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THE VON NEUMANN-JORDAN CONSTANT OF $\pi/2$-ROTATION INVARIANT NORMS ON $\mathbb{R}^2$

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ABSTRACT. In this paper, we study the von Neumann-Jordan constant of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$. We give some estimations of the constant and have a relationship between the constant and a ratio of two certain functions. These results are an extension of existing results of a unitary version of the von Neumann-Jordan constant.

1. Introduction and preliminaries

This paper is concerned with the *von Neumann Jordan constant* of Banach spaces. For a Banach space X, let B_X and S_X be the unit ball and unit sphere, respectively. In connection with the famous work [4] of Jordan and von Neumann concerning inner products, the von Neumann Jordan constant $C_{NJ}(X)$ of X was introduced by Clarkson in [3] as follows:

$$C_{NJ}(X) := \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x, y \in X,\ (x,y) \neq (0,0)\right\}.$$

The constant $C_{NJ}(X)$ can be viewed as a measure of the distortion of B_X from the viewpoint of the parallelogram law. An estimation $1 \leq C_{NJ}(X) \leq 2$ holds for any X. It is known that $C_{NJ}(X) = 1$ if and only if X is a Hilbert space ([4]), and $C_{NJ}(X) < 2$ if and only if X is uniformly nonsquare ([7]). So far many papers were devoted to studying von Neumann-Jordan constant of Banach spaces; see, e.g., [1, 3, 6, 8, 10].

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(a,b)\| = \|(|a|,|b|)\|$ for each $(a,b) \in \mathbb{R}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Typical examples of such norms are the ℓ_p-norms $\|\cdot\|_p$ given by

$$\|(a,b)\|_p := \begin{cases} (|a|^p+|b|^p)^{1/p} & (1 \leq p < \infty) \\ \max\{|a|,|b|\} & (p = \infty). \end{cases}$$

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Let AN_2 be a collection of all absolute normalized norms on $\mathbb{R}^2$. Let Ψ_2 be a family of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for each $t \in [0,1]$. As was shown in [2] and [6], AN_2 is in a one-to-one correspondence with Ψ_2 under an equation $\psi(t) = \|(1-t, t)\|$ for each $t \in [0,1]$. An absolute normalized norm corresponding to $\psi \in \Psi_2$ is denoted by $\|\cdot\|_\psi$; and it satisfies the following equation:

$$\|(a,b)\|_\psi := \begin{cases} (|a|+|b|)\psi\left(\dfrac{|b|}{|a|+|b|}\right) & \big((a,b) \neq (0,0)\big) \\ 0 & \big((a,b) = (0,0)\big). \end{cases}$$

Moreover, a convex function ψ_2 corresponding to the Euclidean norm $\|\cdot\|_2$ is given by

$$\psi_2(t) := \big((1-t)^2 + t^2\big)^{1/2}.$$

It should be noted that $\psi_2(t) = \psi_2(1-t)$ for each $t \in [0,1]$. Furthermore, a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be $\pi/2$-*rotation invariant* if the $\pi/2$-rotation matrix

$$R(\pi/2) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is an isometry on $(\mathbb{R}^2, \|\cdot\|)$, or equivalently, $\|(a,b)\| = \|(-b,a)\|$ for each $(a,b) \in \mathbb{R}^2$.

The purpose of this paper is to study the von Neumann-Jordan constant of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$. Let $\widetilde{\psi}$ be an element of Ψ_2 defined by $\widetilde{\psi}(t) = \psi(1-t)$ for each $\psi \in \Psi_2$, and $\ell^2_{\psi,\widetilde{\psi}}$ the space $\mathbb{R}^2$ endowed with the norm

$$\|(a,b)\|_{\psi,\widetilde{\psi}} := \begin{cases} (|a|+|b|)\psi\left(\dfrac{|b|}{|a|+|b|}\right) & (ab \geq 0) \\ (|a|+|b|)\widetilde{\psi}\left(\dfrac{|b|}{|a|+|b|}\right) & (ab \leq 0). \end{cases}$$

In [5, Theorem 3.2], it was shown that any $\pi/2$-rotation invariant normed space is isometrically isomorphic to some Day-James space of the form $\ell^2_{\psi,\widetilde{\psi}}$. The norm $\|\cdot\|_{\psi,\widetilde{\psi}}$ is also $\pi/2$-rotation invariant for each $\psi \in \Psi_2$ ([5, Proposition 3.4]). Moreover, the von Neumann-Jordan constant is invariant under isometric isomorphisms. Hence for our purpose, it is enough to consider Day-James spaces of the form $\ell^2_{\psi,\widetilde{\psi}}$; and throughout this paper, $\pi/2$-rotation invariant normed spaces are assumed to be $\ell^2_{\psi,\widetilde{\psi}}$ for some $\psi \in \Psi_2$. Henceforth, fix an element ψ in Ψ_2 with $\psi \neq \psi_2$, put the norm $\|\cdot\| = \|\cdot\|_{\psi,\widetilde{\psi}}$ for short, and the space $\ell^2_{\psi,\widetilde{\psi}}$ will be simply denoted by Y_ψ. Under this hypothesis, we obtain some estimations of the von Neumann-Jordan constant.

In the second section, we present keys to the proofs of the von Neumann-Jordan constant in Day-James spaces having relation to $\pi/2$-rotation invariant norms. In the third section, using the keys to the proofs, we get a relationship between the constant and the ratio of two certain functions. These results are an extension of existing results of the unitary version of the von Neumann-Jordan constant ([9]).

2. Auxiliary results on Y_ψ

In this section, we present keys to the proofs of our results in the next section.

Theorem 2.1. *Let $a, b > 0$ and $c \in (0, 1]$. Then the following two statements are equivalent:*

(I) *There exists a pair $x, y \in S_{Y_\psi}$ with $x + cy \neq 0$ satisfying $\|x\|_2 = \|y\|_2 = 1/a$ and $\|x + cy\| = b\|x + cy\|_2$.*

(II) *There exist $r, s, t \in [0, 1]$ such that $\psi(s) = a\psi_2(s)$, $\psi(t) = a\psi_2(t)$, and $\psi(r) = b\psi_2(r)$, where r, s, t satisfy one of the following conditions:*

(a) $r = \dfrac{s\psi(t) + ct\psi(s)}{\psi(t) + c\psi(s)}$.

(b) $\dfrac{s}{\psi(s)} > \dfrac{c(1-t)}{\psi(t)}$ *and* $r = \dfrac{s\psi(t) + c(t-1)\psi(s)}{\psi(t) + c(2t-1)\psi(s)}$.

(c) $\dfrac{s}{\psi(s)} \leq \dfrac{c(1-t)}{\psi(t)}$ *and* $r = \dfrac{(1-s)\psi(t) + ct\psi(s)}{(1-2s)\psi(t) + c\psi(s)}$.

Proof. Suppose that (I) holds. Let x, y be elements of S_{Y_ψ} having the properties set out in (I). Since $R(\pi/2)$ is an isometric isomorphism on Y_ψ, by the definition of the norm of Y_ψ, we have only to consider two kinds of pairs x, y: one pair which x and y use the same norm while another which x and y both use different norms. Thus, we may assume that x is in the first quadrant. The argument separates into two parts according to the position of y.

Case 1: Suppose that both x, y are in the first quadrant. Thus

$$x = \frac{1}{\psi(s)}(1-s, s) \quad \text{and} \quad y = \frac{1}{\psi(t)}(1-t, t) \tag{2.1}$$

for some $s, t \in [0, 1]$. By (I) and the definition of $\|\cdot\|_2$, we obtain $1/a = \|x\|_2 = \psi_2(s)/\psi(s)$ and $1/a = \|y\|_2 = \psi_2(t)/\psi(t)$. Thus we have $\psi(s) = a\psi_2(s)$ and $\psi(t) = a\psi_2(t)$. Next we obtain

$$x + cy = \left(\frac{1-s}{\psi(s)} + c\frac{1-t}{\psi(t)}, \frac{s}{\psi(s)} + c\frac{t}{\psi(t)}\right).$$

It is clear that $x+cy \neq 0$ for all $s,t \in [0,1]$ and $c \in (0,1]$. We have

$$\begin{aligned}\frac{\psi(t)+c\psi(s)}{\psi(s)\psi(t)}\psi(r) &= \left(\left| \frac{1-s}{\psi(s)} + c\frac{1-t}{\psi(t)} \right| + \left| \frac{s}{\psi(s)} + c\frac{t}{\psi(t)} \right| \right) \psi(r) \\ &= \|x+cy\| = b\|x+cy\|_2 = b\frac{\psi(t)+c\psi(s)}{\psi(s)\psi(t)}\psi_2(r),\end{aligned}$$

where r is given by the equation set out in (a). Hence $\psi(r) = b\psi_2(r)$.

Case 2: Suppose that y is in the fourth quadrant. Thus

$$y = \frac{1}{\psi(t)}\big(t, -(1-t)\big) \tag{2.2}$$

for some $t \in [0,1]$ (and x has the same form as (2.1) in Case 1). As in the preceding paragraph, it follows that $\psi(t) = a\psi_2(t)$ since $1/a = \|y\|_2 = \psi_2(1-t)/\psi(t) = \psi_2(t)/\psi(t)$. We obtain

$$x+cy = \left(\frac{1-s}{\psi(s)} + c\frac{t}{\psi(t)}, \frac{s}{\psi(s)} - c\frac{1-t}{\psi(t)} \right).$$

We note that $x+cy \neq 0$. Now, we put

$$\begin{aligned}&\left| \frac{s}{\psi(s)} - c\frac{1-t}{\psi(t)} \right| \left(\frac{1-s}{\psi(s)} + c\frac{t}{\psi(t)} + \left| \frac{s}{\psi(s)} - c\frac{1-t}{\psi(t)} \right| \right)^{-1} \\ &= \begin{cases} \dfrac{s\psi(t)+c(t-1)\psi(s)}{\psi(t)+c(2t-1)\psi(s)} =: r_1 & \left(\dfrac{s}{\psi(s)} > \dfrac{c(1-t)}{\psi(t)} \right) \quad \cdots \text{(i)} \\ \dfrac{-s\psi(t)+c(1-t)\psi(s)}{(1-2s)\psi(t)+c\psi(s)} =: r_2 & \left(\dfrac{s}{\psi(s)} \leq \dfrac{c(1-t)}{\psi(t)} \right). \quad \cdots \text{(ii)} \end{cases}\end{aligned} \tag{2.3}$$

In the case of $c=1$, it must be $(s,t) \neq (1,0)$, but it can be $(s,t) = (0,1)$. Thus the magnitude correlation of s and t is divided into two cases in (2.3).

In the case of (2.3)(i), $x+cy$ is in the first quadrant. We have

$$\begin{aligned}\frac{\psi(t)+c(2t-1)\psi(s)}{\psi(s)\psi(t)}\psi(r_1) &= \left(\left| \frac{1-s}{\psi(s)} + c\frac{t}{\psi(t)} \right| + \left| \frac{s}{\psi(s)} - c\frac{1-t}{\psi(t)} \right| \right) \psi(r_1) \\ &= \|x+cy\| = b\|x+cy\|_2 = b\frac{\psi(t)+c(2t-1)\psi(s)}{\psi(s)\psi(t)}\psi_2(r_1).\end{aligned}$$

Hence $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (b). In the case of (2.3)(ii), $x+cy$ is in the fourth quadrant. We put

$$1-r_2 = 1 - \frac{-s\psi(t)+c(1-t)\psi(s)}{(1-2s)\psi(t)+c\psi(s)} = \frac{(1-s)\psi(t)+ct\psi(s)}{(1-2s)\psi(t)+c\psi(s)} =: r_2'.$$

Since $\widetilde{\psi}(t) = \psi(1-t)$ for all $t \in [0,1]$, we have

$$\begin{aligned}\frac{(1-2s)\psi(t)+c\psi(s)}{\psi(s)\psi(t)}\psi(r_2') &= \left(\left| \frac{1-s}{\psi(s)} + c\frac{t}{\psi(t)} \right| + \left| \frac{s}{\psi(s)} - c\frac{1-t}{\psi(t)} \right| \right) \widetilde{\psi}(r_2) \\ &= \|x+cy\| = b\|x+cy\|_2 = b\frac{(1-2s)\psi(t)+c\psi(s)}{\psi(s)\psi(t)}\psi_2(r_2').\end{aligned}$$

Hence $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (c). This completes the proof of (I) $\Rightarrow$ (II).

For the converse, let r, s, t be elements of $[0,1]$ satisfying one of the three conditions set out in (II). If r, s, t satisfy (a), then the elements $x = \psi(s)^{-1}(1-s, s)$ and $y = \psi(t)^{-1}(1-t, t)$ have the desired properties. Similarly, in the cases of (b) and (c), it is enough to consider $x = \psi(s)^{-1}(1-s, s)$ and $y = \psi(t)^{-1}\big(t, -(1-t)\big)$. The proof is complete. $\square$

Theorem 2.2. *Let $a, b > 0$ and $c \in (0,1]$. Then the following two statements are equivalent:*

(I) *There exists a pair $x, y \in S_{Y_\psi}$ with $x - cy \neq 0$ satisfying $\|x\|_2 = \|y\|_2 = 1/a$ and $\|x - cy\| = b\|x - cy\|_2$.*

(II) *There exist $r, s, t \in [0,1]$ such that $\psi(s) = a\psi_2(s)$, $\psi(t) = a\psi_2(t)$, and $\psi(r) = b\psi_2(r)$, where r, s, t satisfy one of the following conditions:*

(a1) $$\frac{1-s}{\psi(s)} \geq \frac{c(1-t)}{\psi(t)}, \quad \frac{s}{\psi(s)} \geq \frac{ct}{\psi(t)}, \quad \textit{and} \quad r = \frac{s\psi(t) - ct\psi(s)}{\psi(t) - c\psi(s)}.$$

(a2) $r = \dfrac{(1-s)\psi(t) + c(t-1)\psi(s)}{(1-2s)\psi(t) + c(2t-1)\psi(s)}$ *satisfying one of the following conditions:*

(1) $$\frac{1-s}{\psi(s)} > \frac{c(1-t)}{\psi(t)} \quad \textit{and} \quad \frac{s}{\psi(s)} < \frac{ct}{\psi(t)}.$$

(2) $$\frac{1-s}{\psi(s)} < \frac{c(1-t)}{\psi(t)} \quad \textit{and} \quad \frac{s}{\psi(s)} > \frac{ct}{\psi(t)}.$$

(b) $$\frac{1-s}{\psi(s)} \leq \frac{ct}{\psi(t)} \quad \textit{and} \quad r = \frac{(s-1)\psi(t) + ct\psi(s)}{(2s-1)\psi(t) + c\psi(s)}.$$

(c) $$\frac{1-s}{\psi(s)} > \frac{ct}{\psi(t)} \quad \textit{and} \quad r = \frac{s\psi(t) + c(1-t)\psi(s)}{\psi(t) + c(1-2t)\psi(s)}.$$

Proof. Suppose that (I) holds. Let x, y be elements of S_{Y_ψ} having the properties set out in (I). Suppose that x is in the first quadrant.

Case 1: Suppose that both x, y are in the first quadrant. Thus we have (2.1) for some $s, t \in [0,1]$. By (I), we obtain $1/a = \|x\|_2 = \psi_2(s)/\psi(s)$ and $1/a = \|y\|_2 = \psi_2(t)/\psi(t)$. Thus we have $\psi(s) = a\psi_2(s)$ and $\psi(t) = a\psi_2(t)$. Next we obtain

$$x - cy = \left(\frac{1-s}{\psi(s)} - c\frac{1-t}{\psi(t)}, \frac{s}{\psi(s)} - c\frac{t}{\psi(t)}\right).$$

We note that $x - cy \neq 0$. Now, we put

$$
\begin{aligned}
&\left|\frac{s}{\psi(s)} - c\frac{t}{\psi(t)}\right| \left(\left|\frac{1-s}{\psi(s)} - c\frac{1-t}{\psi(t)}\right| + \left|\frac{s}{\psi(s)} - c\frac{t}{\psi(t)}\right|\right)^{-1} \\
&= \begin{cases}
\dfrac{s\psi(t) - ct\psi(s)}{\psi(t) - c\psi(s)} =: r_1 & \left(\dfrac{1-s}{\psi(s)} \geq \dfrac{c(1-t)}{\psi(t)} \text{ and } \dfrac{s}{\psi(s)} \geq \dfrac{ct}{\psi(t)}\right) \cdots \text{(i)} \\
\dfrac{-s\psi(t) + ct\psi(s)}{(1-2s)\psi(t) + c(2t-1)\psi(s)} =: r_2 & \left(\dfrac{1-s}{\psi(s)} > \dfrac{c(1-t)}{\psi(t)} \text{ and } \dfrac{s}{\psi(s)} < \dfrac{ct}{\psi(t)}\right) \cdots \text{(ii)} \\
\dfrac{s\psi(t) - ct\psi(s)}{(2s-1)\psi(t) + c(1-2t)\psi(s)} =: r_3 & \left(\dfrac{1-s}{\psi(s)} < \dfrac{c(1-t)}{\psi(t)} \text{ and } \dfrac{s}{\psi(s)} > \dfrac{ct}{\psi(t)}\right). \cdots \text{(iii)}
\end{cases}
\end{aligned}
\tag{2.4}
$$

We note that if $c = 1$, then we cannot get r_1. Moreover, we must have $s \neq t$ in the case of $c = 1$, but we can choose $s = t$ in the case of $c < 1$.

In the case of (2.4)(i), $x - cy$ is in the first quadrant. For r_1, an argument similar to that in Case 1 of Theorem 2.1, we have $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (a1). In the cases of (2.4)(ii) and (iii), $x - cy$ is in the fourth and second quadrant, respectively. We note that $r_2 = r_3$. We have

$$
1 - r_2 = 1 - \frac{-s\psi(t) + ct\psi(s)}{(1-2s)\psi(t) + c(2t-1)\psi(s)} = \frac{(1-s)\psi(t) + c(t-1)\psi(s)}{(1-2s)\psi(t) + c(2t-1)\psi(s)} =: r_2'.
$$

For r_2', an argument similar to that in Case 2 of Theorem 2.1 shows that $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (a2).

Case 2: Suppose that y is in the fourth quadrant. Then we have (2.2) for some $t \in [0, 1]$ (and x has the same form as in Case 1). As in the preceding paragraph, it follows that $\psi(t) = a\psi_2(t)$ since $1/a = \|y\|_2 = \psi_2(1-t)/\psi(t) = \psi_2(t)/\psi(t)$. We obtain

$$
x - cy = \left(\frac{1-s}{\psi(s)} - c\frac{t}{\psi(t)}, \frac{s}{\psi(s)} + c\frac{1-t}{\psi(t)}\right).
$$

We note that $x - cy \neq 0$. Now, we put

$$
\begin{aligned}
&\left(\frac{s}{\psi(s)} + c\frac{1-t}{\psi(t)}\right) \left(\left|\frac{1-s}{\psi(s)} - c\frac{t}{\psi(t)}\right| + \frac{s}{\psi(s)} + c\frac{1-t}{\psi(t)}\right)^{-1} \\
&= \begin{cases}
\dfrac{s\psi(t) + c(1-t)\psi(s)}{(2s-1)\psi(t) + c\psi(s)} =: r_4 & \left(\dfrac{1-s}{\psi(s)} \leq \dfrac{ct}{\psi(t)}\right) \quad \cdots \text{(i)} \\
\dfrac{s\psi(t) + c(1-t)\psi(s)}{\psi(t) + c(1-2t)\psi(s)} =: r_5 & \left(\dfrac{1-s}{\psi(s)} > \dfrac{ct}{\psi(t)}\right). \quad \cdots \text{(ii)}
\end{cases}
\end{aligned}
\tag{2.5}
$$

In the case of $c = 1$, it must be $(s, t) \neq (0, 1)$, but it can be $(s, t) = (1, 0)$. Thus the magnitude correlation of s and t is divided into two cases of (2.5)(i) and (ii).

In the case of (2.5)(i), $x - cy$ is in the second quadrant. We note that

$$
1 - r_4 = 1 - \frac{s\psi(t) + c(1-t)\psi(s)}{(2s-1)\psi(t) + c\psi(s)} = \frac{(s-1)\psi(t) + ct\psi(s)}{(2s-1)\psi(t) + c\psi(s)} =: r_4'.
$$

For r_4', an argument similar to that in Case 2 of Theorem 2.1 shows that $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (b). In the case of (2.5)(ii), $x - cy$ is in the first quadrant. For r_5, an argument similar to that in Case 1 of Theorem 2.1 shows that $\psi(r) = b\psi_2(r)$, where r is given by the equation set out in (c). This completes the proof of (I) $\Rightarrow$ (II).

For the converse, let r, s, t be elements of $[0, 1]$ satisfying one of the three conditions set out in (II). If r, s, t satisfy (a1) or (a2), then the elements $x = \psi(s)^{-1}(1 - s, s)$ and $y = \psi(t)^{-1}(1 - t, t)$ have the desired properties. Similarly, in the cases of (b) and (c), it is enough to consider $x = \psi(s)^{-1}(1 - s, s)$ and $y = \psi(t)^{-1}(t, -(1 - t))$. The proof is complete. $\square$

In this context, we have the following lemmas.

Lemma 2.3. *Let $b > 0$, $\psi(s) = b\psi_2(s)$ and $\psi(t) = b\psi_2(t)$ for $s, t \in [0, 1]$. Then*

$$\frac{s\psi(t) + t\psi(s)}{\psi(t) + \psi(s)} = \frac{(1 - s)\psi(t) + (t - 1)\psi(s)}{(1 - 2s)\psi(t) + (2t - 1)\psi(s)}.$$

Proof. We have

$$\begin{aligned}
&\{s\psi(t) + t\psi(s)\}\{(1 - 2s)\psi(t) + (2t - 1)\psi(s)\} - \{\psi(t) + \psi(s)\}\{(1 - s)\psi(t) + (t - 1)\psi(s)\}\\
&= \{t(2t - 1) - (t - 1)\}\psi(s)^2 + \{s(1 - 2s) - (1 - s)\}\psi(t)^2\\
&= \{t^2 + (t - 1)^2\}\psi(s)^2 - \{s^2 + (s - 1)^2\}\psi(t)^2\\
&= \psi_2(t)^2\psi(s)^2 - \psi_2(s)^2\psi(t)^2\\
&= \left(\frac{\psi(t)}{b}\right)^2\psi(s)^2 - \left(\frac{\psi(s)}{b}\right)^2\psi(t)^2 = 0.
\end{aligned}$$

$\square$

Lemma 2.4. *Let $b > 0$, $\psi(s) = b\psi_2(s)$ and $\psi(t) = b\psi_2(t)$ for $s, t \in [0, 1]$. Then*

$$\frac{s\psi(t) + (t - 1)\psi(s)}{\psi(t) + (2t - 1)\psi(s)} = \frac{(s - 1)\psi(t) + t\psi(s)}{(2s - 1)\psi(t) + \psi(s)}.$$

Proof. We have

$$\begin{aligned}
&\{s\psi(t) + (t - 1)\psi(s)\}\{(2s - 1)\psi(t) + \psi(s)\} - \{\psi(t) + (2t - 1)\psi(s)\}\{(s - 1)\psi(t) + t\psi(s)\}\\
&= \{s(2s - 1) - (s - 1)\}\psi(t)^2 + \{(t - 1) - t(2t - 1)\}\psi(s)^2\\
&= \{s^2 + (s - 1)^2\}\psi(t)^2 - \{t^2 + (t - 1)^2\}\psi(s)^2\\
&= \psi_2(s)^2\psi(t)^2 - \psi_2(t)^2\psi(s)^2\\
&= \left(\frac{\psi(s)}{b}\right)^2\psi(t)^2 - \left(\frac{\psi(t)}{b}\right)^2\psi(s)^2 = 0.
\end{aligned}$$

$\square$

Lemma 2.5. *Let $b > 0$, $\psi(s) = b\psi_2(s)$ and $\psi(t) = b\psi_2(t)$ for $s, t \in [0, 1]$. Then*

$$\frac{(1-s)\psi(t) + t\psi(s)}{(1-2s)\psi(t) + \psi(s)} = \frac{s\psi(t) + (1-t)\psi(s)}{\psi(t) + (1-2t)\psi(s)}.$$

Proof. By the proof of Lemma 2.4, we have

$$\begin{aligned}&\{(1-s)\psi(t) + t\psi(s)\}\{\psi(t) + (1-2t)\psi(s)\} - \{(1-2s)\psi(t) + \psi(s)\}\{s\psi(t) + (1-t)\psi(s)\}\\ &= \{s^2 + (s-1)^2\}\psi(t)^2 - \{t^2 + (t-1)^2\}\psi(s)^2 = 0.\end{aligned}$$

$\square$

Lemma 2.6 ([9], Lemma 2.2). *If $x, y \in Y_\psi$ are such that $x \pm y \neq 0$ and $\|x\|_2 = \|y\|_2$, then*

$$\frac{\|x+y\|_2}{\|x+y\|} = \frac{\|x-y\|_2}{\|x-y\|}.$$

By Lemmas 2.3, 2.4, 2.5, and 2.6, Theorems 2.1 and 2.2 are reduced to the following theorem in the case of $c = 1$.

Theorem 2.7 ([9], Theorem 2.3). *Let $a, b > 0$. Then the following two statements are equivalent:*

(I) *There exists a pair $x, y \in S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x\|_2 = \|y\|_2 = 1/a$, $\|x + y\| = b\|x + y\|_2$ (and $\|x - y\| = b\|x - y\|_2$).*

(II) *There exist $r, s, t \in [0, 1]$ such that $\psi(s) = a\psi_2(s)$, $\psi(t) = a\psi_2(t)$, and $\psi(r) = b\psi_2(r)$, where r, s, t satisfy one of the following conditions:*

(a) $s \neq t$ *and* $r = \dfrac{s\psi(t) + t\psi(s)}{\psi(t) + \psi(s)}$.

(b) $(s, t) \neq (1, 0)$, $s + t \geq 1$, *and* $r = \dfrac{s\psi(t) + (t-1)\psi(s)}{\psi(t) + (2t-1)\psi(s)}$.

(c) $(s, t) \neq (0, 1)$, $s + t \leq 1$, *and* $r = \dfrac{(1-s)\psi(t) + t\psi(s)}{(1-2s)\psi(t) + \psi(s)}$.

Proof. By Lemma 2.6, if $\|x + y\| = b\|x + y\|_2$, then $\|x - y\| = b\|x - y\|_2$. We note that the function $t \mapsto t/\psi_2(t)$ is strictly increasing. Now we consider Theorems 2.1 and 2.2 in the case of $c = 1$.

Case (a): It is clear that Theorem 2.2(II)(a1) cannot exist. Since $\psi(s) = a\psi_2(s)$ and $\psi(t) = a\psi_2(t)$, we have

$$\frac{s}{\psi(s)} - \frac{t}{\psi(t)} = \frac{1}{a}\left(\frac{s}{\psi_2(s)} - \frac{t}{\psi_2(t)}\right)$$

and

$$\frac{1-s}{\psi(s)} - \frac{1-t}{\psi(t)} = \frac{1}{a}\left(\frac{1-s}{\psi_2(s)} - \frac{1-t}{\psi_2(t)}\right) = \frac{1}{a}\left(\frac{1-s}{\psi_2(1-s)} - \frac{1-t}{\psi_2(1-t)}\right).$$

These imply that if $s < t$, then $s\psi(s)^{-1} < t\psi(t)^{-1}$ and $(1-s)\psi(s)^{-1} > (1-t)\psi(t)^{-1}$. Similarly, if $s > t$, then $s\psi(s)^{-1} > t\psi(t)^{-1}$ and $(1-s)\psi(s)^{-1} < (1-t)\psi(t)^{-1}$. Thus conditions (1) and (2) in Theorem 2.2(II)(a2) if and only if $s \neq t$. Hence, by Lemma 2.3, Theorems 2.1(II)(a) and 2.2(II)(a2) are reduced to Theorem 2.7(II)(a).

Cases (b) and (c): We have

$$\frac{s}{\psi(s)} - \frac{1-t}{\psi(t)} = \frac{1}{a}\left(\frac{s}{\psi_2(s)} - \frac{1-t}{\psi_2(1-t)}\right) \quad \text{and} \quad \frac{1-s}{\psi(s)} - \frac{t}{\psi(t)} = \frac{1}{a}\left(\frac{1-s}{\psi_2(1-s)} - \frac{t}{\psi_2(t)}\right).$$

These imply that if $s+t \geq 1$, then $s\psi(s)^{-1} \geq (1-t)\psi(t)^{-1}$ and $(1-s)\psi(s)^{-1} \leq t\psi(t)^{-1}$. Thus, by Lemma 2.4, Theorems 2.1(II)(b) and 2.2(II)(b) are reduced to Theorem 2.7(II)(b). Moreover, if $s+t \leq 1$, then $s\psi(s)^{-1} \leq (1-t)\psi(t)^{-1}$ and $(1-s)\psi(s)^{-1} \geq t\psi(t)^{-1}$. Thus, by Lemma 2.5, Theorems 2.1(II)(c) and 2.2(II)(c) are reduced to Theorem 2.7(II)(c). $\square$

3. Geometric constants of Y_ψ

In this section, we consider the von Neumann-Jordan constant $C_{NJ}(Y_\psi)$. In relation to the norm $\|\cdot\|_{\psi,\widetilde{\psi}}$, it is known that the following lemmas. In what follows we write $\varphi \leq \psi$ if $\varphi(t) \leq \psi(t)$ for all $t \in [0,1]$.

Lemma 3.1 ([6], Lemma 3)**.** *Let $\varphi, \psi \in \Psi_2$ and let $\psi \leq \varphi$. Then*

$$\|\cdot\|_\psi \leq \|\cdot\|_\varphi \leq \max_{0\leq t\leq 1} \frac{\varphi(t)}{\psi(t)} \|\cdot\|_\psi.$$

Lemma 3.2 ([9], Lemma 2.1)**.** *Let $\varphi, \psi \in \Psi_2$. Then*

$$\|\cdot\|_{\varphi,\widetilde{\varphi}} \leq \max_{0\leq t\leq 1} \frac{\varphi(t)}{\psi(t)} \|\cdot\|_{\psi,\widetilde{\psi}}.$$

We note that $\|\cdot\|_2 = \|\cdot\|_{\psi_2} = \|\cdot\|_{\psi_2,\widetilde{\psi_2}}$. This, together with the preceding lemma, shows that $M_2^{-1}\|\cdot\|_2 \leq \|\cdot\| \leq M_1\|\cdot\|_2$, where

$$M_1 := \max_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)} \quad \text{and} \quad M_2 := \max_{0\leq t\leq 1} \frac{\psi_2(t)}{\psi(t)}.$$

Now we consider von Neumann-Jordan constant $C_{NJ}(Y_\psi)$ when $\psi \leq \psi_2$. As an application of Theorems 2.1 and 2.2, we have the following results.

Theorem 3.3. *Suppose that $\psi \neq \psi_2$ and $\psi \leq \psi_2$. Then*

$$C_{NJ}(Y_\psi) \leq \max_{0\leq t\leq 1} \frac{\psi_2(t)^2}{\psi(t)^2} (= M_2^2).$$

Moreover, $C_{NJ}(Y_\psi) = M_2^2$ if and only if there exist $r_1, s_1, t_1, r_2, s_2, t_2 \in [0,1]$ such that $\psi_2(s_i)/\psi(s_i) = \psi_2(t_i)/\psi(t_i) = M_2$ and $\psi(r_i) = \psi_2(r_i)$ for $i = 1, 2$, where r_1, s_1, t_1

satisfy one of the following conditions (A1)*,* (B1)*, and* (C1)*, and* r_2, s_2, t_2 *satisfy one of the following conditions* (A2)*,* (A3)*,* (B2)*, and* (C2) *for some* $c \in (0, 1]$*:*

(A1) $r_1 = \dfrac{s_1\psi(t_1) + ct_1\psi(s_1)}{\psi(t_1) + c\psi(s_1)}$.

(B1) $\dfrac{s_1}{\psi(s_1)} > \dfrac{c(1-t_1)}{\psi(t_1)}$ *and* $r_1 = \dfrac{s_1\psi(t_1) + c(t_1-1)\psi(s_1)}{\psi(t_1) + c(2t_1-1)\psi(s_1)}$.

(C1) $\dfrac{s_1}{\psi(s_1)} \leq \dfrac{c(1-t_1)}{\psi(t_1)}$ *and* $r_1 = \dfrac{(1-s_1)\psi(t_1) + ct_1\psi(s_1)}{(1-2s_1)\psi(t_1) + c\psi(s_1)}$.

(A2) $\dfrac{1-s_2}{\psi(s_2)} \geq \dfrac{c(1-t_2)}{\psi(t_2)}$, $\dfrac{s_2}{\psi(s_2)} \geq \dfrac{ct_2}{\psi(t_2)}$, *and* $r_2 = \dfrac{s_2\psi(t_2) - ct_2\psi(s_2)}{\psi(t_2) - c\psi(s_2)}$.

(A3) $r_2 = \dfrac{(1-s_2)\psi(t_2) + c(t_2-1)\psi(s_2)}{(1-2s_2)\psi(t_2) + c(2t_2-1)\psi(s_2)}$ *satisfying one of the following conditions:*

(1) $\dfrac{1-s_2}{\psi(s_2)} > \dfrac{c(1-t_2)}{\psi(t_2)}$ *and* $\dfrac{s_2}{\psi(s_2)} < \dfrac{ct_2}{\psi(t_2)}$.

(2) $\dfrac{1-s_2}{\psi(s_2)} < \dfrac{c(1-t_2)}{\psi(t_2)}$ *and* $\dfrac{s_2}{\psi(s_2)} > \dfrac{ct_2}{\psi(t_2)}$.

(B2) $\dfrac{1-s_2}{\psi(s_2)} \leq \dfrac{ct_2}{\psi(t_2)}$ *and* $r_2 = \dfrac{(s_2-1)\psi(t_2) + ct_2\psi(s_2)}{(2s_2-1)\psi(t_2) + c\psi(s_2)}$.

(C2) $\dfrac{1-s_2}{\psi(s_2)} > \dfrac{ct_2}{\psi(t_2)}$ *and* $r_2 = \dfrac{s_2\psi(t_2) + c(1-t_2)\psi(s_2)}{\psi(t_2) + c(1-2t_2)\psi(s_2)}$.

Proof. For each $x, z \in Y_\psi$ with $(x, z) \neq (0, 0)$, we have

$$\begin{aligned}
\|x+z\|^2 + \|x-z\|^2 &\leq \|x+z\|_2^2 + \|x-z\|_2^2 \\
&= 2(\|x\|_2^2 + \|z\|_2^2) \\
&\leq 2M_2^2(\|x\|^2 + \|z\|^2) \qquad (3.1)
\end{aligned}$$

by Lemmas 3.1 and 3.2. This implies that $C_{NJ}(Y_\psi) \leq M_2^2$.

Next we consider restatements of $C_{NJ}(Y_\psi) = M_2^2$. We note that

$$C_{NJ}(Y_\psi) = \sup\left\{\frac{\|x+cy\|^2 + \|x-cy\|^2}{2(\|x\|^2 + \|cy\|^2)} : x, y \in S_{Y_\psi},\ 0 < c \leq 1\right\}.$$

The set $S_{Y_\psi} \times S_{Y_\psi}$ with the product topology is compact and the function

$$S_{Y_\psi} \times S_{Y_\psi} \ni (x, y) \mapsto \frac{\|x+cy\|^2 + \|x-cy\|^2}{2(\|x\|^2 + \|cy\|^2)}$$

is continuous for all $c \in (0,1]$. Thus $C_{NJ}(Y_\psi) = M_2^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm cy \neq 0$ satisfying

$$\frac{\|x+cy\|^2 + \|x-cy\|^2}{2(\|x\|^2 + \|cy\|^2)} = M_2^2;$$

for this, we note that if $x + cy = 0$ or $x - cy = 0$ then $M_2 = 1$, which contradicts $\psi \neq \psi_2$. Moreover, by (3.1) with $z = cy$ for $c \in (0,1]$, $C_{NJ}(Y_\psi) = M_2^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm cy \neq 0$ satisfying $\|x+cy\| = \|x+cy\|_2$, $\|x-cy\| = \|x-cy\|_2$, $\|x\|_2 = M_2\|x\| = M_2$, and $\|y\|_2 = M_2\|y\| = M_2$. By adding Theorems 2.1 and 2.2 with $a = M_2^{-1}$ and $b = 1$, we have that $C_{NJ}(Y_\psi) = M_2^2$ if and only if there exist $r_1, s_1, t_1, r_2, s_2, t_2 \in [0,1]$ such that $\psi(s_i) = M_2^{-1}\psi_2(s_i)$, $\psi(t_i) = M_2^{-1}\psi_2(t_i)$, and $\psi(r_i) = \psi_2(r_i)$ for $i = 1,2$, where $r = r_1,\ s = s_1,\ t = t_1$ satisfy one of the conditions (a), (b), and (c) in Theorem 2.1, and $r = r_2,\ s = s_2,\ t = t_2$ satisfy one of the conditions (a1), (a2), (b), and (c) in Theorem 2.2. This completes the proof. $\square$

Moreover, the case of $\psi \geq \psi_2$ is as follows.

Theorem 3.4. *Suppose that $\psi \neq \psi_2$ and $\psi \geq \psi_2$. Then*

$$C_{NJ}(Y_\psi) \leq \max_{0 \leq t \leq 1} \frac{\psi(t)^2}{\psi_2(t)^2} (= M_1^2).$$

In particular, $C_{NJ}(Y_\psi) = M_1^2$ if and only if there exist $r_1, s_1, t_1, r_2, s_2, t_2 \in [0,1]$ such that $\psi(s_i)/\psi_2(s_i) = \psi(t_i)/\psi_2(t_i) = 1$ and $\psi(r_i) = M_1\psi_2(r_i)$ for $i = 1,2$, where r_1, s_1, t_1 satisfy one of the following conditions (A1), (B1), *and* (C1), *and r_2, s_2, t_2 satisfy one of the following conditions* (A2), (A3), (B2), *and* (C2) *for some $c \in (0,1]$ in Theorem 3.3.*

Proof. For each $x, z \in Y_\psi$ with $(x,z) \neq (0,0)$, we have

$$\begin{aligned} \|x+z\|^2 + \|x-z\|^2 &\leq M_1^2(\|x+z\|_2^2 + \|x-z\|_2^2) \\ &= 2M_1^2(\|x\|_2^2 + \|z\|_2^2) \\ &\leq 2M_1^2(\|x\|^2 + \|z\|^2) \end{aligned}$$

by Lemmas 3.1 and 3.2. This implies that $C_{NJ}(Y_\psi) \leq M_1^2$.

Now, an argument similar to that in the proof of Theorem 3.3 shows that $C_{NJ}(Y_\psi) = M_1^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm cy \neq 0$ for $c \in (0,1]$ satisfying $\|x+cy\| = M_1\|x+cy\|_2$, $\|x-cy\| = M_1\|x-cy\|_2$, $\|x\|_2 = \|x\| = 1$, and $\|y\|_2 = \|y\| = 1$. Hence Theorems 2.1 and 2.2 (applied for $a = 1$ and $b = M_1$) complete the proof. $\square$

If $c = 1$, then Theorems 3.3 and 3.4 are reduced to the following theorems of the *modified von Neumann-Jordan constant* $C'_{NJ}(Y_\psi)$ defined by

$$C'_{NJ}(Y_\psi) := \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_{Y_\psi}\right\} \left(\le C_{NJ}(Y_\psi)\right).$$

Theorem 3.5 ([9], Theorem 3.1). *Suppose that $\psi \neq \psi_2$ and $\psi \le \psi_2$. Then*

$$C'_{NJ}(Y_\psi) \le C_{NJ}(Y_\psi) \le \max_{0\le t\le 1} \frac{\psi_2(t)^2}{\psi(t)^2} (= M_2^2).$$

In particular, $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exist $r, s, t \in [0,1]$ such that $\psi_2(s)/\psi(s) = \psi_2(t)/\psi(t) = M_2$ and $\psi(r) = \psi_2(r)$, where r, s, t satisfy one of the following conditions:

(a) $s \neq t \quad \textit{and} \quad r = \dfrac{s\psi(t) + t\psi(s)}{\psi(t) + \psi(s)}$.

(b) $(s,t) \neq (1,0),\ s + t \ge 1, \quad \textit{and} \quad r = \dfrac{s\psi(t) + (t-1)\psi(s)}{\psi(t) + (2t-1)\psi(s)}$.

(c) $(s,t) \neq (0,1),\ s + t \le 1, \quad \textit{and} \quad r = \dfrac{(1-s)\psi(t) + t\psi(s)}{(1-2s)\psi(t) + \psi(s)}$.

Proof. By the definition of $C'_{NJ}(Y_\psi)$ and Theorem 3.3, we have $C'_{NJ}(Y_\psi) \le C_{NJ}(Y_\psi) \le M_2^2$. Moreover, an argument similar to that in the proof of Theorem 3.3 shows that $C'_{NJ}(Y_\psi) = M_2^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x \pm y\| = \|x \pm y\|_2$, $\|x\|_2 = M_2\|x\| = M_2$, and $\|y\|_2 = M_2\|y\| = M_2$. Hence Theorem 2.7 (applied for $a = M_2^{-1}$ and $b = 1$) completes the proof. □

Theorem 3.6 ([9], Theorem 3.4). *Suppose that $\psi \neq \psi_2$ and $\psi \ge \psi_2$. Then*

$$C'_{NJ}(Y_\psi) \le C_{NJ}(Y_\psi) \le \max_{0\le t\le 1} \frac{\psi(t)^2}{\psi_2(t)^2} (= M_1^2).$$

In particular, $C'_{NJ}(Y_\psi) = M_1^2$ if and only if there exist $r, s, t \in [0,1]$ such that $\psi(s)/\psi_2(s) = \psi(t)/\psi_2(t) = 1$ and $\psi(r) = M_1\psi_2(r)$, where r, s, t satisfy one of the following three conditions (a)–(c) *in Theorem* 3.5.

Proof. By the definition of $C'_{NJ}(Y_\psi)$ and Theorem 3.4, we have $C'_{NJ}(Y_\psi) \le C_{NJ}(Y_\psi) \le M_1^2$. Moreover, an argument similar to that in the proof of Theorem 3.4 shows that $C'_{NJ}(Y_\psi) = M_1^2$ if and only if there exists a pair $(x,y) \in S_{Y_\psi} \times S_{Y_\psi}$ with $x \pm y \neq 0$ satisfying $\|x \pm y\| = M_1\|x \pm y\|_2$, $\|x\|_2 = \|x\| = 1$, and $\|y\|_2 = \|y\| = 1$. Hence Theorem 2.7 (applied for $a = 1$ and $b = M_1$) completes the proof. □

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