) \qquad (22)\\
&\le \ln\left(\frac{(M^2+m^2)^2}{4m^2M^2}\right).
\end{aligned}$$

3) From (20) we have the following upper bound

$$0 \le \tilde{D}_{KL}(V,T) \le \frac{(M^2-m^2)^2}{4m^2M^2}. \tag{23}$$

Utilising the elementary inequality $\ln x \le x - 1$, $x > 0$, we have that

$$\ln\left(\frac{(M^2+m^2)^2}{4m^2M^2}\right) \le \frac{(M^2-m^2)^2}{4m^2M^2},$$

which shows that (22) is better than (23).

Acknowledgement. The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the paper.

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Received October 13, 2016
Revised February 15, 2018