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TOPOLOGICAL LINEAR SUBSPACE OF $L_0(\Omega, \mu)$ FOR THE INFINITE MEASURE μ

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ABSTRACT. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We shall characterize the maximal topological linear subspace M_∞ of $L_0(\Omega, \mathcal{A}, \mu)$ in the case where $\mu(\Omega) = +\infty$. M_∞ is the truncated L_∞ space which is open and closed in $L_0(\Omega, \mathcal{A}, \mu)$. In the case where $\Omega = \mathbf{N}$(natural numbers), $\mu(A) = \sharp A =$ the cardinal number of A, the maximal linear subspace of $L_0(\mathbf{N}, \mu)$ is ℓ_∞.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $L_0 := L_0(\Omega, \mathcal{A}, \mu)$ be the space of all real valued μ-measurable functions on $(\Omega, \mathcal{A}, \mu)$. The topology of L_0 is given by the following translation invariant metric:

$$d_0(f, g) = \inf_{\alpha > 0} \arctan \{\alpha + \mu(\{\omega \in \Omega \mid |f(\omega) - g(\omega)| > \alpha\})\}$$

for $f, g \in L_0$. Then $d_0(f_n, f) \to 0$ if and only if f_n converges to f *in measure*. The metric space (L_0, d_0) is a topological additive group but not necessarily a topological linear space in the case where $\mu(\Omega) = +\infty$. In fact, the scalar multiplication in not necessarily continuous. We remark that if $\mu(\Omega) < +\infty$, then (L_0, d_0) is a topological linear space.

The aim of this paper is to characterize the maximal topological linear subspace M_∞ of (L_0, d_0) set theoretically and topologically in the case where $\mu(\Omega) = +\infty$. Similar problems are considered for the Shepp sequence space which is a topological metric additive group [2, 3, 4, 5]. We show that M_∞ is the truncated L_∞ space given in Section 3. Furthermore M_∞ is the open and closed subset of (L_0, d_0).

As a special case, we consider the case where $\Omega = \mathbf{N}$(natural numbers), $\mu(A) = \sharp A =$ the cardinal number of A. Then the maximal topological linear subspace of

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$L_0(\mathbf{N})$ is the well-known ℓ_∞, the Banach space of all bounded sequences, and ℓ_∞ is open and closed in $L_0(\mathbf{N})$.

In the case where $\mu(\Omega) = +\infty$, the metric topology d_0 of the convergence in measure is very strong. (L_0, d_0) induces the truncated L_∞ metric on the maximal topological linear subspace M_∞. If $f \notin M_\infty$, then on the one-dimensional subspace $I\!Rf := \{tf \mid t \in I\!R\}$, (L_0, d_0) induces the discrete topology. In particular (L_0, d_0) is not separable even if the measure space $(\Omega, \mathcal{A}, \mu)$ is separable.

2. The metric on $L_0(\Omega, \mu)$

Definition 2.1. ([1, Chapter 7]) Let $\{f_n\} \subset L_0$ be a sequence of μ-measurable real functions. Then $\{f_n\}$ converges in measure to a μ-measurable function $f \in L_0$ if and only if for every positive $\varepsilon > 0$,

$$\mu\left(|f_n - f| > \varepsilon\right) = \mu\left(\{\omega \in \Omega \mid |f_n(\omega) - f(\omega)| > \varepsilon\}\right) \to 0 (n \to +\infty).$$

The convergence in measure on L_0 is characterized by the metric $d_0(f, g)$:

$$d_0(f, g) = \inf_{\alpha > 0} \arctan\left\{\alpha + \mu\left(|f - g| > \alpha\right)\right\},$$

where $\arctan x : [0, +\infty] \to [0, \frac{\pi}{2}]$ is the inverse function of $\tan\theta : [0, \frac{\pi}{2}] \to [0, +\infty]$.

Lemma 2.1. ([1, 7.1, Excercise 9]) *d_0 is a metric on L_0. Furthermore we have f_n converges in measure to f if and only if $d_0(f_n, f) \to 0$.*

Lemma 2.2. *(L_0, d_0) is a topological additive group.*

Proof. The metric d_0 is translation invariant, that is,

$$d_0(f + h, g + h) = d_0(f, g), \quad d_0(-f, 0) = d_0(f, 0) \quad \text{for} \quad f, g, h \in L_0.$$

So that the group operation

$$L_0 \times L_0 \to L_0, \quad (f, g) \to f - g$$

is continuous. In fact, if $d_0(f_n, f) \to 0$ and $d_0(g_n, g) \to 0$, then we have $d_0(f_n - g_n, f - g) = d_0(f_n - f, g_n - g) \leqq d_0(f_n, f) + d_0(g_n, g) \to 0 (n \to +\infty)$. □

Remark. In the case where $\mu(\Omega) < +\infty$, the following metric determines also the same topology on L_0:

$$\inf_{\alpha > 0}\left\{\alpha + \mu(|f - g| > \alpha)\right\}.$$

3. Truncated L_∞ subspace M_∞ of L_0

For a subset $D \in \mathcal{A}$, the restricted L_∞ seminorm, $\|f\|_{L_\infty(D)}$ is defined by

$$\|f\|_{L_\infty(D)} = \mathrm{Min}\{r \mid \mu(\{\omega \in D \mid |f(\omega)| > r\}) = 0\} = \mathrm{ess.sup}\{f(\omega) \mid \omega \in D\}.$$

Define $|f|_\infty$ by

$$|f|_\infty = \inf_{A \in \mathcal{A}} \left\{ \mu(A) + \|f\|_{L_\infty(\Omega \setminus A)} \right\}.$$

Then $|f|_\infty$ is characterized as follows.

Lemma 3.1.

$$|f|_\infty = \inf_{\alpha > 0} \{\alpha + \mu(|f| > \alpha)\}.$$

Proof. First we show the inequality $|f|_\infty \leqq \inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\}$. If $\inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\} = +\infty$, the inequality is clear. So assume that $k = \inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\} < +\infty$. Then for every $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon) > 0$ such that $\alpha + \mu(|f| > \alpha) < k + \varepsilon$. It follows that $\mu(|f| > \alpha) < k + \varepsilon - \alpha$. We set $A = \{\omega \in \Omega \mid |f(\omega)| > \alpha\}$, then we have

$$\mu(A) < k + \varepsilon - \alpha, \quad \text{and} \quad \|f\|_{L_\infty(\Omega \setminus A)} \leqq \alpha,$$

and it follows that

$$\mu(A) + \|f\|_{L_\infty(\Omega \setminus A)} < k + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have $|f|_\infty \leqq k = \inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\}$.

Next we show the converse inequality $|f|_\infty \geqq \inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\}$. If $|f|_\infty = +\infty$, then the inequality is clear. Assume $\ell = |f|_\infty < +\infty$. Then for every $\varepsilon > 0$, there exists $A = A(\varepsilon) \in \mathcal{A}$ such that $\mu(A) + \|f\|_{L_\infty(\Omega \setminus A)} < \ell + \varepsilon$, which implies $\|f\|_{L_\infty(\Omega \setminus A)} < \ell + \varepsilon - \mu(A)$. Consequently we have $\mu(\{\omega \in \Omega \setminus A \mid |f| > \ell + \varepsilon - \mu(A)\}) = 0$, and

$$\begin{aligned} \ell + \varepsilon &= \ell + \varepsilon - \mu(A) + \mu(A) + \mu(\{\omega \in \Omega \setminus A \mid |f| > \ell + \varepsilon - \mu(A)\}) \\ &\geqq \ell + \varepsilon - \mu(A) + \mu(\{\omega \in \Omega \mid |f| > \ell + \varepsilon - \mu(A)\}). \end{aligned}$$

So that we have $\inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\} \leqq \ell + \varepsilon$. Letting $\varepsilon \downarrow 0$, it follows that $\inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\} \leqq \ell = |f|_\infty$. □

Definition 3.1. We call the set $M_\infty = \{f \in L_0 \mid |f|_\infty < +\infty\}$ the truncated L_∞ space and $|f|_\infty$ the truncated L_∞ metric on M_∞.

Remark. By Lemma 3.1, we have $M_\infty = \{f \in L_0 \mid \inf_{\alpha>0} \{\alpha + \mu(|f| > \alpha)\} < +\infty\} = \{f \in L_0 \mid d_0(f, 0) < \frac{\pi}{2}\}$.

Theorem 3.2. *$|f - g|_\infty$ is a translation invariant metric on M_∞. The metric topology $|f - g|_\infty$ is equivalent to the induced topology from (L_0, d_0) on M_∞.*

Proof. Let $f, g \in M_\infty$. Then for every $\varepsilon > 0$ there exists $A, B \in \mathcal{A}$ such that

$$\mu(A) + \|f\|_{L_\infty(\Omega\setminus A)} < |f|_\infty + \varepsilon,\ \mu(B) + \|g\|_{L_\infty(\Omega\setminus B)} < |g|_\infty + \varepsilon.$$

This implies

$$\begin{aligned} \mu(A\cup B) + \|f-g\|_{L_\infty(\Omega\setminus A\cup B)} &\leq \{\mu(A) + \|f\|_{L_\infty(\Omega\setminus A)}\} + \{\mu(B) + \|g\|_{L_\infty(\Omega\setminus B)}\} \\ &< |f|_\infty + |g|_\infty + 2\varepsilon. \end{aligned}$$

Consequently we have $|f-g|_\infty \leq |f|_\infty + |g|_\infty + 2\varepsilon$. Letting $\varepsilon \downarrow 0$, we have the triangle inequality. By Lemma 3.1, two metrics $|f-g|_\infty$ and d_0 define the same topology on M_∞. In fact we have $|f-g|_\infty < \varepsilon$ if and only if $d_0(f,g) < \arctan\varepsilon$. □

Lemma 3.2. *M_∞ is a linear subspace of L_0.*

Proof. For $f, g \in M_\infty$, it follows that $f-g \in M_\infty$ since $|f-g|_\infty \leqq |f|_\infty + |g|_\infty < +\infty$ (Theorem 3.2). This means that M_∞ is an additive group. For every real number c and $f \in M_\infty$, by

$$\begin{aligned} |cf|_\infty &= \inf_{A\in\mathcal{A}} \left\{\mu(A) + \|cf\|_{L_\infty(\Omega\setminus A)}\right\} \\ &= \inf_{A\in\mathcal{A}} \left\{\mu(A) + |c|\|f\|_{L_\infty(\Omega\setminus A)}\right\} \\ &\leqq (|c| \vee 1)|f|_\infty < +\infty, \end{aligned}$$

where $a \vee b = \mathrm{Max}\{a, b\}$, we have $cf \in M_\infty$. □

Remark. We have $L_\infty(\Omega,\mu) \subset M_\infty$. If $\mu(\Omega) < +\infty$ then we have $M_\infty = L_0$.

Lemma 3.3. *M_∞ is an open and closed subset of (L_0, d_0).*

Proof. By Lemma 3.1 we have $M_\infty = \{f \in L_0 \mid d_0(f, 0) < \frac{\pi}{2}\}$. So that M_∞ is an open subset of (L_0, d_0).

Let $f \in L_0$ be arbitrary element in the closure $\overline{M}_\infty$ of M_∞ in (L_0, d_0). Then there exist $f_n \in M_\infty, n = 1, 2, \cdots$, such that $d_0(f_n, f) \to 0 (n \to +\infty)$, in particular, there exists N such that $d_0(f_N, f) < \frac{\pi}{2}$. By the definition of d_0, there exists an $\alpha > 0$ such that $\alpha + \mu(|f_N - f| > \alpha) < +\infty$. By Lemma 3.1, it follows that $|f_N - f|_\infty < +\infty$ and consequently we have $f = (f - f_N) + f_N \in M_\infty$. □

Theorem 3.3. *Assume $f \notin M_\infty$. Then the metric d_0 induces the discrete topology on the one-dimensional subspace $I\!\!Rf = \{tf \mid t \in I\!\!R\}$.*

Proof. We show that for every $s, t \in I\!\!R$ with $s \neq t$, $d_0(sf, tf) = \frac{\pi}{2}$. Since $f \notin M_\infty$, for every $A \in \mathcal{A}$ it holds that $\mu(A) + \|f\|_{L_\infty(\Omega\setminus A)} = +\infty$. If there exists $R > 0$ such that

$$\mu(|s-t||f| > R) < +\infty$$

then putting $B = \{|s-t||f| > R\}$, it follows that $\mu(B) < +\infty$ and $|s-t||f| \leqq R$ on $\Omega \backslash B$, which implies

$$\|f\|_{L_\infty(\Omega\backslash B)} \leqq \frac{R}{|s-t|} < +\infty.$$

Consequently it follows that $\mu(B) + \|f\|_{L_\infty(\Omega\backslash B)} < +\infty$, which contradicts to $f \notin M_\infty$. So that for every $R > 0$, we have $\mu(|s-t||f| > R) = \infty$ and hence

$$d_0(sf, tf) = \inf_{\varepsilon>0} \arctan\{\varepsilon + \mu(|sf - tf| > \varepsilon)\} = \arctan(+\infty) = \frac{\pi}{2}.$$

□

Example. For the Lebesgue measure μ on $\Omega = I\!R$, the function $f(x) = x$ does not belong to M_∞ by Lemma 3.1.

Lemma 3.4. *Assume that $f \in M_\infty$. Then for every $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that*

$$\mu(A) < \varepsilon, \ \mu(A) + \|f\|_{L_\infty(\Omega\backslash A)} < +\infty.$$

Proof. By the condition $|f|_\infty < +\infty$, there exists $B \in \mathcal{A}$ such that

$$\mu(B) + \|f\|_{L_\infty(\Omega\backslash B)} < +\infty.$$

Now we set $B_n := \{\omega \in B \mid |f(\omega)| \geq n\}$. Since $B \supset B_n \downarrow \phi$ and $\mu(B) < +\infty$, there exists a natural number N such that $\mu(B_N) < \varepsilon$. Then we have

$$\begin{aligned} \|f\|_{L_\infty(\Omega\backslash B_N)} &\leq \|f\|_{L_\infty(\Omega\backslash B)} + \|f\|_{L_\infty(B\backslash B_N)} \\ &\leq \|f\|_{L_\infty(\Omega\backslash B)} + N < +\infty. \end{aligned}$$

So that the subset $A := B_N$ satisfies the required properties. □

Theorem 3.4. *M_∞ is the maximal topological linear subspace of L_0.*

Proof. (1) M_∞ is a topological additive group by Lemma 2.2 and Theorem 3.2.
(2) The continuity of the scalar multiplication is proved as follows. Assume $|t_n - t_0| \to 0$, $|f_n - f_0|_\infty \to 0$. We shall prove that $|t_n f_n - t_0 f_0|_\infty \to 0$. Since $\{t_n\}$ is a bounded real sequence, we can assume also $|t_n| \leq K < +\infty$. We have

$$|t_n f_n - t_0 f_0|_\infty \leq |t_n(f_n - f_0)|_\infty + |(t_n - t_0) f_0|_\infty.$$

The first term is estimated as

$$\begin{aligned} |t_n(f_n - f_0)|_\infty &= \inf_{A\in\mathcal{A}} \{\mu(A) + \|t_n(f_n - f_0)\|_{L_\infty(\Omega\backslash A)}\} \\ &\leq \inf_{A\in\mathcal{A}} \{\mu(A) + K\|f_n - f_0\|_{L_\infty(\Omega\backslash A)}\} \\ &\leq (K \vee 1) \inf_{A\in\mathcal{A}} \{\mu(A) + \|f_n - f_0\|_{L_\infty(\Omega\backslash A)}\} \\ &= (K \vee 1)|f_n - f_0|_\infty \to 0. \end{aligned}$$

We show the second term also converges to 0. For every $\varepsilon > 0$ by Lemma 3.4 there exists $A_\varepsilon \in \mathcal{A}$ such that

$$\mu(A_\varepsilon) < \varepsilon, \ \mu(A_\varepsilon) + \|f_0\|_{L_\infty(\Omega\setminus A_\varepsilon)} < +\infty.$$

So that we have

$$\begin{aligned} |(t_n - t_0)f_0|_\infty &\leq \mu(A_\varepsilon) + \|(t_n - t_0)f_0\|_{L_\infty(\Omega\setminus A_\varepsilon)} \\ &= \mu(A_\varepsilon) + |t_n - t_0|\|f_0\|_{L_\infty(\Omega\setminus A_\varepsilon)}. \end{aligned}$$

It follows that $\lim_{n\to+\infty} |(t_n - t_0)f_0|_\infty \leq \mu(A_\varepsilon) \leq \varepsilon$, that is, $\lim_{n\to+\infty} |(t_n - t_0)f_0|_\infty = 0$.
(3) The maximality of M_∞: Let S be a topological linear subspace of (L_0, d_0). We show that $S \subset M_\infty$. Let $f \in S$, then by the continuity of the scalar multiplication in S, we have

$$\frac{1}{n} f \to 0 \text{ in } S \ \text{ (and in } L_0).$$

By Lemma 3.3 there exists N such that $\frac{1}{N} f \in M_\infty$, which implies $f \in M_\infty$. □

Remark. Assume $f \in L_0$. Then we have $f \in M_\infty$ if and only if $\frac{1}{n} f \to 0$ in (L_0, d_0).

Theorem 3.5. $L_\infty(\Omega) = M_\infty$ *if and only if* $\inf\{\mu(A) \mid \mu(A) > 0\} > 0$.

Proof. Assume that $L_\infty(\Omega) = M_\infty$. Assume also that $\inf\{\mu(A) \mid \mu(A) > 0\} = 0$, that is, there exists $A_n \in \mathcal{A}$ such that $\mu(A_n) > 0, \mu(A_n) \to 0$. We can assume that $\{A_n\}$ is disjoint, $\mu(A_n) > 0$, $\sum_n \mu(A_n) < +\infty$. We consider the mapping

$$\varphi : I\!\!R^\infty \to L_0(\Omega, \mu), \quad \varphi(\mathbf{a}) := \sum_n a_n \chi_{A_n}(\omega).$$

By $\sum_n \mu(A_n) < +\infty$, we have $\varphi(I\!\!R^\infty) \subset M_\infty$. On the other hand for $\mathbf{a} = \{a_n\}, a_n := n$, we have $\varphi(\mathbf{a}) \notin L_\infty(\Omega)$, so that $L_\infty(\Omega) \neq M_\infty$ ($L_\infty(\Omega)$ is a proper subset of M_∞).
Conversely assume that $\alpha := \inf\{\mu(A) \mid \mu(A) > 0\} > 0$. Take arbitrary $f \in M_\infty$. Then for every $0 < \varepsilon < \alpha$, there exists $A_\varepsilon \in \mathcal{A}$ such that

$$\mu(A_\varepsilon) < \varepsilon, \ \|f\|_{L_\infty(\Omega\setminus A_\varepsilon)} < +\infty.$$

Since $0 < \varepsilon < \alpha$, we have $\mu(A_\varepsilon) = 0$ and $\|f\|_{L_\infty(\Omega)} < +\infty$, which shows $f \in L_\infty(\Omega)$. □

4. $L_0(\mathbf{N})$

In this section we consider the case where $\Omega = \mathbf{N}$(natural numbers), $\mu(A) = \sharp A =$ the cardinal number of A. Denote by $L_0(\mathbf{N})$ for $L_0(\mathbf{N}, \mu)$. Remark that $L_0(\mathbf{N}) =$

$I\!\!R^\infty$(= the set of all real sequences) as a set. For $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in L_0(\mathbf{N})$ the metric d_0 is given by

$$d_0(\mathbf{a}, \mathbf{b}) = \inf_{\alpha>0} \arctan\{\alpha + \sharp\{k \mid |a_k - b_k| > \alpha\}\}.$$

In this case it follows that $M_\infty = \ell_\infty$.

Lemma 4.1. *The basis of neighborhoods of* 0 *in* $L_0(\mathbf{N})$ *is given by the following subsets:*

$$V_\varepsilon = \{\mathbf{a} \in L_0(\mathbf{N}) \mid \|\mathbf{a}\|_{\ell_\infty} < \varepsilon\}, \quad 0 < \varepsilon < 1.$$

Proof. Assume $0 < \varepsilon < 1$. We have

$$d_0(\mathbf{a}, \mathbf{0}) < \varepsilon \iff \inf_{\alpha>0} \arctan\{\alpha + \sharp\{k \mid |a_k| > \alpha\}\} < \varepsilon.$$

If $\arctan \sharp\{k \mid |a_k| > \alpha\}\} < \varepsilon < 1$ then $\sharp\{k \mid |a_k| > \alpha\} = 0$. So that in this case, we have $|a_k| \leq \alpha$ for every k and $\mathbf{a} \in \ell_\infty$. Furthermore we have

$$\inf_{\alpha>0} \arctan\{\alpha + \sharp\{k \mid |a_k| > \alpha\}\} = \|\mathbf{a}\|_{\ell_\infty},$$

which shows

$$d_0(\mathbf{a}, \mathbf{0}) < \varepsilon \iff d_0(\mathbf{a}, \mathbf{0}) = \|\mathbf{a}\|_{\ell\infty} < \varepsilon.$$

□

Lemma 4.2. ℓ_∞ *is an open and closed subset of* $(L_0(\mathbf{N}), d_0)$.

Proof. See Lemma 3.3. □

Theorem 4.1. *Assume* $\mathbf{a} \notin \ell_\infty$. *Then the metric* d_0 *induces the discrete topology on the one-dimensional subspace* $I\!\!R\mathbf{a} = \{t\mathbf{a} \mid t \in I\!\!R\}$.

Proof. Since $\mathbf{a} \notin \ell_\infty$, for every $\varepsilon > 0$ and every $s \neq t$, $\sharp\{k \mid |s - t||a_k| > \varepsilon\} = +\infty$. Consequently it holds that $d_0(s\mathbf{a}, t\mathbf{a}) = \arctan(+\infty) = \frac{\pi}{2}$. □

Theorem 4.2. ℓ_∞ *is the maximal topological linear subspace of* $L_0(\mathbf{N})$.

Proof. See Theorem 3.4. □

Remark. The convergence of s sequence $\{\mathbf{a}^{(n)}\}$ in $(L_0(\mathbf{N}), d_0)$ is as follows:

$\mathbf{a}^{(n)} \to \mathbf{a}^{(0)}$

$\iff$

$\exists N$; $\mathbf{a}^{(n)} - \mathbf{a}^{(0)} \in \ell_\infty(n \geq N)$ and $\|\mathbf{a}^{(n)} - \mathbf{a}^{(0)}\|_{\ell_\infty} \to 0(n \to +\infty)$.

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