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AN L^1-THEORY FOR SCALAR CONSERVATION LAWS WITH MULTIPLICATIVE NOISE ON A PERIODIC DOMAIN

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ABSTRACT. We study the Cauchy problem for a multi-dimensional scalar conservation law with a multiplicative noise. Our aim is to give the well-posedness of an L^1-solution characterized by a kinetic formulation under appropriate assumptions. In particular, we focus on the existence of such a solution.

1. Introduction

In this paper we study a scalar conservation law with a stochastic forcing of the following type

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t) \quad \text{in} \quad \mathbb{T}^N \times (0,T), \tag{1.1}$$

with the initial condition

$$u(\cdot,0) = u_0(\cdot) \quad \text{in} \quad \mathbb{T}^N, \tag{1.2}$$

where $\mathbb{T}^N$ is the N-dimensional torus and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. More precisely, $(\mathscr{F}_t)$ is a complete right-continuous filtration and $W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k$ with $(\beta_k)_{k\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)$ and $(e_k)_{k\geq 1}$ a complete orthonormal system in a separable Hilbert space H (cf. [3] for example).

In the deterministic case (i.e. $\Phi \equiv 0$), the problem (1.1), (1.2) has been extensively studied by many authors [15], [16], [17], [19]. It is well known that a smooth solution is constant along characteristic curves, which can intersect each other and shocks can occur. Consequently classical solutions do not exist in general on the whole interval $[0,T]$ and distributional solutions are not unique. In order to obtain the well posedness of deterministic scalar conservation laws, Kružkov [15] introduced

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the notion of entropy solution. On the other hand, Lions, Perthame and Tadmor [16] introduced the notion of kinetic formulation and kinetic solution. There are several papers concerning kinetic or entropy solutions for deterministic degenerate parabolic equations, see [1], [2], [12], [13].

To perturb a stochastic term is natural for applications, which appears in wide variety of fields as physics, engineering and others. The Cauchy problem for the stochastic equation has been studied in [5], [6], [9], [11]. On the other hand, Kobayasi and the author [14] proved the uniqueness and the existence of kinetic solution, and the author [18] proved the equivalence between kinetic solutions and entropy solutions to the Cauchy-Dirichlet problem for stochastic scalar conservation laws.

It seems that the L^1-setting is the most suitable to scalar conservation laws, but not L^p or L^∞. The reason is that solutions to scalar conservation laws have L^1-contraction property. Besides that, using averaging lemmas for example, we may get the $W^{r,1}$-regularity of a solution for sufficiently small $r > 0$. If we get this regularity, then by the compactness argument we would obtain the long time behavior or invariant measure of such a solution as in [7] which studied the case of an additive noise (i.e. Φ independent of u). For the above reasons, we develop in this paper the L^1-framework.

Our purpose of this paper is to give a proof of existence of L^1-kinetic solutions to the initial value problem (1.1), (1.2) under appropriate assumptions. The assumption of stochastic term

$$G^2(x,\xi) = \sum_{k=1}^{\infty} |g_k(x,\xi)|^2 \le C(1+|\xi|^2) \tag{1.3}$$

is used by many authors [4], [5], [6], [7], [10], [14], [18]. To deal with the stochastic term in the $L^1(\mathbb{T}^N)$-framework, we use a slightly strong assumption (1.4) below instead of (1.3). Even if we use the assumption (1.4) instead of the assumption (1.3), the equation (1.1), (1.2) includes many important cases such as the finite stochastic integral case, the case of the fuction G^2 with compact support, and so on. Moreover, the assumption (1.4) gives a better result as a stochastic process, that is, a solution u belongs to $L^2(\Omega \times [0,T]; L^1(\mathbb{T}^N))$ (see Proposition 3.1).

We now give the precise assumptions in this paper:

(A_1) The flux function $A : \mathbb{R} \to \mathbb{R}^N$ is of class C^1 and its derivatives denoted by $a = (a_1, \dots, a_N)$ have at most polynomial growth.

(A_2) For each $z \in L^2(\mathbb{T}^N)$, $\Phi(z) : H \to L^2(\mathbb{T}^N)$ is defined by $\Phi(z)e_k = g_k(\cdot, z(\cdot))$, where $g_k \in C(\mathbb{T}^N \times \mathbb{R})$ satisfies the following conditions:

$$|g_k(x,\xi)| \le C_k(1+|\xi|), \tag{1.4}$$

$$\sum_{k=1}^{\infty} |g_k(x,\xi) - g_k(y,\zeta)|^2 \le C\Big(|x-y|^2 + |\xi-\zeta| r(|\xi-\zeta|)\Big) \tag{1.5}$$

for every $x, y \in \mathbb{T}^N$, $\xi, \zeta \in \mathbb{R}$. Here, $\{C_k\}$ is a sequence satisfying $\sum_{k=1}^{\infty} C_k^2 < +\infty$ and r is a continuous non-decreasing function on $\mathbb{R}_+$ with $r(0) = 0$.

In this paper, we denote by C a constant. Its value may change from one line to another. We sometimes precise its dependence on some parameters.

This paper is organized as follows. In Section 2, we first introduce the notion of kinetic solutions to the problem (1.1), (1.2) by using the kinetic formulation and then state the main result. We also give some property of kinetic measure which is used to prove the uniqueness. In Section 3, we prove the existence of kinetic solutions and the equi-integrability condition (3.9) below. The condition (3.9) is used to prove the continuity of trajectories in $L^1(\mathbb{T}^N)$.

2. Preliminaries and the main result

We will give the definition of kinetic solutions and mention the main result in this section. Define

$$f_+(u,\xi) = \begin{cases} 1 & \text{if } \xi < u, \\ 0 & \text{if } \xi \ge u, \end{cases} \quad \text{and} \quad f_-(u,\xi) = \begin{cases} -1 & \text{if } \xi > u, \\ 0 & \text{if } \xi \le u. \end{cases}$$

And also, we will use the notations

$$a \vee b := \max\{a,b\}, \quad a \wedge b := \min\{a,b\}, \quad a^+ := a \vee 0, \quad a^- := a \wedge 0,$$

for $a, b \in \mathbb{R}$.

Definition 2.1 (Kinetic measure)**.** A map m from Ω to the set of non-negative Radon measures over $\mathbb{T}^N \times [0,T) \times \mathbb{R}$ is said to be a kinetic measure if

(i) m is weakly measurable, i.e., for each $\phi \in C_c(\mathbb{T}^N \times [0,T) \times \mathbb{R})$ the map $m(\phi) : \Omega \to \mathbb{R}$ is measurable,

(ii) m vanishes for large ξ in the following sense:

$$\lim_{R\to\infty} \frac{1}{R} \mathbb{E} m(\mathbb{T}^N \times [0,T) \times \{\xi \in \mathbb{R}; R \le |\xi| \le 2R\}) = 0, \tag{2.1}$$

(iii) for all $\phi \in C_c(\mathbb{T}^N \times \mathbb{R})$, the process

$$t \mapsto \int_{\mathbb{T}^N \times [0,t] \times \mathbb{R}} \phi(x,\xi)\, dm(x,s,\xi) \tag{2.2}$$

is predictable,

where $\mathbb{E}X$ denotes the expectation of a random variable X i.e. $\mathbb{E}X = \int_\Omega X\, dP$.

Definition 2.2 (Kinetic solution)**.** Let $u_0 \in L^2(\Omega, \mathscr{F}_0, dP; L^1(\mathbb{T}^N))$ and $u \in L^2(\Omega \times [0,T), \mathcal{P}, dP \otimes dt; L^1(\mathbb{T}^N)) \cap L^2(\Omega; L^\infty(0,T; L^1(\mathbb{T}^N)))$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times [0,T)$ associated to $(\mathcal{F}_t)$. Then u is said to be a kinetic solution to (1.1), (1.2) with initial datum u_0 if there exists a kinetic measure m such that the pair (u,m) satisfies a kinetic formulation: for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, P-a.s., a.e. $t \in [0,T)$,

$$\begin{aligned}
&-\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+(u(x,t),\xi)\varphi(x,\xi)\, d\xi dx + \int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+(u_0(x),\xi)\varphi(x,\xi)\, d\xi dx \\
&\quad + \int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+(u(x,s),\xi) a(\xi)\cdot\nabla\varphi(x,\xi)\, d\xi dx ds \\
&= -\sum_{k=1}^{\infty}\int_0^t\int_{\mathbb{T}^N} g_k(x,u)\varphi(x,u)\, dx d\beta_k(s) \\
&\quad - \frac{1}{2}\int_0^t\int_{\mathbb{T}^N} G^2(x,u)\partial_\xi\varphi(x,u)\, dx ds + \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \partial_\xi\varphi(x,\xi)\, dm.
\end{aligned} \tag{2.3}$$

The above definition concerning the notion of kinetic solution has been introduced in [4]. For the advantage of kinetic solutions as well as kinetic formulations in the stochastic case, we refer to [4], [5], [6], [10]. And also, in the same fashion as in [6, Section 3.3], we can see that the kinetic solution in the sense of Defintion 2.2 is equivalent to the entropy solution which satisfies the inequality (38) in [6]. We are now in a position to state our main result.

Theorem 2.1. *Let $u_0 \in L^2(\Omega, \mathscr{F}_0, dP; L^1(\mathbb{T}^N))$. Under the assumptions (A_1), (A_2), there exists a unique kinetic solution to (1.1), (1.2), which has almost surely continuous orbits in $L^1(\mathbb{T}^d)$. Moreover,*

$$\mathbb{E}\|u_1(t) - u_2(t)\|_{L^1(\mathbb{T}^d)} \le \mathbb{E}\|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{T}^d)} \tag{2.4}$$

for all kinetic solutions u_1, u_2 to (1.1), (1.2) with initial data $u_{1,0}$ and $u_{2,0}$, respectively.

To prove the uniqueness of solutions, we need the following lemma:

Lemma 2.1. *Let ψ be a smooth function from $\mathbb{R}$ to $\mathbb{R}_+$ whose support is a subset of $(-1,1)$ and such that $\int_{\mathbb{R}}\psi = 1$. We set $\psi_\delta(\xi) = \frac{1}{\delta}\psi(\frac{\xi}{\delta})$ for $\delta > 0$ and define the nondecreasing function μ on $\mathbb{R}$ by*

$$\mu(\xi) = \begin{cases} \mathbb{E}m(\mathbb{T}^N \times [0,T) \times [0,\xi)) & \text{if } \ \xi \ge 0 \\ -\mathbb{E}m(\mathbb{T}^N \times [0,T) \times (\xi,0)) & \text{if } \ \xi < 0. \end{cases}$$

Let $\mathbb{D}$ be the set of $\xi \in (0,\infty)$ such that μ is differentiable at $-\xi$ and ξ. Then it holds that

(i) *If* $R \in \mathbb{D}$, *then* $\displaystyle\int_{\mathbb{R}} \psi_\delta(\xi \pm R)\ d\mu(\xi) \to \mu'(\mp R)$ *as* $\delta \downarrow 0$.

(ii) $\displaystyle\liminf_{R\to\infty, R\in\mathbb{D}} \mu'(\pm R) = 0$.

Proof. Since $\mu(\xi \mp R) = \mu(\mp R) + \mu'(\mp R)\xi + o(\xi)$, it follows that

$$\int_{\mathbb{R}} \psi_\delta(\xi \pm R)\ d\mu(\xi) = -\int_{-\delta}^{\delta} \mu(\xi \mp R)\ d\psi_\delta(\xi) = \mu'(\mp R) - \int_{-\delta}^{\delta} o(\xi)\psi_\delta'(\xi)\ d\xi.$$

The last term of the right hand side on the above equality tends to 0 as $\delta \to 0$. To see this take an arbitrary $\varepsilon > 0$. There exists $\delta_0 > 0$ such that if $|\xi| < \delta_0$ then $|o(\xi)| \le \varepsilon|\xi|$. Therefore if $0 < \delta < \delta_0$, then

$$\left|\int_{-\delta}^{\delta} o(\xi)\psi_\delta'(\xi)\ d\xi\right| \le \varepsilon \int_{-\delta}^{\delta} |\xi\psi_\delta'(\xi)|\ d\xi = \varepsilon.$$

Thus we obtain the claim of (i).

Next, let us assume that

$$\liminf_{R\to\infty, R\in\mathbb{D}} \mu'(R) = \alpha > 0.$$

Then there exists $R_0 > 0$ such that $\mu'(R) > \alpha/2$ whenever $R \in \mathbb{D} \cap (R_0, \infty)$. Since the function μ is nondecreasing, we have

$$\begin{aligned}\frac{1}{R}\mathbb{E}m(\mathbb{T}^N \times [0,T) \times [R, 2R)) &= \frac{1}{R}(\mu(2R) - \mu(R))\\ &\ge \frac{1}{R}\int_R^{2R} \mu'(\xi)\ d\xi > \frac{\alpha}{2}.\end{aligned}$$

This contradicts the limit (2.1). □

An L^1-contraction property (2.4) is proved in the similar method as in [14], since a kinetic measure in the sense of Definition 2.1 satisfies the limits (i), (ii) in Lemma 2.1. In particular, the kinetic solution to the problem (1.1), (1.2) is unique.

3. Existence

Let $u_0 \in L^2(\Omega, \mathscr{F}_0, dP; L^1(\mathbb{T}^N))$. For each $n \in \mathbb{N}$, define

$$u_{0,n}(x) = \max\{-n, \min\{u_0, n\}\}.$$

Then by the result of [5] there exist kinetic solutions u_n with initial data $u_{0,n}$ such that $u_n \in L^p(\Omega; C([0,T); L^p(\mathbb{T}^N)))$ for any $p \in [1, \infty)$. Moreover, since u_n satisfy the L^1-contraction property (2.4), $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega \times (0,T) \times \mathbb{T}^N)$. Now we define u as the limit of $\{u_n\}$ in the sense of $L^1(\Omega \times (0,T) \times \mathbb{T}^N)$-norm. We will show in this section that u is a kinetic solution.

Proposition 3.1. *Let u be the limit function defined as above. Then it satisfies the inequality*

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T)} \|u(t)\|_{L^1(\mathbb{T}^N)}^2 \le C(1+\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^N)}^2), \tag{3.1}$$

where C is a constant depending on T. Besides, there exists a mapping m from Ω to the set of non-negative Radon measures over $\mathbb{T}^N \times [0,T) \times \mathbb{R}$ such that m satisfies the conditions (i), (iii) in Definition 2.1 and a pair (u,m) satisfies the kinetic formulation (2.3).

Proof. Let θ, Θ be functions on $\mathbb{R}$ defined by

$$\theta(\xi) = \mathbf{1}_{-1\le\xi\le 0}, \quad \Theta(\xi) = \int_{-1}^{\xi}\int_{-1}^{\zeta} \theta(r)\ drd\zeta,$$

and let $\{\chi_i\}_{i\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be a sequence of functions such that $\chi_i(\xi)=1$ if $|\xi|\le i$, $\chi_i(\xi)=0$ if $|\xi|\ge 2i$, $|\chi_i'(\xi)|\le C/i$ with some constant C. We take $\Theta'(\xi)\chi_i(\xi)$ as a test function in the kinetic formulation (2.3) which approximate solutions u_n satisfy. Note that

$$\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+(u_n(x,t),\xi)\Theta'(\xi)\chi_i(\xi)\ d\xi dx \ge \frac{1}{2}\int_{\mathbb{T}^N} u_n^+(x,t)\wedge 2i\ dx$$

and that by the assumption (A_2)

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\mathbb{T}^N} G^2(x,u_n)(\theta(u_n)\chi_i(u_n)+\Theta'(u_n)\chi_i'(u_n))\ dxds \\
&\quad\le C + \frac{C}{i}\int_0^t\int_{\mathbb{T}^N} |u_n(x,s)|^2\mathbf{1}_{|u_n|\le 2i}\ dxds \\
&\quad\le C + C\int_0^t\int_{\mathbb{T}^N} |u_n(x,s)|\wedge 2i\ dxds,
\end{aligned}$$

where C is a constant depending on T. Then since we know that kinetic measures μ_n corresponding to the kinetic solutions u_n satisfy

$$\mathbb{E}\int_{\mathbb{T}^N\times[0,T)\times\mathbb{R}} |\xi|^p\ d\mu_n(x,t,\xi) \le C_{p,n}, \tag{3.2}$$

for any $p\ge 1$ (see [5, Section 4.1.2]), by letting $i\to+\infty$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^N} u_n^+(x,t)\ dx + \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \theta(\xi)\ d\mu_n \\
&\quad\le C + \|u_0\|_{L^1(\mathbb{T}^N)} + C\int_0^t\int_{\mathbb{T}^N} |u_n(x,s)|\ dxds \\
&\qquad + \sum_{k=1}^{\infty}\int_0^t\int_{\mathbb{T}^N} g_k(x,u_n)\Theta'(u_n)\ dxd\beta_k(s),
\end{aligned} \tag{3.3}$$

for a.e. $t \in [0,T)$. Since μ_n is a positive measure, we can drop $\int \theta \, d\mu_n$. Taking the square and expectation, for a.e. $t \in [0,T)$

$$\begin{aligned}
&\mathbb{E}\left|\int_{\mathbb{T}^N} u_n^+(x,t)\, dx\right|^2 \\
&\qquad \le C + C\mathbb{E}\|u_0\|^2_{L^1(\mathbb{T})} + C\mathbb{E}\int_0^T \left(\int_{\mathbb{T}^N} |u_n(x,s)|\, dx\right)^2 ds.
\end{aligned} \tag{3.4}$$

To get the above inequality, we calculated the stochastic integral term as following: by the Burkholder - Davis - Gundy inequality and the assumption (1.4),

$$\begin{aligned}
&\mathbb{E}\left|\sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^N} g_k(x,u_n)\Theta'(u_n)\, dx d\beta_k(s)\right|^2 \\
&\qquad \le C\mathbb{E}\int_0^T \sum_{k=1}^\infty \left|\int_{\mathbb{T}^N} g_k(x,u_n)\Theta'(u_n)\, dx\right|^2 ds \\
&\qquad \le C\mathbb{E}\int_0^T \left|\int_{\mathbb{T}^N} (1+|u_n(x,s)|)\Theta'(u_n)\, dx\right|^2 ds \\
&\qquad \le C + C\mathbb{E}\int_0^T \left(\int_{\mathbb{T}^N} |u_n(x,s)|\, dx\right)^2 ds.
\end{aligned} \tag{3.5}$$

Note that if we replace f_+ with f_- in (2.3), it also holds. Therefore, in a similar manner, we obtain the inequality (3.4) for u^-. Thus using the Gronwall inequality, we get

$$\mathbb{E}\|u_n(t)\|^2_{L^1(\mathbb{T}^N)} \le C(1+\mathbb{E}\|u_0\|^2_{L^1(\mathbb{T}^N)}). \tag{3.6}$$

Then from Lebesgue's convergence theorem, u also satisfies (3.6).

Next, we show that there exists a mapping m which satisfies the conditions (i), (iii) in Definition 2.1. Fix $R \in \mathbb{N}$. Since $u_n \in L^p(\Omega; C([0,T); L^p(\mathbb{T}^N)))$ for $p \in [1,\infty)$ and μ_n satisfies (3.2), we can take $\varphi_R(\xi) = \xi^+ \wedge R$ (resp. $\varphi_R(\xi) = \xi^- \vee (-R)$) as a test function in the kinetic formulation (2.3) (resp. (2.3) for f_-). Note that $\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+ \varphi_R$ is positive. Taking the square and expectation, for a.e. $t \in [0,T)$,

$$\begin{aligned}
&\mathbb{E}|\mu_n(\mathbb{T}^N \times [0,t) \times [0,R))|^2 \\
&\le C\Bigg(\mathbb{E}\left|\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_+(u_{0,n}(x),\xi)\varphi_R(\xi)\, d\xi dx\right|^2 \\
&\qquad + \mathbb{E}\left|\sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^N} g_k(x,u_n(x,s))\varphi_R(u_n(x,s))\, dx d\beta_k(s)\right|^2 \\
&\qquad\qquad + \mathbb{E}\left|\int_0^t \int_{\mathbb{T}^N} G^2(x,u_n(x,s))\partial_\xi \varphi_R(u_n(x,s))\, dx ds\right|^2\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Bigg(R^2\mathbb{E}\|u_{0,n}^+\|_{L^1(\mathbb{T}^N)}^2+\mathbb{E}\int_0^T\left|\int_{\mathbb{T}^N}G^2(x,u_n(x,s))\varphi_R(u_n(x,s))\,dx\right|^2ds\\
&\qquad\qquad+\mathbb{E}\left|\int_0^T\int_{\mathbb{T}^N}(1+|u_n(x,s)|^2)\partial_\xi\varphi_R(u_n(x,s))\,dxds\right|^2\Bigg)\\
&\leq C\left(R^2\mathbb{E}\|u_{0,n}^+\|_{L^1(\mathbb{T}^N)}^2+1+\mathbb{E}\int_0^T\|u_n(s)\|_{L^1(\mathbb{T}^N)}^2ds\right)\\
&\leq C_R(1+\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^N)}^2) \qquad (3.7)
\end{aligned}
$$

$$
\left(\text{resp.}\quad \mathbb{E}|\mu_n(\mathbb{T}^N\times[0,t)\times(-R,0])|^2\leq C_R(1+\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^N)}^2)\right).
$$

Thus for each $R\in\mathbb{N}$, $\{\mu_n\}_{n\in\mathbb{N}}$ is bounded in $L_w^2(\Omega;\mathcal{M}_b(A_R))$, where $A_R=\mathbb{T}^N\times[0,T)\times(-R,R)$, $\mathcal{M}_b(A_R)$ is the space of finite Borel measures on A_R with the total variational norm and $L_w^2(\Omega;\mathcal{M}_b(A_R))$ is the space of weakly measurable mappings μ such that $\mathbb{E}\|\mu\|_{\mathcal{M}_b(A_R)}^2$. Since $\mathcal{M}_b(A_R)$ is the topological dual space of $C_c(A_R)$ and is separable, the space $L_w^2(\Omega;\mathcal{M}_b(A_R))$ is the topological dual space of $L^2(\Omega;C_c(A_R))$ (see [8, Theorem 8.20.3]). Thus, for any $R\in\mathbb{N}$ there exists a mapping $m_R\in L_w^2(\Omega;\mathcal{M}_b(A_R))$ such that up to subsequence, $\mu_n\rightharpoonup m_R$ in $L_w^2(\Omega;\mathcal{M}_b(A_R))$-weak. By a diagonal argument, we obtain for $R\in\mathbb{N}$ $m_R=m_{R+1}$ in $L_w^2(\Omega;\mathcal{M}_b(A_R))$ and the convergence in all the spaces $L_w^2(\Omega;\mathcal{M}_b(A_R))$-weak of a single subsequence still denoted (μ_n). Let us then set $m=m_R$ on A_R, a.s. The conditions (i), (iii) in Definition 2.1 and kinetic formulation (2.3) are stable by weak convergence, hence they are satisfied by m.

Finally, letting $n\to\infty$ in (3.3) and taking square, supremum w.r.t. t on $[0,T)$ and expectation, we obtain

$$
\begin{aligned}
&\mathbb{E}\operatorname*{ess\,sup}_{t\in[0,T)}\|u(t)\|_{L^1(\mathbb{T}^N)}^2\\
&\qquad\leq C+C\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^N)}^2+C\int_0^T\|u(s)\|_{L^1(\mathbb{T}^N)}^2\,ds\\
&\qquad\quad+C\mathbb{E}\operatorname*{ess\,sup}_{t\in[0,T)}\left|\sum_{k=1}^\infty\int_0^T\int_{\mathbb{T}^N}g_k(x,u)\Theta'(u)\,dxd\beta_k(s)\right|. \qquad (3.8)
\end{aligned}
$$

With (3.6) for u in hand, in a same spirit as (3.5), we get the inequality (3.1). $\square$

Proposition 3.2. *Let u be the limit function defined in the beginning of this section and let m be defined in the proof of Proposition 3.1. Then m satisfies the condition (ii) in Definition 2.1. Therefore together with the result of Proposition 3.1, m is a kinetic measure and u is a kinetic solution to (1.1), (1.2). Moreover, u satisfies the* *following equi-integrability condition:*

$$\lim_{R\to\infty} \mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T)} \left(\int_{\mathbb{T}^N} (u(x,t) \mp R)^{\pm} \, dx \right)^2 = 0. \tag{3.9}$$

In particular, P-a.s., $u \in C([0,T); L^1(\mathbb{T}^N))$.

Proof. For $R > 0$, let θ_R, Θ_R be functions on $\mathbb{R}$ defined by

$$\theta_R(u) = \frac{1}{R}\mathbf{1}_{R<u<2R}, \quad \Theta_R(u) = \int_0^u \int_0^r \theta_R(s) \, dsdr. \tag{3.10}$$

Take $\Theta'_R(\xi)$ as a test function in (2.3). Note that

$$(\xi - 2R)^+ \le \Theta_R(\xi) \le (\xi - R)^+ \tag{3.11}$$

and that by the assumption (A_2)

$$\int_0^T \int_{\mathbb{T}^N} G^2(x,u)\theta_R(u) \, dxds \le C \int_0^T \int_{\mathbb{T}^N} (1+|u|)\mathbf{1}_{R\le u} \, dxds. \tag{3.12}$$

Then we have for a.e. $t \in [0,T)$

$$\begin{aligned}
&\int_{\mathbb{T}^N} (u(x,t) - 2R)^+ \, dx + \frac{1}{R} m(\mathbb{T}^N \times [0,t] \times [R,2R]) \\
&\quad \le \int_{\mathbb{T}^N} (u_0(x) - R)^+ \, dx + \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^N} g_k(x,u)\Theta'_R(u) \, dxd\beta_k(s) \\
&\qquad + C \int_0^t \int_{\mathbb{T}^N} (1+|u|)\mathbf{1}_{R\le u} \, dxds
\end{aligned} \tag{3.13}$$

Then dropping the first term in the left hand side, taking expectation and letting $t \uparrow T$, we obtain

$$\begin{aligned}
&\frac{1}{R}\mathbb{E} m(\mathbb{T}^N \times [0,T) \times [R,2R]) \\
&\qquad \le \mathbb{E} \int_{\mathbb{T}^N} (u_0(x) - R)^+ \, dx + C\mathbb{E} \int_0^T \int_{\mathbb{T}^N} (1+|u|)\mathbf{1}_{R\le u} \, dxds.
\end{aligned} \tag{3.14}$$

Since the right hand side apparently goes to 0 as $R \to \infty$, m is a kinetic measure. On the other hand, dropping the second term in the left hand side in (3.13) and taking square, supremum w.r.t. t on $[0,T)$ and expectation, we have

$$\begin{aligned}
&\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T)} \left(\int_{\mathbb{T}^N} (u(x,t) - 2R)^+ \, dx \right)^2 \\
&\quad \le C\mathbb{E} \left(\int_{\mathbb{T}^N} (u_0(x) - R)^+ \, dx \right)^2 + C\mathbb{E} \int_0^T \left(\int_{\mathbb{T}^N} (1+|u|)\mathbf{1}_{R\le u} \, dx \right)^2 ds
\end{aligned} \tag{3.15}$$

Here we calculated the stochastic integral term in a same spirit as (3.5). Since by Proposition 3.1, in particular, $u \in L^2(\Omega \times [0,T); L^1(\mathbb{T}^N))$.

Finally, the equi-integrability condition (3.9) yields the pathwise continuity of kinetic solutions as in [7, Appendix, A.3]. Thus we obtain the conclusion. □

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