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THE EQUIVALENCE OF GYROCOMMUTATIVE GYROGROUPS AND K-LOOPS

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ABSTRACT. It is known that gyrocommutative gyrogroups and K-loops are equivalent. This is a self-contained paper that presents the equivalence.

1. Introduction

Both gyrocommutative gyrogroups and K-loops are non-commutative nor non-associative generalization of commutative groups. In [4], Sabinin, Sabinina and Sbitneva show that every gyrocommutative gyrogroup is just a left Bol loop with Bruck identity. It is well known that a left Bol loop is a K-loop if and only if it has the Bruck property. The paper [4] requires some knowledge of left Bol loops.

There is a possibility that these algebraic systems are defined by a way different depending on literatures. In this paper, the definition of gyrogroup is in accordance with [9] and of K-loop is in accordance with [3]. In section 2, we describe the definitions and some properties of gyrogroups and K-loops for the proof. The descriptions of gyrogroups are in accordance with [9] and of K-loops are in accordance with [3]. In section 3, we prove that K-loops and gyrocommutative gyrogroups are equivalent. The main part of the proof is in accordance with [4].

This paper is self-contained and a patchwork of [3], [9], [4]. The equivalence of these algebraic systems is a fundamental and important fact for who will study gyrogroup or K-loop theory. This paper would be instructive for them.

A referee of the paper kindly recommended the following historical comments.

“For the theory of K-loops readers may consult with Kiechle's book [3]. Not unexpectedly, according to Kiechle [3, pp. 169-170], the term “K-loop” with K named after Karzel was coined by Ungar in 1989 [8] to describe the algebraic structure that later became known as a gyrocommutative gyrogroup. For different purposes, the term “K-loop” was already in use earlier by Soĭkis, in 1970 [6] and later, but independently, by Basarab, in 1992 [1]. Unlike the term “K-loop” that Ungar

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coined, the “ K ” in each of the terms “ K-loop ” coined by Soĭkis and by Basarab does not refer to “ Karzel ” . The early history of K-loops with “ K ” named after Karzel is unfolded in [5, p. 142]. ”

2. Definitions and some notations

A magma $(S, \circ)$ is a set with binary operation $\circ : S \times S \to S;\ (a, b) \mapsto a \circ b$. An automorphism φ of a magma $(S, \circ)$ is a bijective self-map of S, $\varphi : S \to S$, which preserves its magma operation, $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$ for any $a, b \in S$. A magma $(S, \circ)$ is called a groupoid if it contains an identity element e, that is $a \circ e = e \circ a = a$ for any $a \in S$. Such an element is necessarily unique. Let a be an element of a groupoid $(S, \circ)$. An element $b \in S$ is called a left (right) inverse of a if $b \circ a = e$ $(a \circ b = e)$. If b is the uniquely determined left and right inverse of a, then b is called the inverse of a. Note that if b is the inverse of a, then a is the inverse of b.

Let $(S, \circ)$ be a magma, then for each $a \in S$, the map

$$\lambda_a : S \to S;\ x \mapsto a \circ x$$

is called the left translation, and the map

$$\varrho_a : S \to S;\ y \mapsto y \circ a$$

is called the right translation.

Definition 2.1 (K-loop)**.** A groupoid $(L, +)$ is a K-loop if it satisfies the following axioms.

(K1) For any $a, b \in L$, the equation $a + x = b$ has the unique solution $x \in L$.
(K2) For any $a, b \in L$, the equation $y + a = b$ has the unique solution $y \in L$.
(K3) $a + (b + (a + c)) = (a + (b + a)) + c$ for any $a, b, c \in L$.
(K4) Any element a of L has the inverse $-a$ and

$$-(a + b) = (-a) + (-b)$$

for any $a, b \in L$.

Proposition 2.1. *Let* $(L, +)$ *be a groupoid.*

- *The condition* (K1) *is equivalent to the following condition* (K1)’*.*

(K1)’ *Any left translation* λ_a *is bijective.*

- *The condition* (K2) *is equivalent to the following condition* (K2)’*.*

(K2)’ *Any right translation* ϱ_a *is bijective.*

- *The condition* (K3) *is equivalent to the both conditions* (K3)’ *and* (K3)”*.*

(K3)’ $\lambda_a \lambda_b \lambda_a = \lambda_{a+(b+a)}$ *for any* $a, b \in L$*.*
(K3)” $\lambda_a \varrho_{a+c} = \varrho_c \lambda_a \varrho_a$ *for any* $a, b \in L$*.*

Proposition 2.2. *Let $(L,+)$ be a K-loop. Then $\lambda_a^{-1} = \lambda_{-a}$ for any $a \in L$. That is $-a + (a+x) = x$ for any $a, x \in L$.*

Proof. Since Proposition 2.1, $(L,+)$ satisfies the condition (K3)'. Therefore we have $\lambda_a \lambda_{-a} \lambda_a = \lambda_{a+(-a+a)} = \lambda_a$. Thus $\lambda_a \lambda_{-a}$ is the identity map on L and hence $\lambda_a^{-1} = \lambda_{-a}$. □

Definition 2.2 (autotopism)**.** Let $(L,+)$ be a groupoid and α, β, γ be bijections of L. A triple (α, β, γ) is called an autotopism if

$$\alpha(x) + \beta(y) = \gamma(x+y)$$

for any $x, y \in L$. $\operatorname{Top} L$ denotes the set of all autotopisms of L.

Proposition 2.3. *Let $(L,+)$ be a groupoid with the identity e.*

- *If*
$$(\alpha_1, \beta_1, \gamma_1) \circ (\alpha_2, \beta_2, \gamma_2) = (\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)$$
for any $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in \operatorname{Top} L$, then $(\operatorname{Top} L, \circ)$ is a group with the identity (id_L, id_L, id_L) and the inverse
$$(\alpha, \beta, \gamma)^{-1} = (\alpha^{-1}, \beta^{-1}, \gamma^{-1})$$
of $(\alpha, \beta, \gamma) \in \operatorname{Top} L$.
- *If $(\alpha, \beta, \gamma) \in \operatorname{Top} L$ and $\alpha = \gamma$, then*
$$(\alpha, \beta, \gamma) = (\lambda_{\alpha(e)}\beta, \beta, \lambda_{\alpha(e)}\beta).$$

Proof. It is clear that $(\operatorname{Top} L, \circ)$ is a group. Let $(\alpha, \beta, \gamma) \in \operatorname{Top} L$ and $\alpha = \gamma$. By the definition of an autotopism, we have

$$\alpha(e) + \beta(y) = \gamma(e+y) = \gamma(y)$$

for any $y \in L$. Hence

$$\alpha(y) = \gamma(y) = \lambda_{\alpha(e)}\beta(y)$$

for any $y \in L$. □

Definition 2.3 ((gyrocommutative) gyrogroup)**.** A magma $(G, \oplus)$ is a gyrogroup if it satisfies the following axioms.

(G1) There is a left identity $0 \in G$, that is $0 \oplus a = a$ for any $a \in G$.

(G2) There is a left identity $0^* \in G$ such that every $a \in G$ has an element $\ominus a \in G$ satisfying $\ominus a \oplus a = 0^*$.

(G3) For any $a, b, c \in G$, there is a unique element $\operatorname{gyr}[a,b]c \in G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \operatorname{gyr}[a,b]c.$$

(G4) For any $a, b \in G$, the map $\operatorname{gyr}[a,b]$, $c \mapsto \operatorname{gyr}[a,b]c$, is an automorphism of $(G, \oplus)$.

(G5) $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$ for any $a, b \in G$.

A gyrogroup $(G, \oplus)$ is gyrocommutative if the following (G6) is also satisfied.

(G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ for any $a, b \in G$.

Proposition 2.4. *Let $(G, \oplus)$ be a gyrogroup. For any elements $a, b, c \in G$, we have:*

(g1) $a \oplus b = a \oplus c \Leftrightarrow b = c$.
(g2) $\mathrm{gyr}[0, a] = id_G$ *for any left identity* 0.
(g3) $\mathrm{gyr}[\ominus a, a] = id_G$.
(g4) 0^* *is the identity of* $(G, \oplus)$.
(g5) *A left identity is necessarily unique.*
(g6) $\ominus a$ *is a right inverse of* a.
(g7) $\ominus a$ *is the (unique left and right) inverse of* a.
(g8) $\ominus(\ominus a) = a$.
(g9) $\ominus a \oplus (a \oplus b) = b$ *(the left cancellation law).*
(g10) $\lambda_a^{-1} = \lambda_{\ominus a}$.
(g11) $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus c)\}$, *that is,*

$$\mathrm{gyr}[a, b] = \lambda_{(a \oplus b)}^{-1} \lambda_a \lambda_b.$$

(g12) $\mathrm{gyr}[a, b](\ominus c) = \ominus \mathrm{gyr}[a, b]c$.

Proof. (g1): Since $\mathrm{gyr}[\ominus a, a]$ is a bijection, we have

$$\begin{aligned} & a \oplus b = a \oplus c \\ \Leftrightarrow \ & \ominus a \oplus (a \oplus b) = \ominus a \oplus (a \oplus c) \\ \Leftrightarrow \ & (\ominus a \oplus a) \oplus \mathrm{gyr}[\ominus a, a]b = (\ominus a \oplus a) \oplus \mathrm{gyr}[\ominus a, a]c \\ \Leftrightarrow \ & \mathrm{gyr}[\ominus a, a]b = \mathrm{gyr}[\ominus a, a]c \\ \Leftrightarrow \ & b = c. \end{aligned}$$

(g2): For any $x \in G$, we have

$$a \oplus x = (0 \oplus a) \oplus x = 0 \oplus (a \oplus \mathrm{gyr}[0, a]x) = a \oplus \mathrm{gyr}[0, a]x.$$

By (g1), we have $x = \mathrm{gyr}[0, a]x$ and hence $\mathrm{gyr}[0, a] = id_G$.
(g3): Since the condition (G5), we have

$$\mathrm{gyr}[\ominus a, a] = \mathrm{gyr}[\ominus a \oplus a, a] = \mathrm{gyr}[0, a] = id_G.$$

(g4): For any $x \in G$, by (g3), we have

$$\begin{aligned} \ominus x \oplus (x \oplus 0^*) &= (\ominus x \oplus x) \oplus \mathrm{gyr}[\ominus x, x]0^* \\ &= 0^* \oplus \mathrm{gyr}[\ominus x, x]0^* \\ &= 0^* \oplus 0^* \\ &= 0^* \\ &= \ominus x \oplus x. \end{aligned}$$

Hence, by (g1), $x \oplus 0^* = x$ for any $x \in G$. Thus,

$$x \oplus 0^* = 0^* \oplus x = 0^*.$$

(g5): For any left identity 0, we have $0 = 0 + 0^* = 0^*$.

(g6): By (g3) and 0 is the identity, we have

$$\begin{aligned} \ominus a \oplus (a \oplus (\ominus a)) &= (\ominus a \oplus a) \oplus \mathrm{gyr}[\ominus a, a](\ominus a) \\ &= 0 \oplus (\ominus a) \\ &= \ominus a \\ &= \ominus a \oplus 0. \end{aligned}$$

By (g1), $(a \oplus (\ominus a)) = 0$.

(g7): Suppose x and y are left inverses of a. Since (g6), x and y are also right inverses of a, $a \oplus x = 0 = a \oplus y$. By (g1), we have $x = y$.

(g8): It is clear since $\ominus x$ is the inverse of x for any $x \in G$.

(g9): By (g3), we have

$$\ominus a \oplus (a \oplus b) = (\ominus a \oplus a) \oplus \mathrm{gyr}[\ominus a, a]b = b.$$

(g10): By (g8) and (g9), $\lambda_{\ominus a}\lambda_a = \lambda_a \lambda_{\ominus a} = id_G$.

(g11): By (G3) and (g9), we have

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus c)\}.$$

Hence, by (g10),

$$\mathrm{gyr}[a, b] = \lambda_{\ominus(a \oplus b)}\lambda_a\lambda_b = \lambda^{-1}_{a \oplus b}\lambda_a\lambda_b.$$

(g12): Since $\mathrm{gyr}[a, b]$ is an automorphism of $(G, \oplus)$,

$$\mathrm{gyr}[a, b](\ominus c) \oplus \mathrm{gyr}[a, b](c) = \mathrm{gyr}[a, b]0 = 0$$

Hence, $\mathrm{gyr}[a, b](\ominus c) = \ominus \mathrm{gyr}[a, b]c$. □

Lemma 2.1. *Let $(G, \oplus)$ be a gyrogroup. Then*

$$\mathrm{gyr}[a, \ominus \mathrm{gyr}[a, b]b] \, \mathrm{gyr}[a, b] = id_G.$$

for any $a, b \in G$.

Proof. For any $x \in G$, we have

$$\begin{aligned} & a \oplus \operatorname{gyr}[a \oplus b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b]x \\ = \;& (a \oplus (b \ominus b)) \oplus \operatorname{gyr}[a \oplus b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b]x \\ = \;& ((a \oplus b) \ominus \operatorname{gyr}[a,b]b) \oplus \operatorname{gyr}[a \oplus b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b]x \\ = \;& (a \oplus b) \oplus (\ominus \operatorname{gyr}[a,b]b \oplus \operatorname{gyr}[a,b]x) \\ = \;& (a \oplus b) \oplus \operatorname{gyr}[a,b](\ominus b \oplus x) \\ = \;& a \oplus (b \oplus (\ominus b \oplus x)) \\ = \;& a \oplus x. \end{aligned}$$

It implies that

$$\operatorname{gyr}[a \oplus b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b] = id_G$$

by (g1). Hence

$$\begin{aligned} id_G &= \operatorname{gyr}[a \oplus b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b] \\ &= \operatorname{gyr}[(a \oplus b) \ominus \operatorname{gyr}[a,b]b, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b] \\ &= \operatorname{gyr}[a \oplus (b \ominus b), \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b] \\ &= \operatorname{gyr}[a, \ominus \operatorname{gyr}[a,b]b] \operatorname{gyr}[a,b] \end{aligned}$$

by (G5), (G3) and (g12). □

Proposition 2.5. *Let $(G, \oplus)$ be a gyrogroup. Then for any $a, b \in G$, we have:*

(LL) *The equation $a \oplus x = b$ has a unique solution $x = \ominus a \oplus b$.*
(RL) *The equation $y \oplus a = b$ has a unique solution $y = b \ominus \operatorname{gyr}[b,a]a$.*

Proof. (LL): Let $x = \ominus a \oplus b$. By (g9), we have

$$a \oplus x = a \oplus (\ominus a \oplus b) = b.$$

Hence x is a solution of the equation $a \oplus x = b$. If $x' \in G$ satisfies the equation $a \oplus x' = b$, then

$$a \oplus x = a \oplus x'$$

and hence $x = x'$ by (g1).

(RL): Let y be a solution of $y \oplus a = b$. Then

$$\begin{aligned} y &= y \oplus (a \ominus a) \\ &= (y \oplus a) \ominus \operatorname{gyr}[y,a]a \\ &= (y \oplus a) \ominus \operatorname{gyr}[y \oplus a, a]a \\ &= b \ominus \operatorname{gyr}[b,a]a. \end{aligned}$$

Conversely, if $y = b \ominus \mathrm{gyr}[b,a]a$, then

$$\begin{aligned} b &= b \oplus (\ominus \mathrm{gyr}[b,a]a \oplus \mathrm{gyr}[b,a]a) \\ &= (b \ominus \mathrm{gyr}[b,a]a) \oplus \mathrm{gyr}[b, \ominus \mathrm{gyr}[b,a]]\, \mathrm{gyr}[b,a]a \\ &= (b \ominus \mathrm{gyr}[b,a]a) \oplus a \\ &= y \oplus a \end{aligned}$$

by Lemma 2.1. □

Lemma 2.2. *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$\mathrm{gyr}[a,b](\ominus b \ominus a) = \ominus(a \oplus b)$$

for any $a, b \in G$.

Proof. By (g11) and (g9), we have

$$\begin{aligned} \mathrm{gyr}[a,b](\ominus b \ominus a) &= \ominus\,(a \oplus b) \oplus (a \oplus (b \oplus (\ominus b \ominus a))) \\ &= \ominus\,(a \oplus b). \end{aligned}$$

□

Proposition 2.6. *Let* $(G, \oplus)$ *be a gyrogroup. Then* $(G, \oplus)$ *is gyrocommutative if and only if it possesses the automorphic inverse property,*

(G5)' $\ominus(a \oplus b) = \ominus a \ominus b$ *for any* $a, b \in G$.

Proof. If $(G, \oplus)$ is gyrocommutative, then

$$\begin{aligned} \mathrm{gyr}[a,b](\ominus(\ominus b \ominus a)) &= \ominus\, \mathrm{gyr}[a,b](\ominus b \ominus a) \\ &= a \oplus b \\ &= \mathrm{gyr}[a,b](b \oplus a) \end{aligned}$$

for any $a, b \in G$, by (g12) and Lemma 2.2. It implies that $\ominus(b \oplus a) = \ominus b \ominus a$.

Conversely, if $(G, \oplus)$ possesses the automorphic inverse property, then

$$\begin{aligned} a \oplus b &= \ominus\, \mathrm{gyr}[a,b](\ominus b \ominus a) \\ &= \mathrm{gyr}[a,b](\ominus(\ominus b \ominus a)) \\ &= \mathrm{gyr}[a,b](b \oplus a) \end{aligned}$$

for any $a, b \in G$, by Lemma 2.2 and (g12). □

3. Equivalence of gyrocommutative gyrogroups and K-loops

Theorem 3.1. *A magma is a gyrocommutative gyrogroup if and only if it is a K-loop.*

Proof. First, we show that a gyrocommutative gyrogroup is a K-loop. Let $(G, \oplus)$ be a gyrocommutative gyrogroup.

(K1): By (LL) of Proposition 2.5.

(K2): By (RL) of Proposition 2.5.

(K3): Let $a, b \in G$. Put $w = a \oplus b$ and $q = \ominus a$ then the map $(a, b) \mapsto (w, q)$ is a bijective self-map of $G \times G$ and

$$\begin{aligned} \mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b] &\iff \lambda^{-1}_{(a\oplus b)\oplus b}\lambda_{a\oplus b}\lambda_b = \lambda^{-1}_{a\oplus b}\lambda_a\lambda_b \\ &\iff \lambda^{-1}_{(a\oplus b)\oplus b} = \lambda^{-1}_{a\oplus b}\lambda_a\lambda^{-1}_{a\oplus b} \\ &\iff \lambda_{(a\oplus b)\oplus b} = \lambda_{a\oplus b}\lambda_{\ominus a}\lambda_{a\oplus b} \\ &\iff \lambda_{w\oplus(q\oplus w)} = \lambda_w\lambda_q\lambda_w. \end{aligned}$$

By the condition (G5), $(G, \oplus)$ satisfies the condition (K3)'. Hence $(G, \oplus)$ satisfies the condition (K3) by Proposition 2.1.

(K4): By Proposition 2.4, any $a \in G$ has the inverse $\ominus a$. By Proposition 2.6, we have

$$\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b)$$

for any $a, b \in G$.

Next, we show that a K-loop is a gyrocommutative gyrogroup. Let $(L, +)$ be a K-loop.

(G1): Since $(L, +)$ is a groupoid, $(L, +)$ has the identity e.

(G2): By the condition (K4), any $a \in L$ has the inverse $-a$.

(G3): Let $a, b, c \in L$. By the condition (K1), the equation $(a+b)+x = a+(b+c)$ has a unique solution x. Let $\delta[a, b] = \lambda^{-1}_{a+b}\lambda_a\lambda_b$. Then we have $\lambda_{a+b}\delta[a, b] = \lambda_a\lambda_b$. Hence $(a + b) + \delta[a, b](c) = a + (b + c)$. Thus $x = \delta[a, b](c)$ is the unique solution of $(a + b) + x = a + (b + c)$.

(G4): Let $a, x, y \in L$. Put $v = -x$ and $w = x + y$ then

$$\begin{aligned} \lambda_a\varrho_a(x) + \lambda_a^{-1}(y) &= \lambda_a\varrho_a(-v) + \lambda_a^{-1}(v + w) \\ &= \{a + (-v + a)\} + \lambda_a^{-1}(v + w) \\ &= a + \{-v + (a + \lambda_a^{-1}(v + w))\} \\ &= \lambda_a(w) \\ &= \lambda_a(x + y) \end{aligned}$$

by the condition (K3). It implies that $\tau_a = (\lambda_a\varrho_a, \lambda_a^{-1}, \lambda_a) \in \mathrm{Top}\, L$ for any $a \in L$. Therefore $\tau_{a+b} \circ \tau_a^{-1} \circ \tau_b^{-1} \in \mathrm{Top}\, L$ for any $a, b \in L$ by Proposition 2.3. Put $\alpha =$

$\lambda_{a+b}\lambda_a^{-1}\lambda_b^{-1}$. The first component of $\tau_{a+b} \circ \tau_a^{-1} \circ \tau_b^{-1}$ is

$$\begin{aligned}
& \lambda_{a+b}\varrho_{a+b}(\lambda_a\varrho_a)^{-1}(\lambda_b\varrho_b)^{-1} \\
= & \lambda_{a+b}\varrho_{a+b}\varrho_a^{-1}\lambda_a^{-1}\varrho_b^{-1}\lambda_b^{-1} \\
= & \lambda_{a+b}\varrho_{a+b}(\varrho_b\lambda_a\varrho_a)^{-1}\lambda_b^{-1} \\
= & \lambda_{a+b}\varrho_{a+b}(\lambda_a\varrho_{a+b})^{-1}\lambda_b^{-1} \\
= & \lambda_{a+b}\varrho_{a+b}\varrho_{a+b}^{-1}\lambda_a^{-1}\lambda_b^{-1} \\
= & \lambda_{a+b}\lambda_a^{-1}\lambda_b^{-1} \\
= & \alpha
\end{aligned}$$

by (K3)". The second component is $\lambda_{a+b}^{-1}\lambda_a\lambda_b = \delta[a,b]$. The third component is $\lambda_{a+b}\lambda_a^{-1}\lambda_b^{-1} = \alpha$. Thus, we have $(\alpha, \delta[a,b], \alpha) \in \mathrm{Top}\, L$. We have

$$\alpha(e) = (a+b) + (-a + (-b+e)) = (a+b) + (-a-b) = e$$

by the condition (K4). Hence $(\delta[a,b], \delta[a,b], \delta[a,b]) \in \mathrm{Top}\, L$ by Proposition 2.3. It implies that $\delta[a,b]$ is an automorphism of $(L,+)$.

(G5): Let $a, b \in L$. Put $x = -b$ and $y = b + a$ then the map $(a,b) \mapsto (x,y)$ is a bijective self-map of $G \times G$ and

$$\begin{aligned}
\lambda_{a+(b+a)} = \lambda_a\lambda_b\lambda_a & \iff \lambda_{(x+y)+y} = \lambda_{(x+y)}\lambda_{-x}\lambda_{(x+y)} \\
& \iff \lambda_{(x+y)+y}^{-1} = \lambda_{(x+y)}^{-1}\lambda_x\lambda_{(x+y)}^{-1} \\
& \iff \lambda_{(x+y)+y}^{-1}\lambda_{(x+y)}\lambda_y = \lambda_{(x+y)}^{-1}\lambda_x\lambda_y \\
& \iff \delta[x+y,y] = \delta[x,y].
\end{aligned}$$

Since $(L,+)$ satisfies the condition (K3)', we have $\delta[x+y,y] = \delta[x,y]$ for any $x, y \in L$.

(G6): Since $(L,+)$ satisfies the conditions (G1) to (G5), $(L,+)$ is a gyrogroup. Since $(L,+)$ satisfies the condition (K4), Proposition 2.6 asserts that $(L,+)$ satisfies the condition (G6). □

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