Nihonkai Math. J.
Vol.28(2017), 105–116

COMMENTS ON SOME RECENT EXISTENCE THEOREMS OF BEST PROXIMITY POINTS FOR KANNAN-TYPE AND CHATTERJEA-TYPE MAPPINGS

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Abstract. In 2013, Basha, Shahzad and Jeyaraj proved two existence theorems of best proximity points for Kannan-type and Chatterjea-type mappings. In this paper, in order to clarify the mathematical structure of these theorems, we improve these theorems in the aspects of both statements and proofs. Indeed, we give very simple proofs of these theorems. We also discuss the best possibility on the numbers that appear in these theorems.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. We let (X, d) be a metric space and let A and B be nonempty subsets of X. Let T be a mapping from A into B and let S be a mapping from B into A. Define $d(A, B) \in \mathbb{R}$ and a function d^* from $X \times X$ into $[0, \infty)$ by

$$d(A, B) = \inf\{d(x, u) : \ x \in A, \ u \in B\}$$

and

$$d^*(a, b) = d(a, b) - d(A, B)$$

for any $a, b \in X$.

A point $x \in A$ is said to be a *best proximity point* of T if $d^*(x, Tx) = 0$ holds. Also, a point $u \in B$ is said to be a *best proximity point* of S if $d^*(Su, u) = 0$ holds. It is obvious that best proximity points of T are minimizers of the problem: $\min\{d(x, Tx) : x \in A\}$. Similarly, best proximity points of S are minimizers of the problem: $\min\{d(Su, u) : u \in B\}$.

2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47J25, 90C30.
Key words and phrases. Best proximity point, fixed point, Kannan-type mapping, Chatterjea-type mapping.
The author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

We human beings have studied the existence of best proximity points; see [5, 6, 7, 9, 11, 12] and others. Recently, in 2013, Basha, Shahzad and Jeyaraj in [1] proved two existence theorems, Theorems 5 and 7 below, of best proximity points for Kannan-type and Chatterjea-type mappings. Also, in [1], we obtain Kannan's and Chatterjea's fixed point theorems, Theorems 1 and 2 below, as corollaries of Theorems 5 and 7.

In this paper, in order to clarify the mathematical structure of Theorems 5 and 7, we improve Theorems 5 and 7 in the aspects of both statements and proofs. Indeed, we give a very simple proof of Theorem 5 by using Theorem 1 conversely. Similarly we prove Theorem 7 by using Theorem 2 conversely. We also discuss the best possibility on the numbers that appear in Theorems 5 and 7.

2. Preliminaries

Kannan [8] in 1969 proved the following, very interesting fixed point theorem. For the sake of completeness, we give the proof of (iv), for we are usually conscious of (i)–(iii), however we are not usually conscious of (iv).

Theorem 1 (Kannan [8]). *Let (Y, d) be a metric space and let U be a Kannan mapping on Y, that is, there exists $k \in [0, 1/2)$ such that*

$$d(Ua, Ub) \le k\, d(Ua, a) + k\, d(b, Ub) \tag{1}$$

holds for any $a, b \in Y$. Then the following hold:

(i) *$\{U^n a\}$ is a Cauchy sequence for any $a \in Y$.*
(ii) *U has at most one fixed point.*
(iii) *If Y is complete, then U has a unique fixed point.*
(iv) *If U has a fixed point c, then $\{U^n a\}$ converges to c for any $a \in Y$.*

Proof. We have by (i)

$$\begin{aligned}\limsup_{n\to\infty} d(U^n a, c) &= \limsup_{n\to\infty} d(U^n a, Uc)\\ &\le \lim_{n\to\infty} k\left(d(U^n a, U^{n-1}a) + d(c, Uc)\right) = 0\end{aligned}$$

for any $a \in Y$. □

Chatterjea [3] in 1972 also proved the following, very interesting fixed point theorem. As Theorem 1, we give the proof of (iv). See also [2, 4, 10].

Theorem 2 (Chatterjea [3]). *Let (Y, d) be a metric space and let U be a Chatterjea mapping on Y, that is, there exists $k \in [0, 1/2)$ such that*

$$d(Ua, Ub) \le k\, d(Ua, b) + k\, d(a, Ub) \tag{2}$$

holds for any $a, b \in Y$. Then (i)–(iv) of Theorem 1 hold.

Proof. Since

$$d(U^{n+1}a, c) = d(U^{n+1}a, Uc) \le k\, d(U^{n+1}a, c) + k\, d(U^n a, Uc)$$
$$= k\, d(U^{n+1}a, c) + k\, d(U^n a, c),$$

we have

$$d(U^{n+1}a, c) \le \frac{k}{1-k}\, d(U^n a, c)$$

for $n \in \mathbb{N}$. Since $k/(1-k) \in [0, 1)$, $\{d(U^n a, c)\}$ converges to 0. $\square$

Definition 3 ([1])**.** The pair (S, T) is said to *form a weak K-cyclic contraction* if there exists $k \in [0, 1/2)$ such that

$$d^*(STx, Tx) \le k\, d^*(x, Tx) + k\, d^*(STx, Tx)$$

for all $x \in A$.

Remark. It is obvious that the pair (S, T) forms a weak K-cyclic contraction iff there exists $r \in [0, 1)$ such that

$$d^*(STx, Tx) \le r\, d^*(x, Tx)$$

for any $x \in A$.

We introduce a weaker concept than Definition 3.

Definition 4. The pair (S, T) is said to satisfy *Condition* (b2b) if the following hold for any $x \in A$:

(a) $d^*(x, Tx) = 0$ implies $d^*(STx, Tx) = d^*(x, Tx)$.
(b) $d^*(x, Tx) > 0$ implies $d^*(STx, Tx) \ne d^*(x, Tx)$.

3. Kannan-type

In this section, we consider the following Kannan-type theorem.

Theorem 5 ([1])**.** *Assume that X is complete and A and B are closed. Assume also the following*:

(a) *S is nonexpansive, that is, $d(Su, Sv) \le d(u, v)$ for any $u, v \in B$.*
(b) *There exists $k \in [0, 1/2)$ such that $d(Tx, Ty) \le k\, d(STx, x) + k\, d(y, STy)$ holds for any $x, y \in A$.*
(c) *The pair (S, T) forms a weak K-cyclic contraction with k that appears in* (b)*.*

Define a sequence $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ by $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Sa_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then the following hold:

(i) *There exist $z \in A$ and $w \in B$ satisfying $d(z, Tz) = d(A, B)$, $d(Sw, w) = d(A, B)$ and $d(z, w) = d(A, B)$.*

(ii) $\{a_{2n}\}$ *and* $\{a_{2n+1}\}$ *converge to some best proximity points in* A *and* B *of* T *and* S*, respectively.*

(iii) *If* $x, y \in A$ *are best proximity points in* A *of* T*, then* $d(x, y) \leq (2+4k)\, d(A, B)$ *holds.*

We improve Theorem 5 in the aspects of both statement and proof.

Theorem 6. *Assume that either* A *or* B *is complete. Assume also* (a) *and* (b) *of Theorem 5 and the following*:

(c)' *The pair* (S, T) *satisfies Condition* (b2b).

Define a sequence $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ *as in Theorem 5. Then the following hold*:

(i) ST *and* TS *are Kannan mappings on* A *and* B*, respectively.*

(ii) ST *and* TS *have unique fixed points* $z \in A$ *and* $w \in B$*, respectively.*

(iii) z *and* w *are best proximity points in* A *and* B *of* T *and* S*, respectively, which satisfy* $Tz = w$ *and* $Sw = z$*.*

(iv) $\{a_{2n}\}$ *and* $\{a_{2n+1}\}$ *converge to* z *and* w*, respectively.*

(v) *If* $x \in A$ *is a best proximity point of* T*, then* $d(z, x) \leq (2 + 2k)\, d(A, B)$ *and* $d(x, w) \leq (1 + 2k)\, d(A, B)$ *hold.*

(vi) *If* $x, y \in A$ *are best proximity points of* T*, then* $d(x, y) \leq (2 + 4k)\, d(A, B)$ *holds.*

Proof. We first show (i). For $x, y \in A$ and $u, v \in B$, we have

$$\begin{aligned} d(STx, STy) &\leq d(Tx, Ty) \\ &\leq k\, d(STx, x) + k\, d(y, STy) \end{aligned}$$

and

$$\begin{aligned} d(TSu, TSv) &\leq k\, d(STSu, Su) + k\, d(Sv, STSv) \\ &\leq k\, d(TSu, u) + k\, d(v, TSv), \end{aligned}$$

thus, ST and TS are Kannan. We will prove (ii), considering the following two cases:

- A is complete.
- B is complete.

In the first case, by Theorem 1 (iii), ST has a unique fixed point $z \in A$. Since

$$TS(Tz) = T(STz) = Tz,$$

$w := Tz$ is a fixed point of TS. By Theorem 1 (ii), w is a unique fixed point of TS. In the second case, by Theorem 1 (iii), TS has a unique fixed point $w \in B$. Since $STSw = Sw$, $z := Sw$ is a fixed point of ST. By Theorem 1 (ii), z is a unique fixed point of ST. Let us prove (iii). We have already shown $Tz = w$ and $Sw = z$. Since

$STz = z$, $d^*(z, Tz) = 0$ holds from Definition 4 (b). Thus, z is a best proximity point in A of T. So, w is a best proximity point in B of S. We have proved (iii). By Theorem 1 (iv), we obtain (iv). In order to show (v) and (vi), we let $x, y \in A$ be best proximity points of T. We note

$$d(A, B) = d(x, Tx) = d(STx, Tx) = d(y, Ty) = d(STy, Ty) \tag{3}$$

by Definition 4 (a). Then we have

$$\begin{aligned} d(x, w) &\le d(x, Tx) + d(Tz, Tx) \\ &\le d(A, B) + k\, d(STz, z) + k\, d(x, STx) \\ &\le d(A, B) + k\, d(x, Tx) + k\, d(STx, Tx) \\ &\le (1 + 2\, k)\, d(A, B) \end{aligned}$$

and hence

$$d(z, x) \le d(z, w) + d(x, w) \le (2 + 2\, k)\, d(A, B),$$

thus, (v) holds. We finally have by (v)

$$d(x, y) \le d(x, w) + d(y, w) \le (2 + 4\, k)\, d(A, B),$$

thus, (vi) holds. □

Remark.

- We do not use Definition 4 (a) when we prove (i)–(iv).
- Examples 11 and 13 below tell that all the numbers that appear in (v) and (vi) of Theorem 6 are best possible.
- It is interesting that though $z \in A$ and $w \in B$,

  $$\sup\{d(x, w) : x \in A_0\} < \sup\{d(z, x) : x \in A_0\}$$

  is possible, where A_0 is the set of all best proximity points of T; see Example 11.

4. Chatterjea-type

In this section, we consider the following Chatterjea-type theorem.

Theorem 7 ([1])**.** *Assume that X is complete and A and B are closed. Assume also the following:*

(a) *S is nonexpansive.*
(b) *There exists $k \in [0, 1/2)$ such that $d(Tx, Ty) \le k\, d(STx, y) + k\, d(x, STy)$ holds for any $x, y \in A$.*
(c) *The pair (S, T) forms a weak K-cyclic contraction with k that appears in* (b).

Define a sequence $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ *and* $a_{2n+2} = Sa_{2n+1}$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then the following hold*:

(i) *There exist* $z \in A$ *and* $w \in B$ *satisfying* $d(z, Tz) = d(A, B)$, $d(Sw, w) = d(A, B)$ *and* $d(z, w) = d(A, B)$.

(ii) $\{a_{2n}\}$ *and* $\{a_{2n+1}\}$ *converge to some best proximity points in* A *and* B *of* T *and* S, *respectively.*

(iii) *If* $x, y \in A$ *are best proximity points in* A *of* T, *then* $d(x, y) \le \dfrac{2\,(1+2\,k)}{1-2\,k}\,d(A, B)$ *holds.*

We improve Theorem 7 in the aspects of both statement and proof.

Theorem 8. *Assume that either* A *or* B *is complete. Assume also* (a) *and* (b) *of Theorem 7 and the following*:

(c)' *The pair* (S, T) *satisfies Condition* (b2b).

Define a sequence $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ *as in Theorem 7. Then the following hold*:

(i) ST *and* TS *are Chatterjea mappings on* A *and* B, *respectively.*

(ii) ST *and* TS *have unique fixed points* $z \in A$ *and* $w \in B$, *respectively.*

(iii) z *and* w *are best proximity points in* A *and* B *of* T *and* S, *respectively, which satisfy* $Tz = w$ *and* $Sw = z$.

(iv) $\{a_{2n}\}$ *and* $\{a_{2n+1}\}$ *converge to* z *and* w, *respectively.*

(v) *If* $x \in A$ *is a best proximity point of* T, *then* $d(z, x) \le \dfrac{2-2\,k}{1-2\,k}\,d(A, B)$ *and* $d(x, w) \le \dfrac{1}{1-2\,k}\,d(A, B)$ *hold.*

(vi) *If* $x, y \in A$ *are best proximity points of* T, *then* $d(x, y) \le \dfrac{2}{1-2\,k}\,d(A, B)$ *holds.*

Proof. We first show (i). For $x, y \in A$ and $u, v \in B$, we have

$$\begin{aligned} d(STx, STy) &\le d(Tx, Ty) \\ &\le k\,d(STx, y) + k\,d(x, STy) \end{aligned}$$

and

$$\begin{aligned} d(TSu, TSv) &\le k\,d(STSu, Sv) + k\,d(Su, STSv) \\ &\le k\,d(TSu, v) + k\,d(u, TSv), \end{aligned}$$

thus, ST and TS are Chatterjea. Similarly, we can prove (ii)-(iv) as in the proof of Theorem 6. In order to show (v) and (vi), we let $x, y \in A$ be best proximity points of T. We note (3). Since

$$d(z, STx) \le k\,d(z, x) + k\,d(z, STx),$$

we have

$$d(z, STx) \leq \frac{k}{1-k}\, d(z, x). \tag{4}$$

Hence

$$d(z, x) \leq d(z, STx) + d(STx, x) \leq \frac{k}{1-k}\, d(z, x) + d(STx, x).$$

We obtain

$$d(z, x) \leq \frac{1-k}{1-2\,k} d(STx, x) \leq \frac{2-2\,k}{1-2\,k} d(A, B). \tag{5}$$

We have by (4) and (5)

$$\begin{aligned} d(x, w) &\leq d(x, Tx) + d(Tz, Tx) \\ &\leq d(A, B) + k\, d(STz, x) + k\, d(z, STx) \\ &\leq d(A, B) + k \left(1 + \frac{k}{1-k}\right) d(z, x) \\ &\leq d(A, B) + \frac{k\,(2-2\,k)}{(1-k)\,(1-2\,k)}\, d(A, B) \\ &= \frac{1}{1-2\,k}\, d(A, B), \end{aligned}$$

thus, (v) holds. We finally have by (v)

$$d(x, y) \leq d(x, w) + d(y, w) \leq \frac{2}{1-2\,k}\, d(A, B),$$

thus, (vi) holds. □

Remark.

- As in the proof of Theorem 6, we do not use Definition 4 (a) when we prove (i)–(iv).
- $\dfrac{2}{1-2\,k} < \dfrac{2\,(1+2\,k)}{1-2\,k}$ holds for any $k \in (0, 1/2)$.
- Examples 12 and 13 below tell that all the numbers that appear in (v) and (vi) of Theorem 8 are best possible.
- As in Theorem 6,

  $$\sup\{d(x, w) : x \in A_0\} < \sup\{d(z, x) : x \in A_0\}$$

  is possible; see Example 12.

5. Examples

In this section, we will show that all the numbers that appear in Theorems 6 and 8 are best possible.

Example 9. Let $\sigma, \tau, r \in \mathbb{R}$ satisfy $\sigma, \tau \in (0, \infty)$ and $r \in (0, 1)$. Define a subset Y of the Euclidean space $(\mathbb{R}^1, d)$ by

$$Y = \{0\} \cup \big\{ -\tau\, r^n : n \in \mathbb{N} \cup \{0\}\big\} \cup \big\{\sigma\, r^n : n \in \mathbb{N} \cup \{0\}\big\}.$$

Define a mapping U on Y by $Ua = r\,a$ for $a \in Y$. Then the following hold:

(i) If $r < 1/3$, then U is Kannan with $k = \dfrac{r}{1-r}$.

(ii) U is Chatterjea with $k = \dfrac{r}{1+r}$.

Remark. We note that $r/(1+r) \in (0, 1/2)$ always holds, however, $r \geq 1/3$ implies $r/(1-r) \geq 1/2$.

Proof. We first show (ii). Fix $a, b \in Y$ with $a < b$. Noting $a \leq r\,b$ and $r\,a \leq b$, we have

$$\begin{aligned}
&\frac{r}{1+r}\, d(Ua, b) + \frac{r}{1+r}\, d(a, Ub) = \frac{r}{1+r}\, (b - r\,a + r\,b - a) \\
&= r\,(b - a) = d(Ua, Ub),
\end{aligned}$$

thus, (2) holds with $k = r/(1+r)$. In order to show (i), we assume $r < 1/3$. Then we have

$$\begin{aligned}
&\frac{r}{1-r}\, d(Ua, a) + \frac{r}{1-r}\, d(b, Ub) = \frac{r}{1-r}\, (1-r)\,(|a| + |b|) \\
&= r(|a| + |b|) \geq r\,|a - b| = d(Ua, Ub),
\end{aligned}$$

thus, (1) holds with $k = r/(1-r)$. $\square$

Example 10. Let $r \in (0, 1)$ and put $\sigma := 2/(1-r) \in (2, \infty)$. Define sequences $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$, $\{y_n\}_{n \in \mathbb{N} \cup \{0\}}$, $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ by

$$\begin{aligned}
x_n &= (0, \sigma\, r^n), \quad & y_n &= (0, -\sigma\, r^n), \\
u_n &= (1, \sigma\, r^n), \quad & v_n &= (1, -\sigma\, r^n).
\end{aligned}$$

Put $z = (0, 0)$ and $w = (1, 0)$. Define subsets A, B and X of $\mathbb{R}^2$ by

$$\begin{aligned}
A &= \{z\} \cup \big\{x_n : n \in \mathbb{N} \cup \{0\}\big\} \cup \big\{y_n : n \in \mathbb{N} \cup \{0\}\big\}, \\
B &= \{w\} \cup \{u_n : n \in \mathbb{N}\} \cup \{v_n : n \in \mathbb{N}\}
\end{aligned}$$

and $X = A \cup B$. Define mappings T and S by

$$\begin{aligned}
Tx_n &= u_{n+1}, \qquad & Ty_n &= v_{n+1}, \qquad & Tz &= w, \\
Su_n &= x_n, & Sv_n &= y_n, & Sw &= z.
\end{aligned}$$

Define a function e from $X \times X$ into $[0, \infty]$ by

$$e(a,b) = \begin{cases} 1 & \text{if } (a,b) \in \{(x_0, u_1), (y_0, v_1)\} \\ e(b,a) & \text{if } e(b,a) \text{ is defined by the above} \\ \|a-b\|_1 & \text{otherwise,} \end{cases}$$

where $\| \cdot \|_1$ is the ℓ_1-norm on $\mathbb{R}^2$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$d(a,b) = \min \Big\{ \sum_{j=1}^{n} e(a_j, a_{j+1}) : \ n \in \mathbb{N}, \ a_1 = a, \ a_{n+1} = b, \ a_2, a_3, \cdots, a_n \in X \Big\}. \tag{6}$$

Then the following hold:

(i) A, B and X are complete.
(ii) S is nonexpansive.
(iii) $d(Tx, Ty) \leq \dfrac{r}{1-r} d(STx, x) + \dfrac{r}{1-r} d(y, STy)$ for any $x, y \in A$.
(iv) $d(Tx, Ty) \leq \dfrac{r}{1+r} d(STx, y) + \dfrac{r}{1+r} d(x, STy)$ for any $x, y \in A$.
(v) The pair (S, T) forms a weak K-cyclic contraction for any $k \in [0, 1/2)$.
(vi) x_0, y_0 and z are best proximity points of T.
(vii) $d(A, B) = 1$.
(viii) $d(z, x_0) = \sigma$.
(ix) $d(x_0, w) = \sigma - 1$.
(x) $d(x_0, y_0) = 2\,\sigma - 2$.

Proof. We first note

$$\begin{aligned} d(x_0, x) &= e(x_0, x) = \|x_0 - x\|_1, \\ d(x_0, u) &= e(x_0, u_1) + e(u_1, u) = \|x_0 - u\|_1 - 2, \\ d(y_0, x) &= e(y_0, x) = \|y_0 - x\|_1, \\ d(y_0, u) &= e(y_0, v_1) + e(v_1, u) = \|y_0 - u\|_1 - 2, \\ d(x_0, y_0) &= e(x_0, u_1) + e(u_1, v_1) + e(v_1, y_0) = \|x_0 - y_0\|_1 - 2, \\ d(x, y) &= e(x, y) = \|x - y\|_1, \\ d(u, v) &= e(u, v) = \|u - v\|_1 \end{aligned}$$

for $x, y \in A \setminus \{x_0, y_0\}$ and $u, v \in B$. So, (viii)–(x) hold. (vi) and (vii) are obvious. Since $d^*(STx, Tx) = 0$ for any $x \in A$, (v) holds. (i) is obvious. Since

$$d(Su, Sv) = d(u, v)$$

for any $u, v \in B$, (ii) holds. In order to prove (iii) and (iv), we fix $x = (0, \alpha), y = (0, \beta) \in A$. Then we have

$$T(0, \alpha) = (1, r\,\alpha),$$

$$
\begin{aligned}
&ST(0,\alpha) = (0, r\,\alpha) \in A \setminus \{x_0, y_0\},\\
&d\big(T(0,\alpha), T(0,\beta)\big) = |r\,\alpha - r\,\beta|,\\
&d\big(ST(0,\alpha), (0,\alpha)\big) = |r\,\alpha - \alpha|,\\
&d\big((0,\beta), ST(0,\beta)\big) = |r\,\beta - \beta|,\\
&d\big(ST(0,\alpha), (0,\beta)\big) = |r\,\alpha - \beta|,\\
&d\big((0,\alpha), ST(0,\beta)\big) = |\alpha - r\,\beta|.
\end{aligned}
$$

So, as in the proof of Example 9, we can prove (iii) and (iv). □

Example 11. We assume all the assumptions in Example 10. Additionally we assume $r < 1/3$. Put $k := r/(1-r) \in [0, 1/2)$. Then the following hold:

(i) All the assumptions of Theorem 5 hold. Therefore all the assumptions of Theorem 6 hold.
(ii) x_0, y_0 and z are best proximity points of T.
(iii) $d(A,B) = 1$.
(iv) $d(z, x_0) = 2 + 2\,k$, $d(x_0, w) = 1 + 2\,k$ and $d(x_0, y_0) = 2 + 4\,k$.

Proof. We have shown (i)—(iii) in the proof of Example 10. Since $r = k/(1+k)$, we have

$$\sigma = \frac{2}{1-r} = 2 + 2\,k.$$

Using this, we can prove (iv). □

Example 12. We assume all the assumptions in Example 10. Put $k := r/(1+r) \in [0, 1/2)$. Then the following hold:

(i) All the assumptions of Theorem 7 hold. Therefore all the assumptions of Theorem 8 hold.
(ii) x_0, y_0 and z are best proximity points of T.
(iii) $d(A,B) = 1$.
(iv) $d(z, x_0) = \dfrac{2-2\,k}{1-2\,k}$, $d(x_0, w) = \dfrac{1}{1-2\,k}$ and $d(x_0, y_0) = \dfrac{2}{1-2\,k}$.

Proof. We have shown (i)—(iii) in the proof of Example 10. Since $r = k/(1-k)$, we have

$$\sigma = \frac{2}{1-r} = \frac{2-2\,k}{1-2\,k}.$$

Using this, we can prove (iv). □

We finally show that even in the case where $k = 0$, all the numbers that appear in Theorems 6 and 8 are best possible.

Example 13. Put $r = 0$, $\sigma = 2$ and

$$x_0 = (0, 2), \qquad y_0 = (0, -2), \qquad z = (0, 0), \qquad w = (1, 0).$$

Define subsets A, B and X of $\mathbb{R}^2$ by

$$A = \{x_0, y_0, z\}, \qquad B = \{w\}, \qquad X = A \cup B.$$

Define mappings T and S by

$$Tx_0 = w, \qquad Ty_0 = w, \qquad Tz = w, \qquad Sw = z.$$

Define a function e from $X \times X$ into $[0, \infty]$ by

$$e(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \{(x_0, w), (y_0, w)\} \\ e(b, a) & \text{if } e(b, a) \text{ is defined by the above} \\ \|a - b\|_1 & \text{otherwise.} \end{cases}$$

Define a function d from $X \times X$ into $[0, \infty)$ by (6). Then (i)–(x) of Example 10 hold.

Proof. We only note

$$d(x_0, y_0) = e(x_0, w) + e(y_0, w) = 2$$

and $Tx = Ty$ for any $x, y \in A$. We can prove the conclusion as in the proof of Example 10. $\square$

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Received January 31, 2017