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CYCLIC VECTORS IN FOCK-TYPE SPACES OF SINGLE VARIABLE

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ABSTRACT. This paper mainly considers cyclic vectors in the Fock-type spaces $L^{p,s}_{a,\alpha}(\mathbb{C})$ $(\alpha > 0, p \geq 1, s > 0)$ which consists of all entire functions f such that $|f|^p$ is integrable with respect to the measure $\exp(-\alpha|z|^s)dA(z)$. The case of s not being an integer was done in [9], where cyclic vectors are exactly those non-vanishing entire functions in $L^{p,s}_{a,\alpha}(\mathbb{C})$. In this paper it is shown that for each positive integer s, a function f is cyclic in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if f is non-vanishing and $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$, where $\mathcal{C}$ denotes the polynomial ring. Moreover, the condition that $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$ can not be dropped.

1. Introduction

Let $\mathcal{C}$ be the polynomial ring in one variable. Let $\mathcal{X}$ be a Banach space consisting of some holomorphic functions on a domain in $\mathbb{C}$. In this paper, a function f in $\mathcal{X}$ is called *cyclic* if the norm closure of $\{fq : q \in \mathcal{C}, fq \in \mathcal{X}\}$ equals $\mathcal{X}$. In many cases, it holds that for each $f \in \mathcal{X}$, $f\mathcal{C} \in \mathcal{X}$. Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$. The cyclic problem asks for a Banach space $\mathcal{X}$, when a function in $\mathcal{X}$ is cyclic [1]. This problem roots in the characterization for invariant subspaces for coordinate operators, which usually proves hard even in classical reproducing kernel Hilbert spaces. In the classical Hardy space $H^2(\mathbb{D})$, Beurling proved that a function in $H^2(\mathbb{D})$ is cyclic if and only if it is an $H^2(\mathbb{D})$-outer function, which was extended to general Hardy spaces $H^p(\mathbb{D})$ [4]. A similar statement is valid in the Bergman space $L^2_a(\mathbb{D})$; that is, a function f in $L^2_a(\mathbb{D})$ is cyclic if and only if f is $L^2_a(\mathbb{D})$-outer [7]. However, the definition of $L^2_a(\mathbb{D})$-outer is far more complicated than the traditional definition of $H^2(\mathbb{D})$-outer. Therefore, the cyclic problem usually requires different techniques for distinct Banach spaces. For the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk [11], the cyclic problem is challenging, and to the best knowledge of the authors, not much has been achieved on related topics.

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In this paper, we focus on the cyclic problem on a class of Banach space, Fock-type spaces. Precisely, for $\alpha, s > 0$ and $p \geq 1$, let $L^{p,s}_{a,\alpha}(\mathbb{C})$ denote the Banach space of all entire functions f satisfying

$$\|f\|^p = \int_{\mathbb{C}} |f(z)|^p \exp(-\alpha|z|^s) \frac{dA(z)}{2\pi} < \infty,$$

where dA denotes the area measure on the complex plane $\mathbb{C}$. For $\alpha = 1$ and $p = s = 2$, $L^{p,s}_{a,\alpha}(\mathbb{C})$ is exactly the classical Fock space. For related details, we call the reader's attention to [2, 3, 6, 5, 10].

In [8] K. H. Izuchi studied cyclic vectors in $L^{p,s}_{a,\alpha}(\mathbb{C})$, and proved the following.

Theorem 1.1. *Let f be a function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ satisfying $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$. Then the following are equivalent:*

(i) *f is a non-vanishing function.*
(ii) *$f = \exp(h)$, where $h(z) = \sum_{k=0}^{[s]} c_k z^k$ and for each k, $c_k \in \mathbb{C}$; and in addition $|c_s| < \frac{\alpha}{p}$ if s is an integer.*
(iii) *f is cyclic in $L^{p,s}_{a,\alpha}(\mathbb{C})$.*

It is natural to ask whether the condition $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$ could be removed. In the case of s not being an integer, the proof in [8] shows that Theorem 1.1 holds even if the condition $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$ is removed. However, if s is a sufficiently large integer, this condition can not be dropped. In fact, let $\varphi(z) = \exp(\frac{\alpha}{p} z^s)$ and clearly $\varphi(z)$ is non-vanishing. For $s = 5, 6, \cdots$, one can check that $\varphi\mathcal{C} \not\subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$, and that φ is not cyclic in $L^{p,s}_{a,\alpha}(\mathbb{C})$ [8]. That is, (i) and (iii) are not always equivalent.

As for cyclic vectors, there is some difference between the definition presented here and the definition in [8]. In [8], a function f is cyclic if $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$ and the closure of $f\mathcal{C}$ equals $L^{p,s}_{a,\alpha}(\mathbb{C})$. But in the present paper, recall that a function f in $L^{p,s}_{a,\alpha}(\mathbb{C})$ is *cyclic* if the closure of $\{fq : q \in \mathcal{C}, fq \in L^{p,s}_{a,\alpha}(\mathbb{C})\}$ equals $L^{p,s}_{a,\alpha}(\mathbb{C})$.

The following is our main result and it characterizes cyclic vectors in $L^{p,s}_{a,\alpha}(\mathbb{C})$ for the case of s being an integer.

Theorem 1.2. *For each positive integer s, f is a cyclic vector in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if $f = \exp(h)$ where $h(z) = \sum_{n=0}^{s} c_n z^n$, with $a_n \in \mathbb{C}$ and $|c_s| < \frac{\alpha}{p}$ if and only if f is non-vanishing and $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$.*

Let s be an integer. As mentioned before, for $s \geq 5$ a non-vanishing function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ is not necessarily cyclic. In the case of $s = 1, 2, 3, 4$, f is a cyclic vector in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if f is a non-vanishing function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ [9].

In the next section, we give the proof of Theorem 1.2.

2. Proof of Theorem 1.2

This section mainly presents the proof of Theorem 1.2. For this, some lemmas are needed.

For real numbers a_k, b_k $(1 \leq k \leq s-1)$ and $\varepsilon > 0$, set

$$\gamma = \gamma(\rho, \theta) = -\alpha\rho^s(1 - \cos s\theta) + \sum_{k=1}^{s-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k, \tag{1}$$

and

$$\mathcal{I}(t, \varepsilon) = \int_t^{t+\varepsilon} d\theta \int_0^{+\infty} \exp(\gamma(\rho, \theta))\,\rho\, d\rho < +\infty.$$

Put $\mathcal{I} = \mathcal{I}(0, 2\pi)$.

For $\alpha > 0$, put

$$I(s, j) = \int_0^{2\pi} d\theta \int_0^{+\infty} \exp(-\alpha\rho^s(1 - \cos s\theta))\rho^j\, d\rho.$$

We have the following.

Lemma 2.1. *Suppose $s > 0$ and $j > 0$. Then the following are equivalent.*

(i) $j + 1 < \frac{s}{2}$;
(ii) $I(s, j) < +\infty$;
(iii) $\int_0^1 d\theta \int_0^{+\infty} \exp(-\alpha\rho^s\theta^2)\rho^j\, d\rho < +\infty$;
(iv) *for each* $b \in \mathbb{R}$, $\int_0^1 d\theta \int_0^{+\infty} \exp(-\alpha\rho^s\theta^2 + b\rho^{\frac{s}{2}}\theta)\rho^j\, d\rho < +\infty$.

Proof. For a constant $\lambda > 0$,

$$\int_0^{+\infty} \exp(-\alpha\rho^s\lambda^s)\rho^j\, d\rho = \frac{1}{\lambda^{j+1}} \int_0^{+\infty} \exp(-\alpha\rho^s)\rho^j\, d\rho. \tag{2}$$

Letting $\lambda^s = (1 - \cos s\theta)$ yields that

$$\int_0^{2\pi} d\theta \int_0^{+\infty} \exp(-\alpha\rho^s(1 - \cos s\theta))\rho^j\, d\rho = \int_0^{2\pi} \frac{1}{(1 - \cos s\theta)^{\frac{j+1}{s}}} d\theta \int_0^{+\infty} \exp(-\alpha\rho^s)\rho^j\, d\rho.$$

Thus $\int_0^{2\pi} d\theta \int_0^{+\infty} \exp(-\alpha\rho^s(1 - \cos s\theta))\rho^j\, d\rho < +\infty$ if and only if

$$\int_0^{2\pi} \frac{1}{(1 - \cos s\theta)^{\frac{j+1}{s}}} d\theta < +\infty;$$

this is equivalent to $\frac{2(j+1)}{s} < 1$. This immediately gives (i) $\Leftrightarrow$ (ii).

Substituting θ^2 with λ^s in (2) yields that

$$\int_0^1 d\theta \int_0^{+\infty} \exp(-\alpha\rho^s\theta^2)\rho^j\, d\rho = \int_0^1 \frac{1}{\theta^{\frac{2(j+1)}{s}}} d\theta \int_0^{+\infty} \exp(-\alpha\rho^s)\rho^j\, d\rho,$$

which converges if and only if $\int_0^1 \frac{1}{\theta^{\frac{2(j+1)}{s}}} d\theta < \infty$. Then it follows that (i) $\Leftrightarrow$ (iii).

Note that (iv) $\Rightarrow$ (iii) is trivial, and it remains to prove (iii) $\Rightarrow$ (iv). Without loss of generality, assume that $b > 0$. Since (iii) $\Leftrightarrow$ (i), one gets $j + 1 < \frac{s}{2}$, and then

$$\int_0^1 d\theta \int_0^{+\infty} \exp(-\frac{\alpha}{2}\rho^s\theta^2 + \frac{b^2}{2\alpha})\rho^j d\rho < +\infty.$$

Besides, since $b\rho^{\frac{s}{2}}\theta \leq \frac{\alpha}{2}\rho^s\theta^2 + \frac{b^2}{2\alpha}$ for $0 \leq \theta \leq 1$,

$$-\alpha\rho^s\theta^2 + b\rho^{\frac{s}{2}}\theta \leq -\frac{\alpha}{2}\rho^s\theta^2 + \frac{b^2}{2\alpha},$$

which gives $\int_0^1 d\theta \int_0^{+\infty} \exp(-\alpha\rho^s\theta^2 + b\rho^{\frac{s}{2}}\theta)\rho^j d\rho < +\infty$. Therefore, the proof of Lemma 2.1 is complete. $\square$

Lemma 2.1 has an immediate corollary.

Corollary 2.1. *For positive numbers $\varepsilon_0, \varepsilon, \delta$ and s, we have*

$$\int_0^\delta d\theta \int_0^{+\infty} \exp(-\alpha\rho^s(1 - \cos s\theta)) + \varepsilon_0\rho^\varepsilon)\rho d\rho = \infty.$$

The following plays an important role in the proof of Theorem 1.2.

Lemma 2.2. *Let s be a positive integer, and $h = \sum_{k=0}^s c_k z^k$ with $|c_s| = \frac{\alpha}{p}$. Suppose $f = \exp(h)$ belongs to $L^{p,s}_{a,\alpha}(\mathbb{C})$. If s is an odd number, then $f = f(0)\exp(c\frac{\alpha}{p}z^s)$ for some constant c with $|c| = 1$. If s is even, then*

$$f(z) = f(0)\exp(c(\frac{\alpha}{p}z^s + \beta z^{\frac{s}{2}}))$$

where $|c| = 1$ and $Re\ \beta = 0$.

Proof. Note that f is in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if for each unimodular constant c, $f(cz)$ is in $L^{p,s}_{a,\alpha}(\mathbb{C})$. Without loss of generality, assume $c_s = \frac{\alpha}{p}$ and $c_0 = 0$. Rewrite $c_k = \frac{-a_k - ib_k}{p}$ for $1 \leq k \leq s-1$. Let $z = \rho\exp(i\theta)$ be the polar coordinate of z, $\rho > 0$ and $\theta \in \mathbb{R}$. We have

$$Re\,(pc_k z^k) = (-a_k \cos k\theta + b_k \sin k\theta)\rho^k,$$

and thus

$$\exp(-\alpha|z|^s)\,|\exp(h)|^p = \exp(\gamma),$$

where γ is defined by (1)

$$\gamma = \gamma(\rho, \theta) = -\alpha\rho^s(1 - \cos s\theta) + \sum_{k=1}^{s-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k.$$

First assume that s is odd. In this case, it will be shown that for each k $(1 \le k \le s-1)$, $a_k = b_k = 0$. To see this, assume that not all of a_k and b_k vanish. Let j be the largest integer in $\{1, 2, \cdots, s-1\}$ such that

$$(a_j, b_j) \neq (0, 0).$$

First we have $-a_j \le 0$. Otherwise, $-a_j > 0$; that is,

$$(-a_j \cos j\theta + b_j \sin j\theta)|_{\theta=0} > 0.$$

Then by continuity there is a positive constant ε_0 such that

$$-a_j \cos j\theta + b_j \sin j\theta > 2\varepsilon_0,\ 0 \le \theta \le \varepsilon_0.$$

Then for sufficiently large ρ and $0 \le \theta \le \varepsilon_0$,

$$\gamma = -\alpha\rho^s(1-\cos s\theta) + \sum_{k=1}^{s-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k \ge -\alpha\rho^s(1-\cos s\theta) + \varepsilon_0\rho^j.$$

Then by Corollary 2.1 $\mathcal{I} \ge \mathcal{I}(0, \varepsilon_0) = +\infty$, which is a contradiction. Therefore, $-a_j \le 0$.

Put $\theta_0 = \frac{2\pi}{s}$ and $d = \gcd(j, s)$. Write $j = j'd$ and $s = s'd$. Then it follows that

$$\{\frac{j'm2\pi}{s'} \mod 2\pi : m = 1, \cdots, s'\} = \{\frac{m2\pi}{s'} \mod 2\pi : m = 1, \cdots, s'\},$$

where $s' \ge 3$ because s is odd. In particular, there are integers m_1 and m_2 such that

$$j\frac{m_1 2\pi}{s} = \frac{j'm_1 2\pi}{s'} = \frac{2\pi}{s'} \mod 2\pi,$$

and

$$j\frac{m_2 2\pi}{s} = \frac{(s'-1)2\pi}{s'} \mod 2\pi.$$

Rewrite $\theta_k = \frac{m_k 2\pi}{s}$ for $k = 1, 2$, and one has

$$\cos j\theta_1 < 0, \quad \sin j\theta_1 > 0; \tag{3}$$

and

$$\cos j\theta_2 < 0, \quad \sin j\theta_2 < 0. \tag{4}$$

Since for each sufficiently small $\varepsilon > 0$, $\mathcal{I}(\theta_1, \varepsilon) < \mathcal{I} < \infty$, by similar reasoning as proving $-a_j \le 0$ one deduces that $-a_j \cos j\theta_1 + b_j \sin j\theta_1 \le 0$. Since $-a_j \le 0$, by (3) $b_j \le 0$. Similarly, $-a_j \cos j\theta_2 + b_j \sin j\theta_2 \le 0$, and then by (4) $b_j \ge 0$, forcing $b_j = 0$. Again by

$$-a_j \cos j\theta_2 + b_j \sin j\theta_2 \le 0,$$

we have $a_j = 0$. Thus $(a_j, b_j) = (0, 0)$, which is a contradiction. Therefore, if s is odd, then $f = f(0)\exp(\frac{c\alpha}{p}z^s)$ where $|c|$ is a unimodular constant.

It remains to deal with the case of s being an even number. First we will show that for each k, if $1 \leq k \leq s-1$ and $k \neq \frac{s}{2}$, $a_k = b_k = 0$. To see this, we will first show that

$$a_k = b_k = 0, \quad \frac{s}{2} < k \leq s-1.$$

Assume that for $\frac{s}{2} < k \leq s-1$, not all of a_k and b_k vanish. Let j be the largest integer in $\{\frac{s}{2}+1, \cdots, s-1\}$ such that

$$(a_j, b_j) \neq (0,0).$$

Let $\frac{j'}{s'} = \frac{j}{s}$, where j' and s' are relatively prime integers. Since $j \neq \frac{s}{2}$, $s' \geq 3$. Then by similar discussion of the case of s being an odd number, we deduce that $(a_j, b_j) = (0,0)$ to derive a contradiction. Thus $j \leq \frac{s}{2}$.

It is in order to determine $a_{\frac{s}{2}}$ and $b_{\frac{s}{2}}$, and soon we will have $a_{\frac{s}{2}} = 0$. In fact, since $\mathcal{I}(0,\varepsilon) < \mathcal{I} < \infty$, by Lemma 2.1 $-a_{\frac{s}{2}} \leq 0$. To get $a_{\frac{s}{2}} = 0$, assume conversely that $-a_{\frac{s}{2}} < 0$. With $\theta_0 = \frac{2\pi}{s}$, one gets

$$\cos \frac{s}{2}\theta < -\frac{1}{2}, \ |\theta - \theta_0| < \frac{\pi}{2s},$$

and thus for sufficiently large ρ,

$$\begin{aligned}\gamma &= -\alpha\rho^s(1-\cos s\theta) + \sum_{k=1}^{s-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k \\ &\geq -\alpha\rho^s(1-\cos(s(\theta-\theta_0)) + \frac{1}{2}a_{\frac{s}{2}}\rho^{\frac{s}{2}}, \ |\theta-\theta_0| < \frac{\pi}{2s}.\end{aligned}$$

Hence by Corollary 2.1 $\mathcal{I} \geq \mathcal{I}(\theta_0, \frac{\pi}{2s}) = +\infty$, which is a contradiction. Thus $a_{\frac{s}{2}} = 0$. Then we have

$$\gamma = -\alpha\rho^s(1-\cos s\theta) + b_{\frac{s}{2}}\rho^{\frac{s}{2}} \sin \frac{s}{2}\theta + \sum_{k=1}^{\frac{s}{2}-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k.$$

Next we proceed to prove that $a_k = b_k = 0$, $1 \leq k < \frac{s}{2}$. As θ tends to zero, $\rho^s(1-\cos s\theta) \sim \rho^s \dfrac{s^2}{2}\theta^2$, and then there are positive constants M and δ such that

$$\frac{\alpha}{2}\rho^s(1-\cos s\theta) + M \geq |b_{\frac{s}{2}} \sin \frac{s}{2}\theta\rho^{\frac{s}{2}}|, \ |\theta| < \delta.$$

The above also holds if $|\theta - \theta'| < \delta$ for $\theta' = 0, \frac{2\pi}{s}, \frac{4\pi}{s}, \cdots, \frac{(s-1)2\pi}{s}$. For those θ, we have $\gamma \leq \widetilde{\gamma}$, where

$$\widetilde{\gamma} = -\frac{\alpha}{2}\rho^s(1-\cos s\theta) + \sum_{k=1}^{\frac{s}{2}-1}(-a_k \cos k\theta + b_k \sin k\theta)\,\rho^k + M.$$

Since $\mathcal{I} = \int_0^{2\pi} d\theta \int_0^{+\infty} \exp(\gamma(\rho,\theta))\rho d\rho < +\infty$,

$$\int_0^{2\pi} d\theta \int_0^{+\infty} \exp(\widetilde{\gamma}(\rho,\theta))\rho d\rho < +\infty.$$

Then by similar discussions in the case of s being an odd number, we conclude that $a_k = b_k = 0$ for $1 \le k < \frac{s}{2}$ to finish the proof. $\square$

By [8, Lemma 2.3], each non-vanishing entire function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ is of the form $f = \exp(h)$, where h is a polynomial with order $h \le s$. Then by using Lemmas 2.1 and 2.2, for $s = 1,2,3,4$, there is no non-vanishing function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ for $p \ge 1$ (as mentioned in [9]). For an integer $s \ge 5$, we have the following.

Corollary 2.2. *Let s be an integer with $s \ge 5$. Then the function f in Lemma 2.2 is in $L^{p,s}_{a,\alpha}(\mathbb{C})$, but $\{fq : q \in \mathcal{C}, fq \in L^{p,s}_{a,\alpha}(\mathbb{C})\}$ is of finite dimension. Consequently, f is not cyclic.*

Proof. For $s \ge 5$, by Lemma 2.1 the function f in Lemma 2.2 is in $L^{p,s}_{a,\alpha}(\mathbb{C})$. For a polynomial q of degree k, there is a positive number ε_0 such that $|q(z)| \ge \varepsilon_0 |z|^k$ for sufficiently large $|z|$. Again by Lemma 2.1, if $\deg q + 2 \ge \frac{s}{2}$, then fq is not in $L^{p,s}_{a,\alpha}(\mathbb{C})$. Therefore, $\{fq : q \in \mathcal{C}, fq \in L^{p,s}_{a,\alpha}(\mathbb{C})\}$ is of finite dimension. $\square$

Proof of Theorem 1.2. Since f is a non-vanishing function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ where s is a positive integer. There is an entire function h satisfying $f = \exp(h)$. By the proof of [8, Lemma 2.3], it follows that h is a polynomial. Then it is direct to check that order $h \le s$. Write $h(z) = \sum_{k=0}^{s} c_k z^k$.

If $0 \le |c_0| < \frac{\alpha}{p}$, then by [8, Lemma 2.3] and [8, Theorem 1.1], $\exp(h)$ is a cyclic vector in $L^{p,s}_{a,\alpha}(\mathbb{C})$. If $|c_0| > \frac{\alpha}{p}$, then by direct computations $\exp(h)$ is not in $L^{p,s}_{a,\alpha}(\mathbb{C})$.

It remains to consider the case of $|c_0| = \frac{\alpha}{p}$. In this case, if $s \ge 5$, then by Lemma 2.2 and Corollary 2.2, f is not cyclic. If $1 \le s \le 4$, the comments above Corollary 2.2 show that f is not in $L^{p,s}_{a,\alpha}(\mathbb{C})$. Therefore, f is a cyclic vector in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if $f = \exp(h)$, where $h(z) = \sum_{n=0}^{s} c_n z^n$, with $a_n \in \mathbb{C}$ and $|c_s| < \frac{\alpha}{p}$.

In addition, as above it has been shown that for $1 \le s \le 4$, f is a cyclic vector in $L^{p,s}_{a,\alpha}(\mathbb{C})$ if and only if $f\mathcal{C} \subseteq L^{p,s}_{a,\alpha}(\mathbb{C})$. By Corollary 2.2, this is valid for $s \ge 5$. The proof is complete. $\square$

The proof of Theorem 1.2 also shows that for $s = 1,2,3,4$, each non-vanishing function in $L^{p,s}_{a,\alpha}(\mathbb{C})$ is always a cyclic vector.

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