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THE AUTOMORPHISM THEOREM AND ADDITIVE GROUP ACTIONS ON THE AFFINE PLANE

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ABSTRACT. Due to Rentschler, Miyanishi and Kojima, the invariant ring for a $\mathbf{G}_a$-action on the affine plane over an arbitrary field is generated by one coordinate. In this note, we give a new short proof for this result using the automorphism theorem of Jung and van der Kulk.

1. Introduction

Let k be a field, A a k-domain, $A[T]$ the polynomial ring in one variable over A, and $\sigma : A \to A[T]$ a homomorphism of k-algebras. Then, σ defines an action of the additive group $\mathbf{G}_a = \operatorname{Spec} k[T]$ on $\operatorname{Spec} A$ if and only if the following holds for each $a \in A$, where we write $\sigma(a) = \sum_{i\geq 0} a_i T^i$ with $a_i \in A$, and U is a new variable:

(A1) $a_0 = a$. (A2) $\sum_{i\geq 0} \sigma(a_i)U^i = \sum_{i\geq 0} a_i(T+U)^i$ in $A[T,U]$.

If this is the case, we call σ a $\mathbf{G}_a$-*action* on A. The *ring* $A^\sigma := \{a \in A \mid \sigma(a) = a\}$ *of* σ-*invariants* is equal to $\sigma^{-1}(A)$ by (A1). We say that σ is *nontrivial* if $A^\sigma \neq A$.

Let $k[x_1, x_2]$ be the polynomial ring in two variables over k, and $\operatorname{Aut}_k k[x_1, x_2]$ the automorphism group of the k-algebra $k[x_1, x_2]$. We often express $\phi \in \operatorname{Aut}_k k[x_1, x_2]$ as $(\phi(x_1), \phi(x_2))$. We call $f \in k[x_1, x_2]$ a *coordinate* of $k[x_1, x_2]$ if there exists $g \in k[x_1, x_2]$ such that (f, g) belongs to $\operatorname{Aut}_k k[x_1, x_2]$, that is, $k[f, g] = k[x_1, x_2]$.

The following theorem is a fundamental result for $\mathbf{G}_a$-actions on $k[x_1, x_2]$.

Theorem 1.1. *For every nontrivial* $\mathbf{G}_a$*-action* σ *on* $k[x_1, x_2]$*, there exists a coordinate* f *of* $k[x_1, x_2]$ *such that* $k[x_1, x_2]^\sigma = k[f]$.

This theorem was first proved by Rentschler [13] when $\operatorname{char} k = 0$ in 1968, and then by Miyanishi [11] when k is algebraically closed in 1971. Recently, Kojima [7] proved the general case by making use of Russell-Sathaye [14] (see also [9]).

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For each $f \in k[x_1, x_2]$, we denote by $\deg f$ the total degree of f, and by $\bar{f}$ or $(f)^-$ the highest homogeneous part of f for the standard grading on $k[x_1, x_2]$. The following well-known theorem was first proved by Jung [5] when $\operatorname{char} k = 0$ in 1942. The general case was proved by van der Kulk [8] in 1953 (see also the proof of Makar-Limanov [10] and its modifications by Dicks [3] and Cohn [1, Thm. 8.5]).

Theorem 1.2. *For every $(f_1, f_2) \in \operatorname{Aut}_k k[x_1, x_2]$ with $\deg f_1 \geq 2$ or $\deg f_2 \geq 2$, there exist $(i, j) \in \{(1, 2), (2, 1)\}$, $\alpha \in k^*$ and $l \geq 1$ such that $\bar{f}_i = \alpha \bar{f}_j^{\,l}$*

The purpose of this note is to give a new short proof of Theorem 1.1 based on Theorem 1.2 (cf. §2). We should mention that, if k is an infinite field, Theorem 1.1 can be derived from Theorem 1.2 by a group-theoretic approach (cf. [6]). Our approach is different from this approach, and is valid for an arbitrary k.

Conversely, Theorem 1.2 can be derived easily from Theorem 1.1. This seems known to experts, at least when $\operatorname{char} k = 0$ (cf. e.g. [4, §5.1] for related discussion). For completeness, we also give a proof for this implication (cf. §3).

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2. $\mathbf{G}_a$-action

Recall that, if σ is a nontrivial $\mathbf{G}_a$-action on A, then A has transcendence degree one over A^σ (cf. [12, §1.5]). For each $t \in A^\sigma$, the map $\sigma_t : A \xrightarrow{\sigma} A[T] \ni f(T) \mapsto f(t) \in A$ is an automorphism of the k-algebra A. Actually, we have $\sigma_0 = \mathrm{id}_A$ by (A1), and $\sigma_t \circ \sigma_u = \sigma_{t+u}$ for each $t, u \in A^\sigma$ by (A2).

Now, let us derive Theorem 1.1 from Theorem 1.2. For each $q = \sum_{i \geq 0} q_i T^i \in k[x_1, x_2][T] \setminus \{0\}$ with $q_i \in k[x_1, x_2]$, we define the $\mathbf{Z}^2$-*degree* of q by

$$\deg_{\mathbf{Z}^2} q := \max\{(i, \deg q_i) \in \mathbf{Z}^2 \mid i \geq 0,\ q_i \neq 0\},$$

where $\mathbf{Z}^2$ is ordered lexicographically, i.e., $(a, b) \leq (a', b')$ if and only if $a < a'$, or $a = a'$ and $b \leq b'$. Let σ be any nontrivial $\mathbf{G}_a$-action on $k[x_1, x_2]$. It suffices to find $(f_1, f_2) \in \operatorname{Aut}_k k[x_1, x_2]$ for which $\sigma(f_1)$ or $\sigma(f_2)$ belongs to $k[x_1, x_2]$. Suppose that such an (f_1, f_2) does not exist. Choose $\phi = (f_1, f_2) \in \operatorname{Aut}_k k[x_1, x_2]$ so that $\deg_{\mathbf{Z}^2} \sigma(f_1) + \deg_{\mathbf{Z}^2} \sigma(f_2)$ is minimal, and write $\sigma(f_i) = q_i(T) = \sum_{j=0}^{m_i} q_{i,j} T^j$ for $i = 1, 2$, where $q_{i,j} \in k[x_1, x_2]$ with $q_{i,m_i} \neq 0$. By supposition, we have $m_1, m_2 \geq 1$. Since $k[x_1, x_2]^\sigma \neq k$, we may take $g \in k[x_1, x_2]^\sigma \setminus k$. Then, $\sigma_{g^r} \circ \phi = (q_1(g^r), q_2(g^r))$ belongs to $\operatorname{Aut}_k k[x_1, x_2]$ for each $r \geq 0$ as mentioned above. Since $\deg g \geq 1$, there exists $r_0 > 0$ such that, for each $r \geq r_0$ and $i = 1, 2$, we have $(q_i(g^r))^- = \bar{q}_{i,m_i} \bar{g}^{rm_i}$ and $\deg q_i(g^r) \geq 2$. By Theorem 1.2, for each $r \geq r_0$, there exist $(i, j) \in \{(1, 2), (2, 1)\}$, $\alpha \in k^*$ and $l \geq 1$ such that $\bar{q}_{i,m_i} \bar{g}^{rm_i} = \alpha (\bar{q}_{j,m_j} \bar{g}^{rm_j})^l$. This equality implies that

$$r(m_i - l m_j) \deg g = l \deg q_{j,m_j} - \deg q_{i,m_i}. \tag{2.1}$$

We note that (i, j), α and l above depend on r. By (2.1), we see that $m_i = lm_j$ holds for sufficiently large r. Take such an r. Then, we have $\bar{q}_{i,m_i} = \alpha\bar{q}^l_{j,m_j}$, and hence $\deg(q_{i,m_i} - \alpha q^l_{j,m_j}) < \deg q_{i,m_i}$. Thus, the $\mathbf{Z}^2$-degree of

$$\sigma(f_i - \alpha f^l_j) = q_i(T) - \alpha q_j(T)^l = (q_{i,m_i} - \alpha q^l_{j,m_j})T^{m_i} + (\text{terms of lower degree in } T)$$

is strictly less than that of $\sigma(f_i)$. Since $(f_i - \alpha f^l_j, f_j)$ belongs to $\mathrm{Aut}_k\, k[x_1, x_2]$, this contradicts the minimality of $\deg_{\mathbf{Z}^2} \sigma(f_1) + \deg_{\mathbf{Z}^2} \sigma(f_2)$, completing the proof.

3. Automorphism Theorem

We derive Theorem 1.2 from Theorem 1.1. Let $f = \sum_{i_1,i_2 \geq 0} u_{i_1,i_2} x_1^{i_1} x_2^{i_2}$ be an element of $k[x_1, x_2] \setminus \{0\}$, where $u_{i_1,i_2} \in k$. For each $\mathbf{w} = (w_1, w_2) \in \mathbf{R}^2$, we define

$$\deg_{\mathbf{w}} f := \max\{i_1 w_1 + i_2 w_2 \mid i_1, i_2 \geq 0,\ u_{i_1,i_2} \neq 0\} \quad \text{and} \quad f^{\mathbf{w}} := {\sum}' u_{i_1,i_2} x_1^{i_1} x_2^{i_2},$$

where the sum $\sum'$ is taken over $i_1, i_2 \geq 0$ with $i_1 w_1 + i_2 w_2 = \deg_{\mathbf{w}} f$. We say that f is $\mathbf{w}$-*homogeneous* if $f^{\mathbf{w}} = f$, and *non-univariate* if $f \notin k[x_1] \cup k[x_2]$. We define

$$\mathbf{w}(f) := (\deg_{(0,1)} f, \deg_{(1,0)} f).$$

We remark that $f^{\mathbf{w}(f)}$ is non-univariate if f is non-univariate.

The following lemma is a consequence of Theorem 1.1.

Lemma 3.1. *If σ is a nontrivial $\mathbf{G}_a$-action on $k[x_1, x_2]$, and $f \in k[x_1, x_2]^\sigma$ is non-univariate, then there exist $a, b \in k^*$, $(i, j) \in \{(1, 2), (2, 1)\}$ and $l, m \geq 1$ such that*

$$f^{\mathbf{w}(f)} = a(x_i - bx^l_j)^m. \tag{3.1}$$

Proof. Since f is non-univariate, so is $f^{\mathbf{w}(f)}$ as remarked. By Derksen–Hadas–Makar-Limanov [2, Prop. 2.2], there exists a nontrivial $\mathbf{G}_a$-action τ on $k[x_1, x_2]$ such that $f^{\mathbf{w}(f)}$ belongs to $k[x_1, x_2]^\tau$. By Theorem 1.1, $k[x_1, x_2]^\tau = k[h]$ holds for some coordinate h of $k[x_1, x_2]$. We may assume that h has no constant term. Then, since $f^{\mathbf{w}(f)}$ belongs to $k[h] \setminus k$ and $f^{\mathbf{w}(f)}$ is $\mathbf{w}(f)$-homogeneous, we see that h is $\mathbf{w}(f)$-homogeneous, and $f^{\mathbf{w}(f)} = \alpha h^m$ for some $\alpha \in k^*$ and $m \geq 1$. This implies that h is non-univariate. Since h is a $\mathbf{w}(f)$-homogeneous coordinate, h must have the form $\beta x_i + \gamma x^l_j$ for some $\beta, \gamma \in k^*$ and $l \geq 1$. Therefore, $f^{\mathbf{w}(f)}$ is written as in (3.1). □

Now, we prove Theorem 1.2. Take $\phi = (f_1, f_2) \in \mathrm{Aut}_k\, k[x_1, x_2]$. Set $w_i := \deg f_i$ and $g_i := \phi^{-1}(x_i)$ for $i = 1, 2$. Assume that $w_1 \geq 2$ or $w_2 \geq 2$. Then, there exists $t \in \{1, 2\}$ such that g_t is non-univariate. Note that a nontrivial $\mathbf{G}_a$-action σ on $k[x_1, x_2]$ is defined by $\sigma(g_t) = g_t$ and $\sigma(g_u) = g_u + T$, where $u \neq t$. Since g_t belongs to $k[x_1, x_2]^\sigma$, we may write $g_t^{\mathbf{w}(g_t)}$ as in (3.1) by Lemma 3.1. Set $\mathbf{w} := (w_1, w_2)$. Then, we have

$$\deg_{\mathbf{w}} g_t = m \max\{w_i, lw_j\} \geq \max\{w_1, w_2\} \geq 2 > 1 = \deg x_t = \deg \phi(g_t).$$

This implies that $a(\bar{f}_i - b\bar{f}_j^l)^m = 0$. Therefore, we get $\bar{f}_i = b\bar{f}_j^l$, proving Theorem 1.2.

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