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POINTWISE MULTIPLIERS ON MUSIELAK-ORLICZ SPACES

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Dedicated to Professor Kichi-Suke Saito on his 65th birthday

Abstract. We consider the pointwise multipliers on Musielak-Orlicz spaces. We treat a wide class of Musielak-Orlicz spaces with generalized Young functions which include quasi-normed spaces.

1. Introduction

Let (Ω, μ) be a complete σ-finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Let E_1 and E_2 be subspaces of $L^0(\Omega)$. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2, if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$.

In this paper we consider the pointwise multipliers on Musielak-Orlicz spaces $L^\Phi(\Omega)$. For the definitions and basic properties of Orlicz and Musielak-Orlicz spaces, see [3, 6, 8, 10], etc. For $p \in (0, \infty]$, we denote by $L^p(\Omega)$ the usual Lebesgue spaces. Then it is known that

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega),$$

if $1/p_1 + 1/p_3 = 1/p_2$ ($p_i \in [1, \infty]$, $i = 1, 2, 3$). This result was extended to Orlicz spaces by [4, 5]. Our results in this paper are their further extension. We treat a wide class of Musielak-Orlicz spaces with generalized Young functions which include

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quasi-normed spaces, for example, generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot):\Omega\to(0,\infty]$. For the space $L^{p(\cdot)}(\Omega)$, see [1, 2], for example.

Note that Hölder's inequality implies the inclusion

$$L^{p_3}(\Omega)\subset \mathrm{PWM}(L^{p_1}(\Omega),L^{p_2}(\Omega)),$$

if $1/p_1+1/p_3=1/p_2$. In this paper we also use a generalized Hölder's inequality for Musielak-Orlicz spaces (Proposition 3.4). However, the reverse inclusion is non-trivial. Actually, in our result on Musielak-Orlicz spaces (Theorem 4.1), the difficulty is in the proof of the reverse inclusion.

Our proof method is the same as in [4]. However, we must adapt the method to Musielak-Orlicz spaces. To do this we first investigate the properties of Young functions and their generalization in Section 2. Next, in Section 3, we give several examples and state propositions and lemmas on Musielak-Orlicz spaces with generalized Young functions. Then we state the main results in Section 4 and prove them in Section 5.

2. Young functions and their generalization

Let $\bar{\varPhi}$ be the set of all functions $\Phi:[0,\infty]\to[0,\infty]$ such that

$$\lim_{t\to+0}\Phi(t)=\Phi(0)=0\quad\text{and}\quad\lim_{t\to\infty}\Phi(t)=\Phi(\infty)=\infty. \tag{2.1}$$

Let

$$a(\Phi)=\sup\{t\ge 0:\Phi(t)=0\},\quad b(\Phi)=\inf\{t\ge 0:\Phi(t)=\infty\}.$$

Definition 2.1. A function $\Phi\in\bar{\varPhi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is nondecreasing on $[0,\infty)$ and convex on $[0,b(\Phi))$, and

$$\lim_{t\to b(\Phi)-0}\Phi(t)=\Phi(b(\Phi))\ (\le\infty).$$

We denote by $\varPhi_Y$ the set of all Young functions. Any Young function is neither identically zero nor identically infinity on $(0,\infty)$. We define three subsets of Young functions $\mathcal{Y}^{(i)}$ $(i=1,2,3)$ as

$$\begin{aligned}\mathcal{Y}^{(1)}&=\{\Phi\in\varPhi_Y:b(\Phi)=\infty\},\\ \mathcal{Y}^{(2)}&=\{\Phi\in\varPhi_Y:b(\Phi)<\infty,\ \Phi(b(\Phi))=\infty\},\\ \mathcal{Y}^{(3)}&=\{\Phi\in\varPhi_Y:b(\Phi)<\infty,\ \Phi(b(\Phi))<\infty\}.\end{aligned}$$

Then we have the following properties of $\Phi\in\varPhi_Y$:

(i) If $\Phi \in \mathcal{Y}^{(1)}$, then Φ is absolutely continuous on any closed interval in $[0,\infty)$, by the convexity and nondecreasingness, and Φ is bijective from $[a(\Phi),\infty)$ to $[0,\infty)$.

(ii) If $\Phi \in \mathcal{Y}^{(2)}$, then Φ is absolutely continuous on any closed interval in $[0,b(\Phi))$, and Φ is bijective from $[a(\Phi),b(\Phi))$ to $[0,\infty)$.

(iii) If $\Phi \in \mathcal{Y}^{(3)}$, then Φ is absolutely continuous on $[0,b(\Phi)]$ and Φ is bijective from $[a(\Phi),b(\Phi)]$ to $[0,\Phi(b(\Phi))]$.

Next we recall the generalized inverse of Young function Φ in the sense of O'Neil [9, Definition 1.2]. For a Young function Φ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \inf\{t \ge 0 : \Phi(t) > u\}, \tag{2.2}$$

where $\inf \emptyset = \infty$. Then $\Phi^{-1}(u)$ is finite for all $u \in [0,\infty)$. If Φ is bijective from $[0,\infty)$ to itself, then Φ^{-1} is the usual inverse function of Φ.

We have the following properties of $\Phi \in \mathit{\Phi}_Y$ and its inverse:

(P1) $\Phi(\Phi^{-1}(u)) \le u$ for all $u \in [0,\infty)$ and $t \le \Phi^{-1}(\Phi(t))$ if $\Phi(t) \in [0,\infty)$ (Property 1.3 in [9]).

(P2) $\Phi^{-1}(\Phi(t)) = t$ if $\Phi(t) \in (0,\infty)$.

(P3) If $\Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Phi(\Phi^{-1}(u)) = u$ for all $u \in [0,\infty)$.

(P4) If $\Phi \in \mathcal{Y}^{(3)}$ and $0 < \delta < 1$, then there exists a Young function $\Psi \in \mathcal{Y}^{(2)}$ such that $b(\Phi) = b(\Psi)$ and

$$\Psi(\delta t) \le \Phi(t) \le \Psi(t) \quad \text{for all } t \in [0,\infty).$$

To see (P4) we only set $\Psi = \Phi + \Theta$, where we choose $\Theta \in \mathcal{Y}^{(2)}$ such that $a(\Theta) = \delta\, b(\Phi)$ and $b(\Theta) = b(\Phi)$.

Definition 2.2. Let $\mathit{\Phi}_Y^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(x,\cdot)$ is a Young function for every $x \in \Omega$, and that $\Phi(\cdot,t)$ is measurable on Ω for every $t \in [0,\infty]$. Assume also that, for any subset $A \subset \Omega$ with finite measure, there exists $t \in (0,\infty)$ such that $\Phi(\cdot,t)\chi_A$ is integrable.

Definition 2.3. (i) Let $\mathit{\Phi}_{GY}$ be the set of all $\Phi \in \bar{\mathit{\Phi}}$ such that $\Phi((\cdot)^{1/\ell})$ is in $\mathit{\Phi}_Y$ for some $\ell \in (0,1]$.

(ii) Let $\mathit{\Phi}_{GY}^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(\cdot,(\cdot)^{1/\ell})$ is in $\mathit{\Phi}_Y^v$ for some $\ell \in (0,1]$.

For $\Phi, \Psi \in \bar{\mathit{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in (0,\infty).$$

For $\Phi, \Psi : \Omega \times [0,\infty] \to [0,\infty]$, we also write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(x, C^{-1}t) \le \Psi(x,t) \le \Phi(x, Ct) \quad \text{for all } (x,t) \in \Omega \times (0,\infty).$$

Definition 2.4. Let $\bar{\varPhi}_Y$, $\bar{\varPhi}_Y^v$, $\bar{\varPhi}_{GY}$ and $\bar{\varPhi}_{GY}^v$ be the sets of all Φ such that $\Phi \approx \Psi$ for some Ψ in $\varPhi_Y$, $\varPhi_Y^v$, $\varPhi_{GY}$ and $\varPhi_{GY}^v$, respectively.

For $\Phi \in \bar{\varPhi}_{GY}^v$, let

$$a(\Phi; x) = \sup\{t \ge 0 : \Phi(x,t) = 0\}, \quad b(\Phi; x) = \inf\{t \ge 0 : \Phi(x,t) = \infty\}.$$

From the property (P4) we have the following:

(P5) For any $\Phi \in \varPhi_{GY}^v$ and $0 < \delta < 1$, there exists $\Psi \in \varPhi_{GY}^v$ such that $\Psi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for all $x \in \Omega$ and for some $\ell \in (0,1]$, and

$$\Psi(x, \delta t) \le \Phi(x,t) \le \Psi(x,t) \quad \text{for all } (x,t) \in \Omega \times [0,\infty).$$

To see (P5) we only set $\Psi = \Phi + \Theta$, where we choose $\Theta(x,t)$ by the following way: If $\Phi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Theta(x, \cdot) \equiv 0$. If $\Phi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(3)}$, then $\Theta(x, \cdot) \in \mathcal{Y}^{(2)}$ such that $a(\Theta; x) = \delta\, b(\Phi; x)$ and $b(\Theta; x) = b(\Phi; x)$.

At the end of this section we give a lemma.

Lemma 2.1. *Let $\Phi \in \varPhi_{GY}^v$. For a subset $A \subset \Omega$ with $0 < \mu(A) < \infty$, let $\Phi^A(t) = \int_A \Phi(x,t)\, d\mu(x)$. Then $\Phi^A \in \varPhi_{GY}$.*

Proof. By the definition of $\varPhi_{GY}^v$ we have that $\Phi^A(t) < \infty$ for some $t \in (0,\infty)$. Assume that $\Phi(\cdot, (\cdot)^{1/\ell}) \in \varPhi_Y^v$ for some $\ell \in (0,1]$. Then by the properties of Young function and the Lebesgue dominated convergence and monotone convergence theorems, we see that $\Phi^A((\cdot)^{1/\ell})$ is a Young function. $\square$

3. Musielak-Orlicz spaces

In this section we define Musielak-Orlicz spaces $L^\Phi(\Omega)$ for $\Phi \in \bar{\varPhi}_{GY}^v$ and give their properties.

Definition 3.1 (Musielak-Orlicz space)**.** For a function $\Phi \in \bar{\varPhi}_{GY}^v$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(x, c|f(x)|)\, d\mu(x) < \infty \text{ for some } c > 0 \right\},$$

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

Then $|f(x)| < \infty$ a.e. $x \in \Omega$ for all $f \in L^\Phi(\Omega)$. By the assumption in Definition 2.2 all simple functions are in $L^\Phi(\Omega)$. Moreover, $\|\cdot\|_{L^\Phi}$ is a quasi-norm, that is, there exists $\kappa \in [1,\infty)$ such that, for all $f, g \in L^\Phi(\Omega)$ and a scalar c,

(i) $\|f\|_{L^\Phi} \geq 0$, $\|f\|_{L^\Phi} = 0 \Leftrightarrow f = 0$,
(ii) $\|cf\|_{L^\Phi} = |c|\|f\|_{L^\Phi}$,
(iii) $\|f+g\|_{L^\Phi} \leq \kappa(\|f\|_{L^\Phi} + \|g\|_{L^\Phi})$.

If $\Phi \in \mathit{\Phi}_{GY}^v$ and $\Phi(\cdot,(\cdot)^{1/\ell}) \in \mathit{\Phi}_Y^v$, then

(iv) $\|f+g\|_{L^\Phi}^\ell \leq \|f\|_{L^\Phi}^\ell + \|g\|_{L^\Phi}^\ell$.

If $\Phi \in \mathit{\Phi}_Y^v$, then $\|\cdot\|_{L^\Phi}$ is a norm.

The following is clear:

(v) If $g \in L^\Phi(\Omega)$ and $|f(x)| \leq |g(x)|$ a.e. $x \in \Omega$, then $f \in L^\Phi(\Omega)$ and $\|f\|_{L^\Phi} \leq \|g\|_{L^\Phi}$.

The property (v) is called the *lattice property* or *ideal property*.

Let $\Phi \in \mathit{\Phi}_{GY}^v$. Then by the left-continuity of $\Phi(x,t)$ with respect to t and the theory of the Lebesgue integral we have the following:

(vi) If $\liminf_{j\to\infty} \|f_j\|_{L^\Phi} < \infty$ and $\lim_{j\to\infty} f_j = f$ a.e. Ω, then $f \in L^\Phi(\Omega)$ and $\|f\|_{L^\Phi} \leq \liminf_{j\to\infty} \|f_j\|_{L^\Phi}$.
(vii) If $\sup_j \|f_j\|_{L^\Phi} < \infty$, $0 \leq f_1 \leq f_2 \leq \cdots \to f$ a.e. Ω, then $f \in L^\Phi(\Omega)$ and $\lim_{j\to\infty} \|f_j\|_{L^\Phi} = \|f\|_{L^\Phi}$.

The properties (vi) and (vii) are called the *Fatou property*.

Let $\Phi, \Psi \in \bar{\mathit{\Phi}}_{GY}^v$. If $\Phi \approx \Psi$, then $L^\Phi(\Omega) = L^\Psi(\Omega)$ with equivalent quasi-norms. If there exist $t_0, t_1 \in (0,\infty)$ such that

$$\Phi(x,t) = \Psi(x,t) \quad \text{for} \quad (x,t) \in \Omega \times ((0,t_0] \cup [t_1,\infty)),$$

then $\Phi \approx \Psi$.

In the following examples we always interpret $\Phi(x,0) = 0$ and $\Phi(x,\infty) = \infty$ for all $x \in \Omega$.

Example 3.1. Let $p \in (0,\infty]$ and $\Phi(x,t) = t^p$. Then $L^\Phi(\Omega)$ is the usual Lebesgue space $L^p(\Omega)$. Here we use the following interpretation:

$$t^\infty = \begin{cases} 0, & t \in [0,1], \\ \infty, & t \in (1,\infty]. \end{cases}$$

Example 3.2. Let $p \in (0, \infty]$ and

$$\Phi(x,t) = \begin{cases} 1/\exp(1/t^p), & t \in [0,1], \\ \exp(t^p), & t \in (1,\infty]. \end{cases}$$

Here we use the following interpretation:

$$\begin{cases} 1/\exp(1/t^\infty) = 0, & t \in [0,1], \\ \exp(t^\infty) = \infty, & t \in (0,\infty]. \end{cases}$$

Note that, if $p \in (0,\infty)$, then we can choose a convex function E_p such that $E_p(t) = 1/\exp(1/t^p)$ for small t and $E_p(t) = \exp(t^p)$ for large t, that is, $\Phi \in \bar{\varPhi}_Y$. In this case we denote $L^\Phi(\Omega)$ by $\exp(L^p)(\Omega)$.

Example 3.3. Let p be a variable exponent, that is, it is a measurable function defined on Ω valued in $(0,\infty]$, and let $\Phi(x,t) = t^{p(x)}$. If $p_- \equiv \inf_{x\in\Omega} p(x) > 0$, then $\Phi \in \varPhi_{GY}^v$ and $\Phi(x, (\cdot)^{\max(1,1/p_-)}) \in \varPhi_Y^v$. In this case we denote $L^\Phi(\Omega)$ by $L^{p(\cdot)}(\Omega)$ which is a generalized Lebesgue space with variable exponent p.

Example 3.4. Let w be a weight function, that is, it is a measurable function defined on Ω valued in $(0,\infty)$ a.e., and $\int_A w(x)\,d\mu(x) < \infty$ for any $A \subset \Omega$ with finite measure. Let p be a variable exponent, and let

$$\Phi(x,t) = t^{p(x)} w(x).$$

If $\inf_{x\in\Omega} p(x) > 0$, then $\Phi \in \varPhi_{GY}^v$. In this case we denote $L^\Phi(\Omega)$ by $L_w^{p(\cdot)}(\Omega)$.

Example 3.5. Let p be a variable exponent, and let

$$\Phi(x,t) = \begin{cases} 1/\exp(1/t^{p(x)}), & t \in [0,1], \\ \exp(t^{p(x)}), & t \in (1,\infty]. \end{cases}$$

If $\inf_{x\in\Omega} p(x) > 0$, then $\Phi \in \bar{\varPhi}_Y^v$. In this case we denote $L^\Phi(\Omega)$ by $\exp(L^{p(\cdot)})(\Omega)$.

Remark 3.1. In Examples 3.3, 3.4 and 3.5, let

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

If $\sup_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$, then there exists $\Psi \in \varPhi_{GY}^v$ such that $\Phi \approx \Psi$ and $\Psi^A((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for some $\ell \in (0,1]$ and for any $A \subset \Omega$ with $0 < \mu(A) < \infty$, where $\Psi^A(t) = \int_A \Psi(x,t)\,d\mu(x)$. To see this we have only to set

$$\Psi(x,t) = \begin{cases} \Phi(x,t), & x \in \Omega \setminus \Omega_\infty, \\ \Phi(x,t) + \Theta(t), & x \in \Omega_\infty, \end{cases}$$

where we choose $\Theta \in \mathcal{Y}^{(2)}$ such that $a(\Theta) = 1/2$ and $b(\Theta) = 1$.

By using the method in [1, pages 38–40] or [6, pages 35–36], we can prove the following proposition and lemma:

Proposition 3.1. *Let $\Phi \in \bar{\Phi}^v_{GY}$. Then $L^\Phi(\Omega)$ is complete.*

Lemma 3.2. *Let $\Phi \in \bar{\Phi}^v_{GY}$. If a sequence $\{f_j\}$ converges in $L^\Phi(\Omega)$ to f, then there exists a subsequence $\{f_{j(k)}\}$ which converges μ-almost everywhere to f.*

The next lemma follows from Lemma 3.2 and the closed graph theorem, see [7] for example. See also [11, Theorem 1 in page 79] for the closed graph theorem on complete quasi-normed spaces (F-spaces).

Lemma 3.3. *Let $\Phi_1, \Phi_2 \in \bar{\Phi}^v_{GY}$. Then every pointwise multiplier g from $L^{\Phi_1}(\Omega)$ to $L^{\Phi_2}(\Omega)$ is a bounded operator.*

For $\Phi \in \bar{\Phi}_{GY}$, we define its generalized inverse by the same way as (2.2). Then $\Phi^{-1}(u) = (\Psi^{-1}(u))^\ell$ if $\Psi(t) = \Phi(t^{1/\ell})$, and $\Psi^{-1}(u)/C \le \Phi^{-1}(u) \le C\Psi^{-1}(u)$ if $\Psi(t/C) \le \Phi(t) \le \Psi(Ct)$. For $\Phi \in \bar{\Phi}^v_{GY}$, we denote by Φ^{-1} the generalized inverse with respect to t. Then we give a proposition on a generalized Hölder's inequality, which can be proven in the same way as O'Neil [9].

Proposition 3.4. *Let $\Phi_i \in \bar{\Phi}^v_{GY}$, $i = 1, 2, 3$. Assume that there exists a constant $C > 0$ such that*

$$\Phi_1^{-1}(x,t)\Phi_3^{-1}(x,t) \le C\, \Phi_2^{-1}(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{3.1}$$

If $f \in L^{\Phi_1}(\Omega)$ and $g \in L^{\Phi_3}(\Omega)$, then $fg \in L^{\Phi_2}(\Omega)$ and

$$\|fg\|_{L^{\Phi_2}} \le C' \|f\|_{L^{\Phi_1}} \|g\|_{L^{\Phi_3}},$$

where C' is a positive constant dependent only on Φ_i, $i = 1, 2, 3$, and C.

4. Main results

In this section we state the main results. For $\Phi \in \bar{\Phi}^v_{GY}$, we denote by Φ^{-1} the generalized inverse with respect to t.

Theorem 4.1. *Let $\Phi_i \in \bar{\Phi}^v_{GY}$, $i = 1, 2, 3$. Assume that there exists a constant $C > 0$ such that*

$$\frac{1}{C}\Phi_2^{-1}(x,t) \le \Phi_1^{-1}(x,t)\Phi_3^{-1}(x,t) \le C\Phi_2^{-1}(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{4.1}$$

Assume also that there exists $\Psi_3 \in \varPhi_{GY}^v$ *such that*

$$\Phi_3 \approx \Psi_3 \quad \textit{and} \quad \Psi_3^A((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}, \tag{4.2}$$

for some $\ell \in (0,1]$ *and for any* $A \subset \Omega$ *with* $0 < \mu(A) < \infty$, *where* $\Psi_3^A(t) = \int_A \Psi_3(x,t)\,d\mu(x)$. *Then*

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$ *is comparable to* $\|g\|_{L^{\Phi_3}}$.

Remark 4.1. There exists $\Phi \in \varPhi_Y^v$ such that $\Phi(x,\cdot) \in \mathcal{Y}^{(1)}$ for all $x \in \Omega$ and $\Phi^\Omega \in \mathcal{Y}^{(3)}$. Actually, let $\Omega = (0,1)$ be the open interval in the real line with the Lebesgue measure and take Young functions $\Phi(x,t)$ for all $x \in \Omega$ such that $\Phi(x,1) = 1$ and $\Phi(x,1+x) = 2/x$. Then

$$\int_0^1 \Phi(x,1)\,dx = 1, \quad \int_0^1 \Phi(x,1+\epsilon)\,dx \geq \int_0^\epsilon \Phi(x,1+x)\,dx = \infty,$$

for any $\epsilon \in (0,1)$. In this case we can find $\Psi \in \varPhi_Y^v$ such that $\Phi \approx \Psi$ and $\Psi^A \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for any $A \subset (0,1)$. However, it is unknown whether we can take $\Psi \in \varPhi_{GY}^v$ which satisfies (4.2) for any $\Phi \in \bar{\varPhi}_{GY}^v$, in general.

Corollary 4.2. *Let* $\Phi \in \bar{\varPhi}_{GY}^v$. *Then*

$$\mathrm{PWM}(L^{\Phi}(\Omega)) = L^\infty(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{\Phi}(\Omega))$ *is comparable to* $\|g\|_{L^\infty}$.

Next we give three examples of Theorem 4.1, by using the properties in Examples 3.3, 3.4, 3.5, and Remark 3.1.

Example 4.1. Let p_i be variable exponents, $i = 1,2,3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\inf_{x\in\Omega} p_i(x) > 0$, $i = 1,2,3$, $\sup_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)} \quad \text{for} \quad x \in \Omega. \tag{4.3}$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}(\Omega), L^{p_2(\cdot)}(\Omega)) = L^{p_3(\cdot)}(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{p_1(\cdot)}(\Omega), L^{p_2(\cdot)}(\Omega))$ is comparable to $\|g\|_{L^{p_3(\cdot)}}$.

Example 4.2. Let p_i be variable exponents, w_i be weight functions, $i = 1, 2, 3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\inf_{x\in\Omega} p_i(x) > 0$, $i = 1, 2, 3$, $\sup_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}, \quad w_1(x)^{1/p_1(x)} w_3(x)^{1/p_3(x)} = w_2(x)^{1/p_2(x)} \quad \text{for} \quad x \in \Omega. \tag{4.4}$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}_{w_1}(\Omega), L^{p_2(\cdot)}_{w_2}(\Omega)) = L^{p_3(\cdot)}_{w_3}(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{p_1(\cdot)}_{w_1}(\Omega), L^{p_2(\cdot)}_{w_2}(\Omega))$ is comparable to $\|g\|_{L^{p_3(\cdot)}_{w_3}}$.

Example 4.3. Let p_i be variable exponents, $i = 1, 2, 3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\inf_{x\in\Omega} p_i(x) > 0$, $i = 1, 2, 3$, $\sup_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)} \quad \text{for} \quad x \in \Omega. \tag{4.5}$$

Then

$$\mathrm{PWM}(\exp(L^{p_1(\cdot)})(\Omega), \exp(L^{p_2(\cdot)})(\Omega)) = \exp(L^{p_3(\cdot)})(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\exp(L^{p_1(\cdot)})(\Omega), \exp(L^{p_2(\cdot)})(\Omega))$ is comparable to $\|g\|_{\exp(L^{p_3(\cdot)})}$.

Remark 4.2. In Examples 4.1, 4.2 and 4.3, the condition $\sup_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ is not necessary. We only need the condition (4.2).

5. Proof of main results

In this section we prove Theorem 4.1. From Proposition 3.4 it follows that

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) \supset L^{\Phi_3}(\Omega),$$

and that

$$\|g\|_{\mathrm{Op}} \le C\|g\|_{L^{\Phi_3}} \quad \text{for} \quad g \in L^{\Phi_3}(\Omega),$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of g as a pointwise multiplier.

Conversely, let $g \in \mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$. Then g is a bounded operator by Lemma 3.3. In the following we prove that g is in $L^{\Phi_3}(\Omega)$ and that

$$\|g\|_{L^{\Phi_3}} \le C\|g\|_{\mathrm{Op}}. \tag{5.1}$$

If $\Phi_i \approx \Psi_i$, $i = 1,2,3$, then Ψ_i also satisfy (4.1). Hence we may assume that $\Phi_i(\cdot,(\cdot)^{1/\ell_i}) \in \mathit{\Phi}_Y^v$ for some $\ell_i \in (0,1]$, $i = 1,2,3$. Moreover, by (P5) and the assumption, we may assume that $\Phi_2(x,(\cdot)^{1/\ell_2}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for all $x \in \Omega$ and that $\Phi_3^A((\cdot)^{1/\ell_3}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for any $A \subset \Omega$ with $0 < \mu(A) < \infty$, where $\Phi_3^A(t) = \int_A \Phi_3(x,t)\,d\mu(x)$.

To show (5.1) we consider two cases.

Case 1: g is a simple function. In this case $g \in L^{\Phi_3}(\Omega)$. Let

$$g = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad 0 < c_1 < c_2 < \cdots < c_N,$$
$$0 < \mu(A_k) < \infty \ (k = 1,2,\cdots,N), \text{ and } A_j \cap A_k = \emptyset \text{ if } j \neq k.$$

and let

$$\Phi_3^g(t) = \int_\Omega \Phi_3(x,|g(x)|t)\,d\mu(x), \quad \Phi_3^{A_k}(t) = \int_{A_k} \Phi_3(x,t)\,d\mu(x).$$

Then

$$\Phi_3^g(t) = \sum_{k=1}^{N} \Phi_3^{A_k}(c_k t).$$

Then $\Phi_3^g((\cdot)^{1/\ell_3}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ and

$$a(\Phi_3^g) = \min_k a(\Phi_3^{A_k}(\cdot/c_k)), \quad b(\Phi_3^g) = \min_k b(\Phi_3^{A_k}(\cdot/c_k)).$$

Therefore, $\Phi_3^g((\cdot)^{1/\ell_3})$ is continuous and convex on $[0, b(\Phi_3^g))$ and bijective from $[a(\Phi_3^g), b(\Phi_3^g))$ to $[0,\infty)$. Since

$$\|g\|_{L^{\Phi_3}} = \inf\{\lambda > 0 : \Phi_3^g(1/\lambda) \le 1\},$$

we have

$$\Phi_3^g(1/\|g\|_{L^{\Phi_3}}) = 1.$$

That is,

$$\int_\Omega \Phi_3\left(x, \frac{g(x)}{\|g\|_{L^{\Phi_3}}}\right) d\mu(x) = 1. \tag{5.2}$$

Let

$$h(x) = \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{\Phi_3}}}\right).$$

Then h is in $L^1(\Omega)$ and $h(x) < \infty$ a.e. $x \in \Omega$. Let

$$f(x) = \begin{cases} \Phi_1^{-1}(x,h(x)), & 0 < h(x) < \infty, \\ 0, & h(x) = 0. \end{cases}$$

From the property (P1) it follows that $\Phi_1(x, f(x)) \le h(x)$ a.e. $x \in \Omega$ and

$$\int_\Omega \Phi_1(x, f(x))\, d\mu(x) \le \int_\Omega \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{\Phi_3}}}\right) d\mu(x) = 1.$$

That is, $\|f\|_{L^{\Phi_1}} \le 1$. If $0 < h(x) < \infty$, then by the property (P2) and the assumption (4.1),

$$\begin{aligned}
\frac{f(x)g(x)}{\|g\|_{L^{\Phi_3}}} &= \Phi_1^{-1}(x, h(x))\Phi_3^{-1}\left(x, \Phi_3\left(x, \frac{g(x)}{\|g\|_{L^{\Phi_3}}}\right)\right) \\
&= \Phi_1^{-1}(x, h(x))\Phi_3^{-1}(x, h(x)) \\
&\ge C^{-1}\, \Phi_2^{-1}(x, h(x)),
\end{aligned}$$

and hence, by (P3),

$$\Phi_2\left(x, \frac{Cf(x)g(x)}{\|g\|_{L^{\Phi_3}}}\right) \ge \Phi_2(\Phi_2^{-1}(x, h(x))) = h(x).$$

If $h(x) = 0$, then $f(x) = 0$ and $\Phi_2\left(x, \frac{Cf(x)g(x)}{\|g\|_{L^{\Phi_3}}}\right) = 0$. Thus, by (5.2),

$$\begin{aligned}
\int_\Omega \Phi_2\left(x, \frac{Cf(x)g(x)}{\|g\|_{L^{\Phi_3}}}\right) d\mu(x) &\ge \int_\Omega h(x)\, d\mu(x) \\
&= \int_\Omega \Phi_3\left(x, \frac{g(x)}{\|g\|_{L^{\Phi_3}}}\right) d\mu(x) = 1.
\end{aligned}$$

Therefore,

$$\|g\|_{L^{\Phi_3}} \le C\|fg\|_{L^{\Phi_2}} \le C\|g\|_{\mathrm{Op}}\|f\|_{L^{\Phi_1}} \le C\|g\|_{\mathrm{Op}}.$$

That is, we have (5.1).
Case 2: For general g, let $\{g_j\}$ be a sequence of simple functions such that

$$0 \le g_1 \le g_2 \le \cdots \to |g| \text{ a.e. in } \Omega.$$

Then, by the result in Case 1 and the lattice property of $L^{\Phi_2}(\Omega)$ we have

$$\|g_j\|_{L^{\Phi_3}} \le C\|g_j\|_{\mathrm{Op}} \le C\|g\|_{\mathrm{Op}}.$$

By the Fatou property of $L^{\Phi_3}(\Omega)$ we have (5.1).

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